

10.1: LIMITS I: INDETERMINATE FORMS $\frac{0}{0}, \frac{\infty}{\infty}$
need more work!

All derivatives involve $\frac{f}{g}$:
 $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$

Pop #1, 2, 4, 14

(1) L'Hôpital's Rule

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} \quad \left(\begin{array}{c} \rightarrow 0 \\ \rightarrow 0 \end{array} \right) \text{ or } \left(\begin{array}{c} \rightarrow \pm\infty \\ \rightarrow \pm\infty \end{array} \right)$$

(or $x \rightarrow c^-, c^+, \infty, -\infty$)

Then

Your 150 teacher would've slapped you

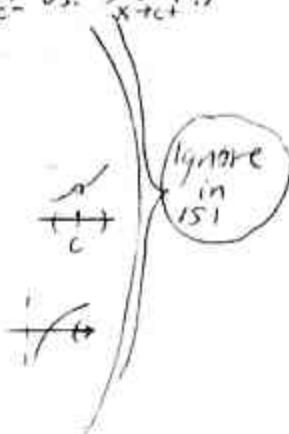
$$= \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)} \quad (\text{not Quotient Rule})$$

Provided

- 1) This limit is not "DNE" ($\infty, -\infty$ OK). ($\lim_{x \rightarrow c^-}$ vs. $\lim_{x \rightarrow c^+}$)
- 2) f, g are differentiable, and $g' \neq 0$
"near where we care" $\begin{matrix} x \rightarrow c & \leftarrow 0+ \\ x \rightarrow \infty & \rightarrow \infty \end{matrix}$

$\lim_{x \rightarrow c}$: "around" (maybe not at) c

$\lim_{x \rightarrow \infty}$: "eventually"



If you get DNE,
you've wasted
your time.
L'H was made
to apply in
1st place

and/or

Repeat until

- 1) you can determine the limit, maybe after manipulation, or
- 2) L'Hôpital's rule can't be applied

10.2: manip
some can't use
L'H

⑧ Exs

Ex (Pop #4)

Leading term
of a poly
determines
its long-term
behavior
($x \rightarrow \pm\infty$)

$$\lim_{x \rightarrow \infty} \frac{\underbrace{2x^3 - x + 3}_{\text{leading term}}}{4x^2 + 1} \leftarrow p_x \quad \leftarrow p_x \quad \left(\frac{\infty}{\infty} \right) \quad \text{Write!}$$

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{6x^2 - 1}{8x} \quad \left(\frac{\infty}{\infty} \right)$$

$\frac{1^2}{0}$? No!

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow \infty} \frac{12x}{8} \rightarrow \infty \quad \text{L'H}$$

$$= \boxed{\infty}$$

$$\text{or } \lim_{x \rightarrow \infty} \frac{6x^2 - 1}{8x} = \lim_{x \rightarrow \infty} \left(\left(\frac{3}{4}x \right)^{\infty} - \left(\frac{1}{8x} \right)^0 \right) \\ = \boxed{\infty}$$

(Discuss Pop #1)

$$\text{Ex (Pop #14)} \quad \lim_{x \rightarrow 0} \frac{\sin x}{x} \quad \left(\frac{0}{0} \right)$$

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow 0} \frac{\cos x}{1} \rightarrow \cos 0 = 1$$

uglier proof on
Ap. 04-20

$$= \boxed{1}$$

Circular reasoning! $D_x(\sin x) = \cos x$

If you forget
what this is...

Prob #12:
 $\lim_{x \rightarrow \infty} \sin x$ due

vs. #15 $\lim_{x \rightarrow \infty} \frac{\sin x}{x} \rightarrow \text{DNE}$ X

Ex $\lim_{x \rightarrow 0^+} \frac{e^{3x}-1}{x} \rightarrow e^0 - 1 = 1 - 1 = 0 \quad (\frac{0}{0})$

$\stackrel{L'H}{=} \lim_{x \rightarrow 0^+} \frac{3e^{3x}}{1} \rightarrow 3e^0 = 3$

= 3

Would splitting work? NO

$$\lim_{x \rightarrow 0^+} \left(\frac{e^{3x}}{x} - \frac{1}{x} \right)$$

$\infty - \infty$ is indet.
 $(\infty + \infty$ is not. $\rightarrow \infty)$

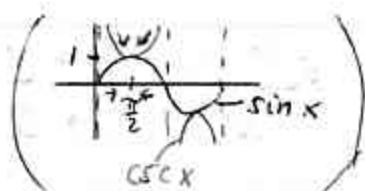
If $\lim_{x \rightarrow 0^+}$, consider $\lim_{x \rightarrow 0^-}$, also.

