9.3: TRIG SUBS

Ⓐ Intro

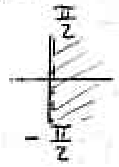
8.2: Proof $D_x(\sin^{-1}x)$ using Imp Diff.

Integrable, furthermore

Ex Show $\int \underbrace{\frac{1}{\sqrt{1-x^2}}} dx = \sin^{-1}x + C$

Domain $= (-1, 1)$ 
are "sin values"

θ in $(-\frac{\pi}{2}, \frac{\pi}{2})$
Range of $\sin^{-1} = [-\frac{\pi}{2}, \frac{\pi}{2}]$

Let $x = \sin\theta$, θ in 

Then, $\theta = ?$

$\Rightarrow \theta = \sin^{-1}x$

Sub: $x = \sin\theta$
$dx = \cos\theta \, d\theta$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \int \frac{1}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta$$

What's $\sqrt{x^2}$?

$\sqrt{\cos^2\theta}$ by Pyth. ID
$= |\cos\theta|$ (bec. $\sqrt{A^2} = |A|$)
$= \cos\theta$ $\quad \cos\theta > 0$

$$= \int \frac{1}{\cancel{\cos\theta}} \cancel{\cos\theta} \, d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

Sub back!

$$= \boxed{\sin^{-1}x + C}$$

Edwards has details, but uses diff ranges.

Area of ellipse Stewart 491

Set up ans on a date or $x = a\cosh\theta$ Stewart 493

Assume "a" real, +.

If the integrand has	Sub
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$
$\sqrt{a^2 + x^2}$ or $x^2 + a^2$	$x = a \tan\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$

Use a Pyth. ID.

Note

Really, sub rule requires $\theta = \sin^{-1}\left(\frac{x}{a}\right)$, $|x| \le a$ relevant to $\int$"

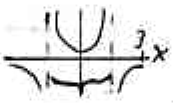
(Arc length!) θ in range of $\sin^{-1}$ $= [-\frac{\pi}{2}, \frac{\pi}{2}]$

Swokowski defines the range of $\csc^{-1}$ to be $[-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$ (p. 433, #32), so $\sin^{-1}\left(\frac{x}{a}\right) = \csc^{-1}\left(\frac{a}{x}\right)$ if $|x| \le a$ and $x \ne 0$.

However, $\sec^{-1}\left(\frac{x}{a}\right) \ne \cos^{-1}\left(\frac{a}{x}\right)$ for $x < -a$, because of the very different ranges of $\sec^{-1}$, $\cos^{-1}$. Also for $\tan^{-1}$, $\cot^{-1}$.)

Ex $\displaystyle\int \frac{1}{1-x^2}\,dx$

Domain = $\{\text{real } x \mid x \ne \pm 1\}$



~~$x = \sin\theta$~~

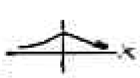
3 is not a sin value!

9.4

Ex $\displaystyle\int \frac{1}{1+x^2}\,dx$

Domain = $\mathbb{R}$

$x = \tan\theta$ ✓



Every real x is a "tan value."

Is 3 a tan value?

$\vdots$

$= \tan^{-1} x + C$

Ex $\displaystyle\int \frac{x}{1+x^2}\,dx$ Easier: $u = 1 + x^2$, $du = 2x\,dx$

In 9.3, we can ignore domain, range, | | issues - they work out!

(B) Exs

Helps w/ll

Ex (#6) $\displaystyle\int \frac{1}{x^3(\sqrt{x^2-25})}\,dx$ $\qquad \cancel{u = x^2 - 25}$?

$7 = (\sqrt{7})^2$

Integrand has $\sqrt{(x)^2-(5)^2}$

Sub: $x = 5\sec\theta$

$dx = 5\sec\theta\tan\theta\,d\theta$

$dx \to d\theta$?? NO!

$$= \int \frac{1}{(5\sec\theta)^3\underbrace{(\sqrt{(5\sec\theta)^2-25})}}(5\sec\theta\tan\theta\,d\theta)$$

can skip
$$\begin{aligned}
&= \sqrt{25\sec^2\theta-25}\\
&= \sqrt{25(\sec^2\theta-1)}\\
&= 5\sqrt{\tan^2\theta} && \text{(by Pyth. ID)}\\
&= 5\tan\theta && \text{(ignore | |)}
\end{aligned}$$

$d\theta$ OK: Just like $\int \frac{du}{u}$ OK

$$= \int \frac{\cancel{5}\sec\theta\,\cancel{\tan\theta}\,d\theta}{125\sec^3\theta\cdot\cancel{5}\,\cancel{\tan\theta}}$$

$$= \frac{1}{125}\int \frac{1}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{125}\int \underbrace{\cos^2\theta}_{\text{Use a PRI.}}\,d\theta$$

$$= \frac{1}{125}\int \frac{1+\cos(2\theta)}{2}\,d\theta$$

$$\left(= \frac{1}{250}\int (1+\cos(2\theta))\,d\theta\right)$$

$$= \frac{1}{250}\left[\theta + \frac{1}{2}\underbrace{\sin(2\theta)}_{\substack{\text{(Use a Double-Angle ID)} \\ = 2\sin\theta\cos\theta}}\right] + C$$

$$= \frac{1}{250}\left[\theta + \sin\theta\cos\theta\right] + C$$

Are we done? No way!

