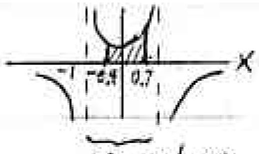
9.4: PARTIAL FRACTION DECOMPOSITIONS (PFDs)

Ⓐ $\int \frac{1}{1-x^2}\,dx$

Ex $\int_{-0.4}^{0.7} \frac{1}{1-x^2}\,dx$



sin values

$x = \sin\theta$ OK

SKIP

$$\text{Let } x = \sin\theta \Rightarrow \theta = \sin^{-1}x$$

in $(-\frac{\pi}{2}, \frac{\pi}{2})$

$$dx = \cos\theta\, d\theta$$

What does this integrand boil down to?

$$= \int_{\sin^{-1}(-0.4)}^{\sin^{-1}(0.7)} \underbrace{\frac{1}{1-\sin^2\theta} \cos\theta}_{=\sec\theta}\, d\theta$$

$$= \left[\ln|\sec\theta + \tan\theta|\right]_{\sin^{-1}(-0.4)}^{\sin^{-1}(0.7)}$$

$\vdots$

$$\approx 1.29095$$

Ex $\int_2^3 \frac{1}{1-x^2}\,dx$

We're stuck! Or Are we?

~~$x = \sin\theta$~~

<u>not</u> sin values

Ex Find an expression for $\int \frac{1}{1-x^2}\,dx$ that will work

$$= \int \underbrace{\frac{1}{(1+x)(1-x)}}_{\text{Find a pfd.}}\,dx$$

Factor

Stewart 497: proper rat'l = Σ (form $\frac{A}{(ax+b)^i}$) (form $\frac{Ax+B}{(\text{quad})^j}$)

$$\frac{1}{\underset{\nwarrow}{(1+x)}\underset{\nearrow}{(1-x)}} = \underbrace{\frac{A}{1+x} + \frac{B}{1-x}}_{\text{Form of pfd}}$$

Distinct linear factors

Find A, B

LCD

$\cdot$ thru by $(1+x)(1-x)$

Who's missing from denom.?

$$1 = A\overbrace{(1-x)} + B\overbrace{(1+x)}^{\text{Who's missing?}} \leftarrow \underline{\text{Basic Eq.}}$$

Method 1 (Expand)

$$1 = A - Ax + B + Bx$$

Combine like terms.

$$1 = (-Ax + Bx) + (A+B)$$

$$\underbrace{1}_{=0x+1} \Rightarrow = \underbrace{(-A+B)}_{\text{set }=0}x + \underbrace{(A+B)}_{\text{set }=1} \leftarrow \text{Equate coeffs. of like powers.}$$

$$\begin{cases} -A + B = 0 \\ A + B = 1 \end{cases}$$

$$2B = 1$$

$$B = \tfrac{1}{2}$$

$$\Rightarrow A = \tfrac{1}{2}$$

Method 2 (Pick convenient values of x.)

Basic Eq. (holds for all real x): $1 = A\underbrace{(1-x)}_{\substack{x=1\\ \text{makes}\\ \text{this } 0}} + B\underbrace{(1+x)}_{x=-1}$

Stewart 448 everywhere original eq. invalid: #67 F, G, Q polys, $\frac{F(x)}{Q(x)} = \frac{G(x)}{Q(x)}$ $\forall x$ except where $Q(x)=0$ $\Rightarrow$ Use continuity to prove $F(x) = G(x)$ $\forall x$

$F(a) = \lim_{x\to a} F(x) = \lim_{x\to a} G(x) = G(a)$

Plug in $x=1$: $1 = \cancel{A(0)} + B(2) \Rightarrow B = \frac{1}{2}$

$x=-1$: $1 = A(2) + \cancel{B(0)} \Rightarrow A = \frac{1}{2}$

pfd: $\dfrac{1}{(1+x)(1-x)} = \dfrac{\frac{1}{2}}{1+x} + \dfrac{\frac{1}{2}}{1-x}$

$$\int \frac{1}{(1+x)(1-x)}dx = \frac{1}{2}\Bigg(\underbrace{\int \frac{1}{1+x}dx}_{\substack{u_1 = 1+x \\ du_1 = dx}} + \underbrace{\int \frac{1}{1-x}dx}_{\substack{u_2 = 1-x \\ du_2 = -dx}}\Bigg)$$