The remedy
 would be to
 recombine \rightarrow
 single fraction
 $\lim_{x \rightarrow 0^+} \frac{e^{3x}+1}{x} \rightarrow 2$
 ∞ (Don't use L'H)

Whenever
trig funs
are defined,
we have
2-sided lims.

Ex $\lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \csc x}{\cos x} \rightarrow \frac{1 - 1}{\cos \frac{\pi}{2}} = 0 \quad (\frac{0}{0})$

$$\begin{array}{l} \text{SIN } \frac{\pi}{2} = 1 \\ \Rightarrow \csc \frac{\pi}{2} = 1 \end{array}$$



L'H $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\csc x \cot x}{-\sin x}$ Rewrite

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{1}{-\sin x} \cdot \frac{\cos x}{\sin x} \cdot \frac{\sin^2 x}{\sin^2 x}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{-\sin^2 x} \rightarrow 0 \rightarrow -[\underbrace{\sin(\frac{\pi}{2})}_1]^3 = -1$$

$$= 0$$

Tell me
what you
think.

Ex $\lim_{x \rightarrow 0} \frac{\cos x}{x} \stackrel{\text{DNE}}{=} \lim_{x \rightarrow 0} \frac{-\sin x}{1}$

$$= 0$$

$$\frac{\cos x}{x} \rightarrow \frac{1}{0} \quad \text{DNE} \text{ (does not apply!)}$$

Correct: $\lim_{x \rightarrow 0^-} \frac{\cos x}{x} \rightarrow \frac{1}{0^-} = -\infty \quad \mid \quad \lim_{x \rightarrow 0^+} \frac{\cos x}{x} \rightarrow \frac{1}{0^+} = \infty$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\cos x}{x} \boxed{\text{DNE}}$$

10.2: LIMITS II: MORE INDET. FORMS(A) Review "Limit Forms"

$$\begin{array}{c} \text{Limit} \\ \frac{0}{0} \\ 0 \\ \frac{0}{\infty} \\ 0 \end{array}$$

Boles don't say what DNE $\frac{0}{0} \leftarrow \frac{\infty}{\infty}$

Ex $\lim_{x \rightarrow \infty} \frac{3}{x}$
 but doesn't $\rightarrow 0^+$ or $\rightarrow 0^-$

$$\begin{array}{ccccc} \frac{3}{0} & \text{need more info} & & \left. \frac{\infty}{0} \right. & \text{similar} \\ \frac{3}{0^+} & \infty & & & \\ \frac{3}{0^-} & -\infty & & & \\ \frac{-3}{0^+} & \infty & & & \end{array}$$

$$\begin{array}{c} \frac{0}{0} \\ \frac{\pm\infty}{\pm\infty} \end{array}$$

indet. (L'H?)

Note $\lim_{x \rightarrow \infty} \frac{3}{\sin x}$ DNE
 but doesn't $\rightarrow 0^+$ by Squeeze Thm.
 $\frac{\sin x}{x} \rightarrow 0$ from x

(B) $0 \cdot \infty$ Indet. Form

\uparrow
or $-\infty$

(Why Indet.?)

$$\begin{array}{l} \text{Ex As } x \rightarrow 0^+, (x) \cdot \left(\frac{1}{x}\right) = 1 \rightarrow 1 \\ \quad x \cdot \frac{1}{x^2} = \frac{1}{x} \rightarrow \infty \\ \quad x^2 \cdot \frac{1}{x} = x \rightarrow 0 \end{array} \rightarrow \text{(different results)}$$

$$\lim_{x \rightarrow 0} (\sin x \cdot \frac{1}{x^2})$$

DNE:

$$\lim_{x \rightarrow 0^+} \text{is } \infty$$

$$\lim_{x \rightarrow 0^-} \text{is } -\infty$$

Trick Rewrite into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.
 Use L'H?

I put in 0.