Go back to x!!

$\underline{\theta = ?}$

$$x = 5\sec\theta$$
$$\frac{x}{5} = \sec\theta$$
$$\theta = \sec^{-1}\left(\frac{x}{5}\right)$$ (Ignore domain, range issues.)

We want alg. exprs. wherever possible.

$\underline{\cos\theta = ?}$ $\quad \cos\left(\sec^{-1}\left(\frac{x}{5}\right)\right)$ not good enough!

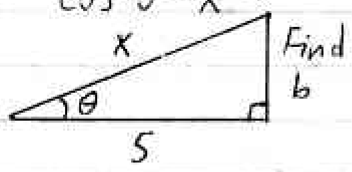
$$\sec\theta = \frac{x}{5}$$
$$\cos\theta = \frac{5}{x}$$

$\underline{\sin\theta = ?}$

Use 6: $x = a\sec\theta$

<u>Method 1</u> (Set up a right triangle.)

$\cos\theta = \frac{5}{x} \Rightarrow$



(You can assume θ is acute.)

Look familiar? Always works! Sometimes w/o tan sub, you intro √

$$5^2 + b^2 = x^2$$
$$b^2 = x^2 - 25$$
$$b = \pm\sqrt{x^2 - 25} \quad (\text{In } 1^{st} \int)$$

Is this $x-5$?

$$\sin\theta = \frac{b}{x} = \frac{\sqrt{x^2-25}}{x}$$

Method 2 (Pyth. ID)

SKIP

$$\sin^2\theta = 1-\cos^2\theta, \text{ Take "+" root.}$$
$$= 1-\left(\frac{5}{x}\right)^2$$
$$= 1-\frac{25}{x^2}$$
$$= \frac{x^2-25}{x^2}$$

Can assume: + + +

$$\sin\theta = \pm\sqrt{\frac{x^2-25}{x^2}}$$
$$= \frac{\sqrt{x^2-25}}{x}$$

$x>0$

$$\frac{1}{250}[\theta + \sin\theta\cos\theta] + C$$

$$= \frac{1}{250}\left[\sec^{-1}\left(\frac{x}{5}\right) + \frac{\sqrt{x^2-25}}{x}\cdot\frac{5}{x}\right] + C$$

Don't you miss u-subs now?

$$= \boxed{\frac{1}{250}\left[\sec^{-1}\left(\frac{x}{5}\right) + \frac{5\sqrt{x^2-25}}{x^2}\right] + C}$$

Don't have to rat'lize denoms.

Helps w/ I, 3 $\int\csc\theta\, d\theta$
10: $\int\sec\theta\, d\theta$

Ex (#18) $\displaystyle\int \frac{1}{x\sqrt{25x^2+16}}\,dx$ ~~$u = 25x^2+16$~~?

Integrand has $\sqrt{(5x)^2+(4)^2}$

sin, tan, sec
Not x =

Sub: $5x = 4\tan\theta$
$$x = \frac{4}{5}\tan\theta$$
$$dx = \frac{4}{5}\sec^2\theta\, d\theta$$

$$= \int \frac{1}{\left(\cancel{\tfrac{4}{5}}\tan\theta\right)\underbrace{\sqrt{(4\tan\theta)^2+16}}} \cdot \cancel{\tfrac{4}{5}}\sec^2\theta \, d\theta$$

can skip

$$\begin{aligned} &= \sqrt{16\tan^2\theta+16} \\ &= 4\sqrt{\tan^2\theta+1} \\ &= 4\sqrt{\sec^2\theta} \quad \text{(by Pyth. ID)} \\ &= 4\sec\theta \quad \text{(ignore | |)} \end{aligned}$$

OK: $\int \frac{du}{d\theta}$: $\int \frac{du}{u}$ OK

$$= \int \frac{\sec^2\theta \, d\theta}{\tan\theta \cdot 4\sec\theta}$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan\theta} \, d\theta$$

What's a strategy for dealing w/mixtures of sec, tan, ...

$$= \frac{1}{4}\int \frac{\frac{1}{\cancel{\cos\theta}}}{\frac{\sin\theta}{\cancel{\cos\theta}}} \, d\theta$$

$$= \frac{1}{4}\int \csc\theta \, d\theta$$

Know Pop Quiz!

$$= \frac{1}{4}\ln|\csc\theta - \cot\theta| + C$$

Go back to x!!

$$5x = 4\tan\theta$$

$$\tan\theta = \frac{5x}{4} \Rightarrow \cot\theta = \frac{4}{5x}$$

$\underline{\csc\theta = ?}$ $\quad \tan\theta = \frac{5x}{4} \Rightarrow$

What's recip of $\csc\theta$?

ln 1st 5 ↗

(or Pyth. Thm.)

$$\sin\theta = \frac{5x}{\sqrt{25x^2+16}}$$

$$\csc\theta = \frac{\sqrt{25x^2+16}}{5x}$$

$$= \boxed{\frac{1}{4}\ln\left|\frac{\sqrt{25x^2+16}}{5x} - \frac{4}{5x}\right| + C} \quad \text{or } \ln\sqrt[4]{\left|\frac{\sim}{5x}\right|} + C$$