CH. 9: ∫ TECHNIQUES

9.1: ∫ BY PARTS (like a product rule for ∫s)

∫ product ≠ product of ∫s

Ex (#2) Evaluate $\underbrace{\int x \sin x \, dx}_{\text{Split into } u,\, dv \text{ — must include } dx}$

Book's general strategy: (Hope du is nice)
Make dv as big/complex as you can?

Strategy: $\int \underbrace{\begin{pmatrix}\text{Easy} \\ \text{to } D_x\end{pmatrix}}_{=u} \underbrace{\begin{pmatrix}\text{Biggest piece} \\ \text{that's easy to } \int\end{pmatrix}}_{=dv}$

x - would you rather ∫ or D_x

$$\int \underbrace{x}_{u} \underbrace{\sin x \, dx}_{dv}$$

Let $u = x$ · Let $dv = \sin x \, dx$ ∫s
$du = dx$ −∫. $v = -\cos x$

Don't need +C

$$\boxed{\int \text{by Parts}: \int u \, dv = uv - \int v \, du}$$

$$\begin{aligned}\int x \sin x \, dx &= -x\cos x - \int -\cos x \, dx \\ &= -x\cos x + \int \cos x \, dx \\ &= \boxed{-x\cos x + \sin x \underbrace{+ C}_{\text{Need!}}}\end{aligned}$$

Do 1

Can D_x to ✓.

What if... $\int \underbrace{x \underbrace{\sin x}_{u}}_{dv} dx$

$u = \sin x \qquad \cdot \qquad dv = x\,dx$

$du = \cos x\, dx \quad -\int. \quad v = \frac{x^2}{2}$

$\frac{1}{2}x^2 \sin x - \underbrace{\int \frac{1}{2} x^2 \cos x\, dx}_{\text{WORSE!}}$

x: You'd rather D_x than $\int$

sin x: OK to D_x or $\int$

Ex $\int \underbrace{x \underbrace{\ln x}_{u}}_{dv} dx$ Hard to $\int \ln x$!!

(We'll do this next.)

$u = \ln x \qquad \cdot \qquad dv = x\,dx$

$du = \frac{1}{x}dx \quad -\int. \quad v = \frac{x^2}{2}$

$$\begin{aligned}
\int x \ln x\, dx &= \tfrac{1}{2}x^2 \ln x - \int \tfrac{x^2}{2}\cdot\tfrac{1}{x}\,dx \\
&= \tfrac{1}{2}x^2 \ln x - \tfrac{1}{2}\int x\,dx \\
&= \tfrac{1}{2}x^2 \ln x - \tfrac{1}{2}\left(\tfrac{x^2}{2}\right) + C \\
&= \boxed{\tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2 + C}
\end{aligned}$$

Do 1

Ex 3, p. 458

Ex $\int \underbrace{\ln x}_{u} \underbrace{dx}_{dv}$

$$u = \ln x \qquad dv = dx$$
$$du = \frac{1}{x}dx \xrightarrow{-\int\cdot} v = x$$

$$\int \ln x \, dx = x \ln x - \int x\left(\tfrac{1}{x}\right)dx$$
$$= x \ln x - \int dx$$
$$= \boxed{x \ln x - x + C}$$

Doesn't help us develop $\int$ by parts

If we ever do

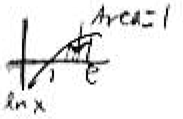
$$u = \qquad dv = \ln x \, dx$$
$$du = -\int\cdot \qquad v = x \ln x - x$$

YUK!

Ex $\int_1^e \ln x \, dx = \left[x \ln x - x\right]_1^e$

$$= [e \underbrace{\ln e}_{=1} - e] - [1 \underbrace{\ln 1}_{=0} - 1]$$
$$= \underbrace{[e - e]}_{=0} - [0 - 1]$$
$$= \boxed{1}$$

What do definite $\int$s corresp. to?



Good news and bad news...

Ex (#18) $\int \underbrace{e^{3x}}_{} \underbrace{\cos(2x)}_{u} \, dx$ ($e^{3x} \, dx$: dv)

CAN SKIP - WE'LL DO A SHORTCUT LATER!

$$u = \cos(2x) \qquad dv = e^{3x}dx$$
$$du = -\sin(2x) \cdot 2 \, dx$$
$$= -2\sin(2x)\,dx \xrightarrow{-\int\cdot} v = \tfrac{1}{3}e^{3x}$$

$$\int e^{3x}\cos(2x)\,dx = \tfrac{1}{3}e^{3x}\cos(2x) - \int\left(\tfrac{1}{3}e^{3x}\right)\left(-2\sin(2x)\right)dx$$

$$= \tfrac{1}{3}e^{3x}\cos(2x) + \tfrac{2}{3}\underbrace{\int e^{3x}\sin(2x)\,dx}_{\bigstar}$$

DO PARTS AGAIN!