Ex (#2) $\lim_{x \rightarrow \frac{\pi}{2}^-} (\tan x) \ln(\sin x)$ $(\infty \cdot 0)$

$$\begin{array}{ccc} & \downarrow & \downarrow \\ & \infty & \ln(\sin \frac{\pi}{2}) \\ & = \ln 1 & = 0 \\ \text{graph} & \nearrow & \end{array}$$

Ⓐ

choice:

∞ or 0

$$\begin{aligned} D_x \left(\frac{1}{\ln(\sin x)} \right) &= \frac{1}{\sin x \cdot \cos x} \\ &= \frac{1}{(\ln(\sin x))^2} \\ &= \frac{\cot x}{(\ln(\sin x))^2} \end{aligned}$$

You've already
analyzed the
limit.

Rewrite $\tan x \ln(\sin x)$ as

$$\frac{\tan x}{\ln(\sin x)} \quad \text{or} \quad \frac{\ln(\sin x)}{\tan x}$$

$$= \frac{\ln(\sin x)}{\cot x} \quad \text{Easier}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\ln(\sin x)}{\cot x} \xrightarrow[\cot x \rightarrow 0]{\ln(\sin x) \rightarrow 0} \left(\frac{0}{0} \right)$$

If $\tan x \rightarrow \infty$,
then $\cot x \rightarrow 0$

$$\stackrel{\text{L'H}}{=} \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\frac{1}{\sin x} \cdot D_x(\sin x)}{-\csc^2 x} \quad \leftarrow \text{Rewrite}$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} \frac{\frac{\cos x}{\sin x}}{-\frac{1}{\sin^2 x}}, \frac{\sin^2 x}{\sin^2 x} \quad \text{or} \quad \lim_{x \rightarrow \frac{\pi}{2}^-} \left(\frac{\cos x}{\sin x} \right) (-\sin^2 x)$$

$$= \lim_{x \rightarrow \frac{\pi}{2}^-} (-\cos x \sin x)$$

$$= -\underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1}$$

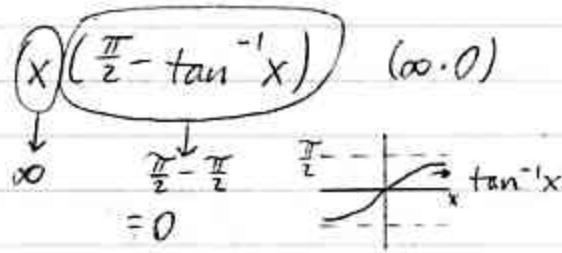
You know
one's 0,
other's 1

Up to 11

$$= \boxed{0}$$

$$\text{Ex (#8)} \lim_{x \rightarrow \infty} x \left(\frac{\pi}{2} - \tan^{-1} x \right) (\infty, 0)$$

~~$\tan x$~~ $\frac{1}{x}$
 $\sqrt{x^2 - \frac{\pi^2}{4}}$
 $\Rightarrow \text{HAs for } \tan^{-1} x$



Could do $\frac{\infty}{\infty}$ form

$$= \lim_{x \rightarrow \infty} \frac{\frac{\pi}{2} - \tan^{-1} x}{\frac{1}{x}} \left(\frac{0}{0} \right)$$

$$\stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} \leftarrow D_x(x^{-1}) = -x^{-2}$$

\therefore by L'H rule
 \therefore by its recip.

$$= \lim_{x \rightarrow \infty} \left(-\frac{1}{1+x^2} \right) (-x^2)$$

$$= \lim_{x \rightarrow \infty} \frac{x^2}{1+x^2} \left(\frac{\infty}{\infty} \right)$$

Could $\div N, D$ by x^2 (MISO)

Just
alg. simplify
 $(Ax+B)$

$$\stackrel{L'H}{=} \lim_{x \rightarrow \infty} \frac{2x}{2x}$$

$$= \boxed{1}$$

⑦ $0^\circ, \infty^\circ, 1^\infty$ Indet. Forms

Yay!

$$\text{Alg. I} \quad \begin{cases} x^\circ = 1 \\ 0^x = 0 \\ 0^\circ ? \end{cases} \quad \text{if } x \neq 0$$

(Could be "2"!)