Guess-and-✓ OK

$$= \frac{1}{2}\left(\int \frac{du_1}{u_1} - \int \frac{du_2}{u_2}\right)$$

Sol'n manual: +K

Do 1, 7, 9, NOT 5

$$= \frac{1}{2}(\ln|u_1| - \ln|u_2|) + \overbrace{K}^{\text{In case you've used } C}$$

diff of logs = log|quotient| Can ✓

$$= \boxed{\frac{1}{2}(\ln|1+x| - \ln|1-x|) + K}$$

$$\boxed{\text{or } \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + K}$$

$$\boxed{\text{or } \ln\sqrt{\left|\frac{1+x}{1-x}\right|} + K}$$

<u>Ex</u> $\int_2^3 \frac{1}{1-x^2}dx$

$$= \left[\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right]_2^3$$

$$= \frac{1}{2}\left(\ln\left|\frac{1+3}{1-3}\right| - \ln\left|\frac{1+2}{1-2}\right|\right)$$

$$= \frac{1}{2}\left(\ln\left|\frac{4}{-2}\right| - \ln\left|\frac{3}{-1}\right|\right)$$

$$= \frac{1}{2}(\ln 2 - \ln 3) \text{ or } \ln\sqrt{\tfrac{2}{3}}$$

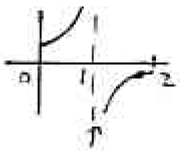
Why is this "−"?

$$\approx -0.202733$$

(<u>Ex</u> $\int_{-0.4}^{0.7} \frac{1}{1-x^2}dx \approx 1.29095$ (again))

Can we use the same technique?

<u>Ex</u> $\int_0^2 \frac{1}{1-x^2}dx$



Ch. 10

(B) Exs

Ex $\displaystyle\int \frac{-x^2-3x+32}{x^3-8x^2+16x}\,dx$ ← degree 2 ← 3.

2 < 3, so proper
If improper, use long ÷.

Factor denom.

$x(x^2-8x+16)$

$x(x-4)^2$

pfd Form: $\dfrac{-x^2-3x+32}{\underset{\text{linear factor}}{x}\,\underbrace{(x-4)^2}_{\substack{\text{repeated linear factors}\\ (x-4)(x-4)}}} = \dfrac{A}{x} + \underbrace{\dfrac{B}{x-4} + \dfrac{C}{(x-4)^2}}_{\text{run up to the power}}$

Find A, B, C

LCD

$\cdot$ thru by $x(x-4)^2$

Who's missing from denom.?

$-x^2-3x+32 = A(x-4)^2 + Bx(x-4) + Cx$

Like a cryptogram

Method 1 (Expand, combine like terms)

$-x^2-3x+32 = \underbrace{(A+B)}_{=-1}x^2 + \underbrace{(-8A-4B+C)}_{=-3}x + \underbrace{(16A)}_{=32}$

I skipped a lot of steps, but this is the overall schematic.

$A=2$

$B=-3$

$C=1$

Method 2 (Plug into x)

$$-x^2-3x+32 = A(x-4)^2+Bx(x-4)+Cx$$

$x=4$: $-16-12+32 = 0 + 0 + 4C$

$$4 = 4C$$

$$C=1$$

$x=0$: $-0-0+32 = A(0-4)^2 + 0 + 0$

$$32 = 16A$$

$$A=2$$

$$\underbrace{-x^2-3x+32}_{-x^2\ldots} = \underbrace{2(x-4)^2}_{2x^2\ldots} + \underbrace{\overset{?}{B}x(x-4)}_{Bx^2\ldots} + x$$

If $x \neq 0$ (arbitrary) $\Rightarrow \div x^2$

Equate x^2 terms:

$$-x^2 = 2x^2+Bx^2$$

$$-1 = 2 + B$$

$$B=-3$$

or Plug in $x=1$, say.
5?

pfd: $$\frac{-x^2-3x+32}{x(x-4)^2} = \frac{2}{x} + \frac{-3}{x-4} + \frac{1}{(x-4)^2}$$

$$\int \frac{-x^2-3x+32}{x(x-4)^2}\,dx = 2\int\frac{1}{x}\,dx - 3\underbrace{\int\frac{1}{x-4}\,dx}_{\substack{u=x-4 \\ \text{or Guess-and-}\checkmark}} + \underbrace{\int\frac{1}{(x-4)^2}\,dx}_{\substack{u=x-4 \\ du=dx}}$$

Even if "u"s diff, don't have to write u_1, u_2 (irritating)

$$= 2\ln|x| - 3\ln|x-4| + \underbrace{\int u^{-2}\,du}$$

$$= \frac{u^{-1}}{-1}+K$$

$$= -\frac{1}{x-4}+K$$

$$= \boxed{\begin{array}{l} 2\ln|x| - 3\ln|x-4| - \frac{1}{x-4} + K \\ \text{or } \ln\frac{x^2}{|x-4|^3} - \frac{1}{x-4} + K \end{array}}$$

Up to 11

$$\text{Ex} \quad \int \frac{2x^4-4x^3+40x^2-71x+82}{x^3-2x^2+16x-32}\,dx \quad \begin{array}{l}\leftarrow \text{deg } 4\\ \leftarrow \text{deg } 3\end{array}$$

Need for 16

$4 \geq 3$, so improper

Long $\div$

$$\begin{array}{r} 2x \\ x^3-2x^2+16x-32 \overline{\smash{\big)}\, 2x^4-4x^3+40x^2-71x+82} \\ \underline{-2x^4 + 4x^3 - 32x^2 + 64x} \\ 8x^2-7x+82 \end{array}$$

subtract (flip signs)

Now, add (combine like terms):

$\leftarrow$ deg 2

$2 < 3$ ✓

I hope you know this...

