9.2: TRIG $\int$s

(A) Review IDs

Pythagorean

$$\sin^2 x + \cos^2 x = 1$$

$\div \sin^2 x$ → $1 + \cot^2 x = \csc^2 x$

$\div \cos^2 x$ → $\tan^2 x + 1 = \sec^2 x$

$(^{\cup}{}_{\cap})^2$

Power-Reducing IDs (PRIs)

sin is bad

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

OK: neg or frac

(B) $\int \sin^m x \cos^n x \, dx$

Lawson +

We like the odd power-hurt ones we love Beat him up

low, + int. exp.

Case 1 (m and/or n is odd, >0)

Rip out

Rip apart the [easier] odd power.
if both odd

- Peel one $\sin x$ or $\cos x$ factor.
- Rewrite the rest using a Pyth. ID.

Let $u = \sin x$ or $\cos x$

Who was <u>not</u> peeled? "Other guy"

Then, $du = \pm$ (peeled) dx.

<u>Ex</u> $\int \sin^3 x \, dx \quad (n=0)$

$$= \int \underbrace{\sin^2 x}_{\text{Use Pyth. ID.}} \cdot \overbrace{\sin x}^{\text{Peel}} \, dx$$

$$= \int (1-\cos^2 x) \sin x \, dx$$

OK:
$u = -\cos x$
$du = \sin x \, dx$
but $u \leftarrow (-\cos x)$ at end

Let $u = \overbrace{\cos x}^{\text{Other guy}}$ "Embedded"

$du = -\sin x \, dx \Rightarrow \sin x \, dx = -du$

$$\left(= -\int (1-\cos^2 x)\underbrace{(-\sin x \, dx)}_{du} \right) \text{can skip}$$

$$= -\int (1-u^2) \, du \qquad \text{Note} \left(= \int (u^2-1) \, du \right)$$

"Switch Rule"

$$= -\left[u - \frac{u^3}{3} \right] + C$$

Am I done?

$$= -u + \frac{1}{3}u^3 + C$$

Sub back

$$= \boxed{-\cos x + \frac{1}{3}\cos^3 x + C}$$

Mathem gives $\frac{1}{12}(-9\cos x + \cos(3x))$ - another form

Ex (#4) $\int \cos^7 x\, dx$

How is $\cos^6 x$ related to $\cos^2 x$?

$$= \int \cos^6 x \cdot \overbrace{\cos x}^{\text{Peel}}\, dx$$

$$= \int \underbrace{(\cos^2 x)}_{\text{Use Pyth. ID.}}{}^3 \cos x\, dx$$

$$= \int (1-\sin^2 x)^3 \cos x\, dx$$

Let $u = \overbrace{\sin x}^{\text{Other guy}}$
$du = \cos x\, dx$

We care about exp!

If had $(1-u)^{10}$, can use $w = 1-u$, but not naturally from Pyth. IDs

$$= \int (1-u^2)^3\, du$$

Binomial Thm.:
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
a: (1), b: $(-u^2)$

Pascal's Δ

Row	
0	1
1	1 1
2	1 2 1
3	1 3 3 1

← coeffs.

$$= \int [(1)^3 + 3(1)^2(-u^2) + 3(1)(-u^2)^2 + (-u^2)^3]\, du$$

$$= \int [1 - \underbrace{3u^2}_{\text{or Guess-and-✓}} + 3u^4 - u^6]\, du$$

$$= u - \cancel{3}\left(\frac{u^3}{\cancel{3}}\right) + 3\left(\frac{u^5}{5}\right) - \frac{u^7}{7} + C$$

Sub back

$$= \boxed{\sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C}$$

80 is numerically irritating, but it doesn't hurt us. 1,000,000 OK

Ex $\int \sin^{\overset{\text{odd}}{5}} x \cos^{80} x \, dx$

$$= \int \sin^4 x \cos^{80} x \cdot \overbrace{\sin x}^{\text{peel}} \, dx$$

$$= \int \underbrace{(\sin^2 x)}_{\text{Use Pyth. ID.}}{}^2 \cos^{80} x \cdot \sin x \, dx$$

$$= \int (1-\cos^2 x)^2 \cos^{80} x \cdot \sin x \, dx$$

Let $u = \cos x$

$du = -\sin x \, dx$

$\Rightarrow \quad \sin x \, dx = -du$

$$= -\int (1-u^2)^2 u^{80} \, du$$

$$= -\int (1-2u^2+u^4) u^{80} \, du$$

$$= -\int (u^{80} - 2u^{82} + u^{84}) \, du$$

$$= -\left[\frac{u^{81}}{81} - 2\left(\frac{u^{83}}{83}\right) + \frac{u^{85}}{85}\right] + C$$

$$= -\frac{1}{81}u^{81} + \frac{2}{83}u^{83} - \frac{1}{85}u^{85} + C$$

Dangerous to flip signs at same time. I guess I can't walk + chew gum at same time.

Sub back.

$$= \boxed{-\frac{1}{81}\cos^{81} x + \frac{2}{83}\cos^{83} x - \frac{1}{85}\cos^{85} x + C}$$

Mathem → ugly constants!

Case 2 (m, n both even, ≥ 0)

Power-Reducing IDs

Use PRIs.