$$\approx x^{\frac{\ln 2}{\ln x}} = x^{\log_x 2}$$

= 2
Larson 7.7.86

Note

$$\lim_{x \rightarrow 0^+} \left(\frac{\ln 2}{1 + \ln x} \right)^{\frac{1}{x}} = 2$$

$$1^\infty \text{ Exs} \quad \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = e \quad (\text{in Cont. Comp. Interest})$$

is this indet? go:
 $x^{1/x}$ problem even?
 $(0^+)^{\infty} = \infty$ ↗
 Larson 7.7.83-4

but $\lim_{x \rightarrow \infty} 1^x = 1$ — different!
 $(0^+)^{\infty} \Rightarrow 0$ think $(\frac{1}{10})^{10000} \approx 0$
 Not indet.

$$\text{Ex (#22)} \quad \lim_{x \rightarrow 0^+} (1+3x)^{\csc x} \quad (1^\infty)$$

Note $\left(\frac{u}{v} \right)^\infty \underset{u \rightarrow \infty, v \rightarrow \infty}{\sim} \infty$

How do we bring this exponent down to Earth?

Like Log Diff
 but only log 0
 a problem in
 Log Diff
 Avoid 1/0 issue,
 at x=0.
 $\ln x^k = k \ln x$

In the end,
 what will
 we use to
 pull off ln?
 Remind yourself!
 (always
 forget at
 the end!)

If "for fun by"
 you forgot, anyway

$$\textcircled{1} \quad \text{Let } y = (1+3x)^{\csc x}$$

$$\textcircled{2} \quad \begin{aligned} \ln y &= \overbrace{\ln (1+3x)}^{\text{as } x \rightarrow 0^+}^{\csc x} \\ \ln y &= \csc x \cdot \underbrace{\ln (1+3x)}_{> 0 \text{ as } x \rightarrow 0^+} \end{aligned}$$

$$\textcircled{3} \quad \lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \csc x \cdot \ln (1+3x) \quad (\infty \cdot 0)$$

@ ∞ \downarrow $\ln 1 = 0$
 Reminder!

$$\lim_{x \rightarrow 0^+} \frac{\ln (1+3x)}{\sin x} \quad (0/0)$$

$$\text{②} \lim_{x \rightarrow 0^+} \frac{\frac{1}{1+3x} \cdot 3}{\cos x}$$

$$\bar{\lim}_{x \rightarrow 0^+} \frac{3}{(1+3x)\cos x} \xrightarrow{(1+3x) \rightarrow 1, \cos 0 = 1} \frac{3}{1} = 3$$

③ NOT DONE!

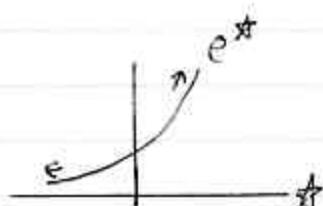
$$\text{④} \lim_{x \rightarrow 0^+} \ln y = 3$$

Continuity of e^y

$$\Rightarrow \lim_{x \rightarrow 0^+} e^{\ln y} = e^3$$

$$\Rightarrow \lim_{x \rightarrow 0^+} y = \boxed{e^3}$$

(Note) If $\ln y \rightarrow \infty$
 $\Rightarrow y \rightarrow \infty$ (" $e^\infty = \infty$ ")



If $\ln y \rightarrow -\infty$
 $\Rightarrow y \rightarrow 0$ (" $e^{-\infty} = 0$ ")

Up to 21

① $\infty - \infty$ Indet form As $x \rightarrow \infty$, $\begin{array}{l} x^2 - x^2 \rightarrow 0 \\ x^2 - x \rightarrow \infty \\ x - x^2 \rightarrow -\infty \end{array}$ \rightarrow different!

$$\text{Ex (#24)} \lim_{x \rightarrow 1^+} \left(\underbrace{\frac{1}{(x-1)}}_{\rightarrow 0^+} - \underbrace{\frac{1}{\ln x}}_{\rightarrow 0^+} \right) \quad (\infty - \infty)$$

To +
We need
a ... LCO.

In num.:
what's missing?