Recall: $4\overline{\smash{)}9}$ with quotient 2, -8, $R1$ $\Rightarrow 2\frac{1}{4}$ $\leftarrow$ remainder, $\leftarrow$ divisor

$$\int\left(2x + \frac{8x^2-7x+82}{x^3-2x^2+16x-32}\right)dx$$

Need for 17

Factor denom. by grouping

$$(x^3-2x^2)+(16x-32)$$

$$x^2(x-2)+16(x-2)$$

$$(x^2+16)(x-2)$$

Otherwise, use $\pm\frac{p}{q}$, synthetic $\div$.

Rational Roots Thm. / Rat'l Zeros Test

(Rat'l roots have form $\pm\frac{p}{q}$ $\leftarrow$ factor of const term, $\leftarrow$ factor of leading coeff.)

$$\int\left(2x + \underbrace{\frac{8x^2-7x+82}{(x^2+16)(x-2)}}_{\text{Find pfd.}}\right)dx$$

pfd form: $$\frac{8x^2-7x+82}{\underbrace{(x^2+16)}_{\text{irreducible quadratic factor}}\underbrace{(x-2)}_{\text{linear}}} = \frac{Ax+B}{x^2+16} + \frac{C}{x-2}$$

no real zeros

Can't factor further over reals

Can't kill C easily

Find A, B, C

- thru LCD

$$8x^2-7x+82 = (Ax+B)(x-2) + C(x^2+16)$$

Hybrid of 2 methods

Plug in $x=2$: $32-14+82 = 0 + 20C$

$100 = 20C$

$C=5$

$$8x^2-7x+82 = \underbrace{(Ax+B)(x-2)+5(x^2+16)}_{\text{Expand, Combine like terms}}$$

$$= \underbrace{(A+5)}_{=8}x^2 + (-2A+B)x + \underbrace{(-2B+80)}_{=82}$$

$A=3$ $\qquad$ $B=-1$

pfd: $$\frac{8x^2-7x+82}{(x^2+16)(x-2)} = \frac{3x-1}{x^2+16} + \frac{5}{x-2}$$

$$\int\left(2x + \frac{8x^2-7x+82}{(x^2+16)(x-2)}\right)dx = \int 2x\,dx + \int\frac{3x-1}{x^2+16}dx + 5\int\frac{1}{x-2}dx$$

split

~~4 x²+16~~

$$= x^2 + \underbrace{\int \frac{3x}{x^2+16}\,dx}_{①} - \underbrace{\int \frac{1}{x^2+16}\,dx}_{②} + 5\ln|x-2|$$

① $3\int \frac{x}{x^2+16}\,dx$ $\qquad u = x^2+16$
$\qquad du = 2x\,dx$

$$= \frac{3}{2}\int \frac{2x}{x^2+16}\,dx$$

or $x\,dx = \frac{1}{2}du$

$$= \frac{3}{2}\int \frac{du}{u}$$

$$= \frac{3}{2}\ln|u| + C_1$$

$$= \frac{3}{2}\ln(\cancel{|}\underbrace{x^2+16}_{\text{always} \geq 0}\cancel{|}) + C_1$$

$$= \frac{3}{2}\ln(x^2+16) + C_1$$

Shortcut p. 437 formula in book easy to forget.

② $\int \frac{1}{x^2+16}\,dx$ $\qquad x = 4\tan\theta$
$\qquad dx = 4\sec^2\theta\,d\theta$

$$= \int \frac{1}{16\tan^2\theta+16} \cdot 4\sec^2\theta\,d\theta$$

$16(\tan^2\theta+1) = 16\sec^2\theta$

$$= \frac{1}{16}\int \frac{1}{\cancel{\sec^2\theta}} \cdot 4\cancel{\sec^2\theta}\,d\theta$$

$$= \frac{1}{4}\int d\theta$$

$$= \frac{1}{4}\theta + C_2$$

Find θ:

$$x = 4\tan\theta$$
$$\frac{x}{4} = \tan\theta$$
$$\theta = \tan^{-1}\left(\frac{x}{4}\right)$$

$$= \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C_2$$

(or p. 437 shortcuts)

$$\int \frac{1}{x^2+16}\,dx = \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)+C$$
$$\int \frac{1}{x\sqrt{x^2-16}}\,dx = \frac{1}{4}\sec^{-1}\left(\frac{x}{4}\right)+C$$
$$\int \frac{1}{\sqrt{16-x^2}}\,dx = \sin^{-1}\left(\frac{x}{4}\right)+C$$

no $\frac{1}{4}$

$$\boxed{x^2 + \frac{3}{2}\ln(x^2+16) - \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + 5\ln|x-2| + K}$$