$u = e^{3x}$ works, too vs Ex. 5

$$\bigstar = \int \underbrace{e^{3x}}_{dv}\underbrace{\sin(2x)}_{u}\underbrace{dx}_{dv}$$

$$u = \sin(2x) \qquad dv = e^{3x}\,dx$$
$$du = 2\cos(2x)\,dx \qquad v = \tfrac{1}{3}e^{3x}$$

$$\bigstar = \tfrac{1}{3}e^{3x}\sin(2x) - \int\left(\tfrac{1}{3}e^{3x}\right)\left(2\cos(2x)\right)dx$$
$$= \tfrac{1}{3}e^{3x}\sin(2x) - \tfrac{2}{3}\int e^{3x}\cos(2x)\,dx$$

≈ semi-circular, self-referential

$$\underbrace{\int e^{3x}\cos(2x)\,dx}_{I} = \tfrac{1}{3}e^{3x}\cos(2x) + \tfrac{2}{3}\Big[\tfrac{1}{3}e^{3x}\sin(2x) - \tfrac{2}{3}\underbrace{\int e^{3x}\cos(2x)\,dx}_{I}\Big]$$

$$I = \tfrac{1}{3}e^{3x}\cos(2x) + \tfrac{2}{9}e^{3x}\sin(2x) - \tfrac{4}{9}I$$

Solve for I

$$\tfrac{13}{9}I = \text{〃} \qquad \text{〃} + C$$
$$I = \tfrac{9}{13}\left[\tfrac{1}{3}e^{3x}\cos(2x) + \tfrac{2}{9}e^{3x}\sin(2x)\right] + C$$

$$I = \boxed{\tfrac{3}{13}e^{3x}\cos(2x) + \tfrac{2}{13}e^{3x}\sin(2x) + \underbrace{C}_{\text{Need!}}}$$

SHORTCUT: TABULAR $\int$ by PARTS ("TURBO METHOD")

<u>Ex</u> $\int x^3 \cos x \, dx$ (Need to use $\int$ by Parts more than once!)

Alternating Signs (Start w/ "+")	Successive Derivs. (poly. :))	Successive Antiderivs. ($\int$) (Don't need dx.)
$+$	(u=) x^3	$\cancel{\cos x}$ cross out later
$-$	$3x^2$	(times) $\sin x$
$+$	$6x$	$(-\cos x)$ so you remember it's "times"
$-$	6	$(-\sin x)$
$+$	0	$\cos x$

$\int$ (can stop table at any line)

$$x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x + \underbrace{\int 0 \cos x \, dx}_{\substack{\int \text{ of last line} \\ = 0 + C}}$$

$$\boxed{x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x + C}$$

<u>Old Ex</u> $\int x \ln x \, dx$

$+$	$\ln x$	$\cancel{x}$
$-$	$\frac{1}{x}$	$\frac{x^2}{2}$

$\int$

$$\frac{1}{2} x^2 \ln x - \int \frac{1}{x} \cdot \frac{x^2}{2} \, dx$$
$$= \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x \, dx$$

etc.

Old Ex $\int e^{3x}\cos(2x)\,dx$

$u = e^{3x}$ works, too vs. Ex 5

$+$	$\cos(2x)$	e^{3x}
$-$	$(-2\sin(2x))$	$\frac{1}{3}e^{3x}$
$+$	$(-4\cos(2x))$	$\frac{1}{9}e^{3x}$

$\int$ (Remainder 1)

Let's stop table here. (We'll see why.)

$\approx$ semi-circular, self-referential

$$\underbrace{\int e^{3x}\cos(2x)\,dx}_{\text{Call "I"}} = \frac{1}{3}e^{3x}\cos(2x) + \frac{2}{9}e^{3x}\sin(2x) - \frac{4}{9}\underbrace{\int e^{3x}\cos(2x)\,dx}_{I}$$

$$I = \frac{1}{3}e^{3x}\cos(2x) + \frac{2}{9}e^{3x}\sin(2x) - \frac{4}{9}I$$

Solve for I.
Add $\frac{4}{9}I$ to both sides.

$$\frac{13}{9}I = (\qquad\qquad) + C$$

$$I = \frac{9}{13}\left[\frac{1}{3}e^{3x}\cos(2x) + \frac{2}{9}e^{3x}\sin(2x)\right] + C$$

$\frac{9}{13}C \Rightarrow$ Just call "C"

$$\boxed{I = \frac{3}{13}e^{3x}\cos(2x) + \frac{2}{13}e^{3x}\sin(2x) + C}$$

Compare w/the long way, which I worked out!
(9.1.3-9.1.4)