Trick: Get 1 fraction
missing from 1st frac.

$$= \lim_{x \rightarrow 1^+} \frac{\ln x - (x-1)}{(x-1)\ln x}$$

(Simplify 1st, then
identify limit form.)

$$= \lim_{x \rightarrow 1^+} \frac{\ln x - x + 1}{(x-1) \ln x} \rightarrow \frac{\overset{=0}{\ln 1} - 1 + 1 = 0}{0 \underset{=0}{\ln 1}} = 0 \quad \left(\frac{0}{0}\right)$$

$$\stackrel{L'H}{=} \lim_{x \rightarrow 1^+} \frac{\frac{1}{x} - 1}{\ln x + (x-1)(\frac{1}{x})} \cdot \frac{x}{x}$$

By Product Rule

$$= \lim_{x \rightarrow 1^+} \frac{1-x}{x \ln x + x-1} \rightarrow 0 \rightarrow \underbrace{1 \ln 1}_{=0} + 1 - 1 = 0 \quad \left(\frac{0}{0}\right)$$

use Product Rule

$$\stackrel{L'H}{=} \lim_{x \rightarrow 1^+} \frac{-1}{\ln x + \underbrace{x(\frac{1}{x})}_{=1} + 1}$$

$$= \lim_{x \rightarrow 1^+} \frac{-1}{\ln x + 2} \rightarrow \frac{-1}{\ln 1 + 2} = 2$$

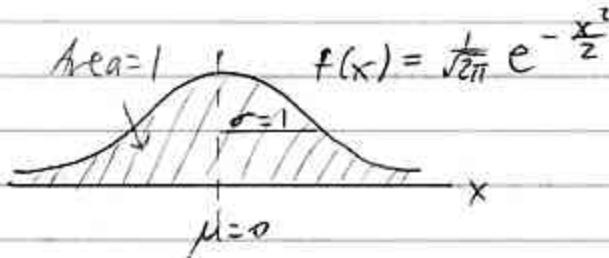
$$= \boxed{-\frac{1}{2}}$$

10.3: IMPROPER \int_5 I: $\int_{+\infty}^{+\infty} f(x) dx$

(A) Intro Exs

Standard Normal Density Func. (Stats)

Tails fall rapidly.
If a gallon of
paint covers
 1 ft.^2 (for a
given thickness).
Area = 1 ft.^2
You won't run
out of paint.



Cartella 104-5
Calc III

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

a real # \Leftrightarrow the \int converges
Otherwise, it diverges (maybe to ∞ or $-\infty$).

Graph doesn't
fall fast enough.



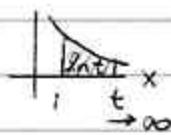
Sketch 503: not
definite
Harper Pitt-15

SKIP;
Proof
is
key.

Why? $\int_1^{\infty} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx$

cont. on $[1, \infty)$ \implies has a value
(for fixed $t \geq 1$)

Remember the defn.



$$= \lim_{t \rightarrow \infty} \ln t$$

by defn of ln t (p. 382)

$$= \infty$$

Proof: pp. 389-90. Key!

Idea of Proof:
 $\ln x > \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$
 $\ln 4 > 1$
 $\ln 4 > M, \forall Q^+$
 $\ln x$ can be made arbitrarily large

Mead

⑧ Exs

Ex (#22) Does $\int_0^\infty xe^{-x} dx$ converge?
If so, find its value.

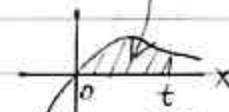
First, find $\int xe^{-x} dx$

$$\int \text{ by Parts: } u = x \quad dv = e^{-x} dx \\ du = dx \quad \underline{\quad} \quad v = -e^{-x}$$

$$\begin{aligned}\int xe^{-x} dx &= -xe^{-x} - \int -e^{-x} dx \\ &= -xe^{-x} + \int e^{-x} dx \\ &= -xe^{-x} - e^{-x} + C\end{aligned}$$

$\int_0^\infty xe^{-x} dx$ ^{tilde us off} $= \lim_{t \rightarrow \infty} \int_0^t xe^{-x} dx$ (if it exists)

cont. on $(0, \infty)$ Definite S



Apply FTC. (^{Fund. Thm.} of Calc.)

$$= \lim_{t \rightarrow \infty} [-xe^{-x} - e^{-x}]_0^t$$

$$= \lim_{t \rightarrow \infty} [(-te^{-t} - e^{-t}) - (0 - e^0)]$$

$$= \lim_{t \rightarrow \infty} \left(-\left(\frac{t}{e^t}\right) - \left(\frac{1}{e^t}\right) + 1 \right)$$

$$= 1$$

$\therefore \int_0^\infty xe^{-x} dx$ converges, and its value is 1.