Ex $\int \cos^4 x \, dx$

$$= \int \underbrace{(\cos^2 x)}_{\text{Use PRI.}}{}^2 \, dx$$

$$= \int \left[\frac{1 + \cos(2x)}{2} \right]^2 dx$$

cos is the ∂_x of what?

$$= \frac{1}{4} \int [1 + 2\cos(2x) + \cos^2(2x)] \, dx$$

$$= \frac{1}{4} \left[x + \underbrace{\sin(2x)}_{\text{Guess-and-✓ OK}} + \int \underbrace{\cos^2(2x) \, dx}_{\text{PRI: } \cos^2 u = \frac{1+\cos(2u)}{2}} \right]$$

$u = (2x)$, $2u = (4x)$

$$= \frac{1}{4} x + \frac{1}{4} \sin(2x) + \frac{1}{4} \int \frac{1 + \cos(4x)}{2} \, dx$$

Try $\sin(4x)$ What's the ∂_x

$$= \frac{1}{4} x + \frac{1}{4} \sin(2x) + \frac{1}{8} \left[x + \frac{1}{4} \sin(4x) \right] + C$$

$$= \frac{1}{4} x + \frac{1}{4} \sin(2x) + \frac{1}{8} x + \frac{1}{32} \sin(4x) + C$$

$$= \boxed{\frac{3}{8} x + \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) + C}$$

World not just made up of sin, cos → tan, cot, Compa-Loompa? [uncertain reading]

We like his power, like use MI.

Ex 5, p. 466: $\int \tan^{\text{odd}} \sec^{\text{odd}}$ csc odd, though

Ⓒ More $\int$s

We like:		$\int \tan^m x \sec^n x \, dx$	$\int \cot^m x \csc^n x \, dx$
m odd, +	$u =$ Peel	$\sec x$ / $\sec x \tan x$	$\csc x$ / $(-) \csc x \cot x$
n even, +	$u =$ Peel	$\tan x$ / $\sec^2 x$	$\cot x$ / $(-) \csc^2 x$

Think: "tan is odd, sec is even"

$u =$ Other guy

$du = \underbrace{\sim}_{(-?)\text{ Peel}} dx$

Use Pyth. ID to make "u"s.

Like Ⓑ (sin, cos Exs from Ⓑ)

If m odd and n even $\int \underbrace{\tan^3 x}_{\text{Like better } 3<6} \sec^6 x \, dx$ Let u = other = sec x

Why we like sec even

Ex $\int \tan^{2/3} x \sec^{6} x \, dx$ (6: even, +)

Let $u = \tan x$.

$du = \sec^2 x \, dx$

$= \int \tan^{2/3} x \underbrace{\sec^4 x}_{\text{Use Pyth. ID to make "u"s ("tan x"s).}} \cdot \overbrace{\sec^2 x}^{\text{Peel}} \, dx$

Use: $\tan^2 x + 1 = \sec^2 x$.

$= \int \tan^{2/3} x (\sec^2 x)^2 \cdot \sec^2 x \, dx$

$= \int \tan^{2/3} x (\tan^2 x + 1)^2 \cdot \sec^2 x \, dx$

$u = \tan x$
$du = \sec^2 x\, dx$

$= \int u^{2/3} (u^2+1)^2\, du$

$= \int u^{2/3} (u^4 + 2u^2 + 1)\, du$

$= \int (u^{14/3} + 2u^{8/3} + u^{2/3})\, du$

$= \dfrac{u^{17/3}}{17/3} + 2\left(\dfrac{u^{11/3}}{11/3}\right) + \dfrac{u^{5/3}}{5/3} + C$

$= \boxed{\frac{3}{17}\tan^{17/3} x + \frac{6}{11}\tan^{11/3} x + \frac{3}{5}\tan^{5/3} x + C}$

Reminder:

		$\int \tan^m x \sec^n x\, dx$	$\int \cot^m x \csc^n x\, dx$
m	$u=$	$\sec x$	$\csc x$
odd, +	Peel	$\sec x \tan x$	$(-)\csc x \cot x$
n	$u=$	$\tan x$	$\cot x$
even, +	Peel	$\sec^2 x$	$(-)\csc^2 x$

Ex $\int \cot^{3} x \csc^7 x\, dx$

odd, +

Let $u = \csc x$.
$du = -\csc x \cot x\, dx$.

$= \int \underbrace{\cot^2 x} \csc^6 x \cdot \overbrace{\csc x \cot x}^{\text{peel}}\, dx$

Use Pyth. ID to make "u"s ("csc x"s)
Use: $1 + \cot^2 x = \csc^2 x$
$\Rightarrow \cot^2 x = \csc^2 x - 1$

Book throws around

$$= \int (\csc^2 x - 1)\csc^6 x \cdot \csc x \cot x \, dx$$

$$u = \csc x$$
$$du = -\csc x \cot x \, dx$$

or $\int (1-u^2) u^6 du$

$$= -\int (u^2 - 1) u^6 \, du$$

$$= -\int (u^8 - u^6) \, du$$

$$= -\left[\frac{u^9}{9} - \frac{u^7}{7}\right] + C$$

$$= -\frac{1}{9}u^9 + \frac{1}{7}u^7 + C$$

PRIs not nice: $\tan^2 x = \frac{1-\cos(2x)}{1+\cos(2x)}$ $\csc^2 x$ = recip of $\sin^2 x$ ID (right side) $\frac{2}{1-\cos(2x)}$

$$= \boxed{-\frac{1}{9}\csc^9 x + \frac{1}{7}\csc^7 x + C}$$

cot

$\int \tan^{n} x \, dx$ (n even, +) Peel $\tan^2 x$. Pyth. ID

csc

Parties

$\int \sec^{n} x \, dx$ (n odd, +) Parts!

If csc cos

If simp expr w/ tan, cot, csc

Trick: $\to$ sin, cos (if have mix of csc, cot, ...)