Recap

Long $\div$ $\Rightarrow$ poly + $\underbrace{\frac{N(x)}{D(x)}}_{\text{proper}}$

Factor denom.

pfd form for

Find $A, B, C \Rightarrow$ get pfd

$\int$ term-by-term

$+K$ at end

Up to 19

Ex $\displaystyle\int \frac{x^3}{(x^2+9)^2}\,dx$ $\leftarrow$ deg 3, $\leftarrow$ deg 4 proper

pfd form: $$\frac{x^3}{\underbrace{(x^2+9)^2}_{\substack{\text{repeated irred.}\\ \text{quadratic factors}\\ (x^2+9)(x^2+9)}}} = \frac{Ax+B}{x^2+9} + \frac{Cx+D}{(x^2+9)^2}$$

Just like w/ repeated linear factors.

run up to the power

Find A, B, C, D

· thru by $(x^2+9)^2$

$$x^3 = \underbrace{(Ax+B)(x^2+9) + (Cx+D)}_{\text{Expand}}$$

$$x^3 = Ax^3 + 9Ax + Bx^2 + 9B + Cx + D$$

$$x^3 = (A)x^3 + (B)x^2 + \underbrace{(9A+C)}_{=0}x + \underbrace{(9B+D)}_{=0}$$

$A=1$ $\quad$ $B=0$ $\quad$ $9(1)+C=0$, $C=-9$ $\quad$ $9(0)+D=0$, $D=0$

$$\text{ptd: } \frac{x^3}{(x^2+9)^2} = \frac{x}{x^2+9} + \frac{-9x}{(x^2+9)^2}$$

$$\int \frac{x^3}{(x^2+9)^2}dx = \int \frac{x}{x^2+9}\,dx - 9\int \frac{x}{(x^2+9)^2}dx$$

$u = x^2+9$

$du = 2x\,dx$

can skip in class

$$= \frac{1}{2}\int \frac{2x}{x^2+9}\,dx - \frac{9}{2}\int \frac{2x}{(x^2+9)^2}dx$$

$$= \frac{1}{2}\int \frac{du}{u} - \frac{9}{2}\int \underbrace{\frac{du}{u^2}}_{u^{-2}du}$$

$$= \frac{1}{2}\ln|u| - \frac{9}{2}\left(\frac{u^{-1}}{-1}\right) + K$$

$$= \frac{1}{2}\ln|\underbrace{x^2+9}_{\text{always} \geq 0}| + \frac{9}{2}\left(\frac{1}{x^2+9}\right) + K$$

$$= \boxed{\frac{1}{2}\ln(x^2+9) + \frac{9}{2x^2+18} + K}$$

© Big Picture

(Assume all coeffs. are real.)

We can $\int$ any rational expression (in x, say) BUT we're limited by our ability to factor polys ($D(x)$)

$$\int \frac{N(x)}{D(x)}\,dx$$ polys in x; $D(x) \neq$ "0"

What if $D(x)$ is an ugly 10^{th}-deg poly?

Maybe use: Factor & cancel
u-sub
Long $\div$ if $\deg(N(x)) \geq \deg(D(x))$

Thm $D(x)$ must be:

① a non-0 constant
$\Rightarrow \int$ poly Easy!

② linear ($\Rightarrow \ln$)

③ irreducible, quadratic $\left(\Rightarrow \underbrace{\int \frac{1}{1+x^2}\,dx}_{\tan^{-1}}, \underbrace{\int \frac{x}{1+x^2}\,dx}_{u\text{-sub}, \ln}, 9.5\right)$

or ④ some product of the above (incl. powers)
Use pfd?

Assume $\frac{N(x)}{D(x)}$ proper

Stewart 497
proper rat'l = Σ form $\frac{A}{(ax+b)^i}$, form $\frac{Ax+B}{(ax^2+bx+c)^j}$
always doable

#64: $\int \frac{1}{x^4+1}\,dx$

$x^4+1 = x^4+2x^2+1-2x^2$
$= (x^2+1)^2 - (\sqrt{2}x)^2$
Factor $A^2 - B^2$

Ex $\int \frac{1}{x^2(x-1)^3(x^2+9)}\,dx$ (Find pfd form.)

Fair for test!

$$= \underbrace{\frac{A}{x} + \frac{B}{x^2}}_{\text{run up}} + \underbrace{\frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}}_{\text{run up}} + \frac{Fx+G}{x^2+9}$$