$\int x^{Area=1}$

HW 10.1, 31
(n int. > 0)
domain if n ≤ 0

$$\textcircled{A} \quad \lim_{t \rightarrow \infty} \frac{t^n}{e^t} = 0$$

$\Rightarrow e^t$ "overwhelms" any poly. func. of t in the long run.

Big-O: papers
over messy
details

Analysis of algorithms

time, memory requirements as input size $\rightarrow \infty$
poly.-time vs. exp'l-time algorithms
"BAD!"

#12: $\int_{-\infty}^3 \frac{x}{(1+x^2)^2} dx$

\curvearrowleft cont. on $(-\infty, 3]$

$$\int \frac{x}{(1+x^2)^2} dx = \dots = -\frac{1}{2(1+x^2)} + C$$

Strategy?

$$\begin{aligned} \text{let } u &= 1+x^2 \\ du &= 2x dx \end{aligned}$$

$$\int_{-\infty}^3 \frac{x}{(1+x^2)^2} dx = \lim_{t \rightarrow -\infty} \int_t^3 \frac{x}{(1+x^2)^2} dx \quad (\text{if exists})$$

$$= \lim_{t \rightarrow -\infty} \left[-\frac{1}{2(1+x^2)} \right]_t^3$$

$$= \lim_{t \rightarrow -\infty} \left(-\frac{1}{20} - \left[-\frac{1}{2(1+t^2)} \right] \right) \xrightarrow{?}$$

$$= \boxed{-\frac{1}{20}, \text{ I converges}}$$

upto 13

Note

Not in book

Method 2 (no one does this, though)

$$\int_{-\infty}^3 \frac{x}{(1+x^2)^2} dx$$

Let $u = 1+x^2$
 $du = 2x dx$
 $\Rightarrow \frac{1}{2} du = x dx$

$$= \lim_{t \rightarrow -\infty} \int_{x=t}^{x=3} \frac{x}{(1+x^2)^2} dx \quad (\text{if exists})$$

$x = t \Rightarrow u = 1+t^2$
 $x = 3 \Rightarrow u = 10$

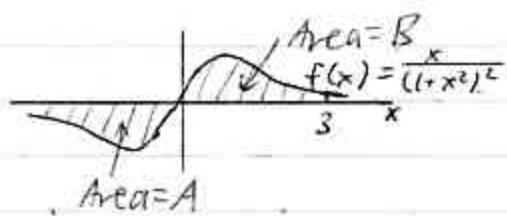
$$= \lim_{t \rightarrow -\infty} \int_{u=1+t^2}^{u=10} \frac{\frac{1}{2} du}{u^2}$$

$$= \lim_{t \rightarrow -\infty} \left[-\frac{1}{2u} \right]_{u=1+t^2}^{u=10}$$

$$= \lim_{t \rightarrow -\infty} \left(-\frac{1}{20} - \left[-\frac{1}{2(1+t^2)} \right]_0 \right)$$

$$= \boxed{-\frac{1}{20}, \text{ so it converges}}$$

In terms of
A & B what is
the value of $\int f(x) dx$?



$$= -\frac{1}{20} \text{ also}$$

$$= \int_{-\infty}^{-3} + 0$$

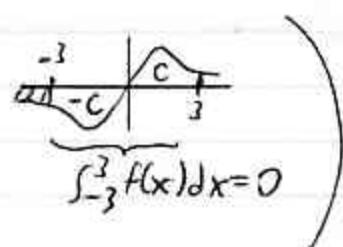
odd func.
I'll mention later

$$\int_{-\infty}^3 f(x) dx = -A + B$$

$$= -\frac{1}{20}$$

Note f odd $\Rightarrow \int_{-\infty}^{-3} f(x) dx = -\frac{1}{20}$, also.

$$\int_3^\infty f(x) dx = \frac{1}{20}$$



Same integrand
as previous ex.

$$\text{Ex } \int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2} dx \quad \textcircled{A}$$

cont. on $(-\infty, \infty)$

$$= \left(\int_{-\infty}^0 \frac{x}{(1+x^2)^2} dx \right) + \left(\int_0^{\infty} \frac{x}{(1+x^2)^2} dx \right)$$

(if both \int s converge)

Note 1 If one or both diverge,
then the original \int diverges.

Note 2 $\int_{-\infty}^0 + \int_0^{\infty}$ OK (Doesn't matter where split)
 \Rightarrow Same result

$$\text{Why? } \int_{-\infty}^{\infty} = \underbrace{\int_{-\infty}^0}_{\int_{-\infty}^0} + \underbrace{\int_0^1 + \int_1^{\infty}}_{\int_0^{\infty}}$$

if converge

+ ok but
converging in \mathbb{R}
 \lim exists
self-contained

$$\textcircled{I} = \lim_{t \rightarrow -\infty} \int_t^0 \frac{x}{(1+x^2)^2} dx \quad (\text{if exists})$$

$$= \lim_{t \rightarrow -\infty} \left[-\frac{1}{2(1+x^2)} \right]_t^0$$

from before

$$= \lim_{t \rightarrow -\infty} \left(-\frac{1}{2} - \left[-\frac{1}{2(1+t^2)} \right]_t^0 \right)$$

$$= -\frac{1}{2}$$

$$\textcircled{II} = \lim_{w \rightarrow \infty} \int_0^w \frac{x}{(1+x^2)^2} dx \quad (\text{if exists})$$

$$= \lim_{w \rightarrow \infty} \left[-\frac{1}{2(1+x^2)} \right]_0^w$$

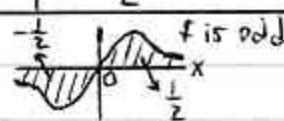
$$= \lim_{w \rightarrow \infty} \left(-\frac{1}{2(1+w^2)} - \left[-\frac{1}{2} \right]_0 \right)$$

$$= \frac{1}{2}$$

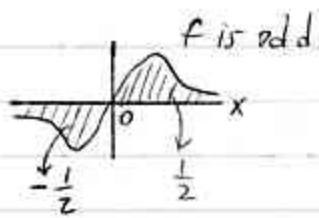
why should this
not be
surprising?
Up to 23

$$\textcircled{I} + \textcircled{II} = -\frac{1}{2} + \frac{1}{2}$$

$$= 0, \int \text{converges}$$



Begs the?
Is this true of
all odd funer? NO!
Can go up to 2?
we'll talk more.



$$\text{Ex } \int_{-\infty}^{\infty} x dx \quad \textcircled{A}$$

$$= \int_{-\infty}^0 x dx + \int_0^{\infty} x dx$$

$$\textcircled{I} = \lim_{t \rightarrow -\infty} \int_t^0 x dx$$

Wait to see if \textcircled{I} diverges! (Then \textcircled{II} div.)
Write this anyway, though! If \textcircled{II} div, you don't
have to work it out.

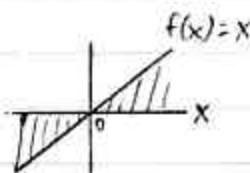
(if \int converges)

You may want to do this
before \textcircled{I} , since ∞ may be
easier to deal with than $-\infty$.

$$= \lim_{t \rightarrow -\infty} \left[\frac{x^2}{2} \right]_t^0$$

$$= \lim_{t \rightarrow -\infty} \underbrace{\left(0 - \frac{t^2}{2} \right)}_{\stackrel{\approx -\infty}{\curvearrowleft}}$$

$\Rightarrow \textcircled{I}$ diverges



$$\Rightarrow \int_{-\infty}^{\infty} x dx \quad \boxed{\text{diverges}} \quad (x \text{ is odd, but answer is } \underline{\text{not } 0})$$

† may be happen.
"div." like
 $x = \frac{1}{t}$

WARNING $\int_{-\infty}^{\infty} f(x) dx$ does not necessarily equal
 $\lim_{t \rightarrow \infty} \int_{-t}^t f(x) dx$.

Dumb P.D.N

Edwards 528
 $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$ div.
 $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$ div.

but $\lim_{t \rightarrow \infty} \int_{-t}^t \frac{1}{1+x^2} dx = \pi$

Me: If $\int_{-\infty}^{\infty}$
converges

$\Rightarrow \lim_{t \rightarrow \infty} \int_{-t}^t = ?$

$\infty - \infty$ but we know: $\int_{-\infty}^{\infty} = 0$

$$\int_{-\infty}^{\infty} x dx \quad \underline{\text{WRONG}} \quad \lim_{t \rightarrow \infty} \int_{-t}^t x dx$$

$$= \lim_{t \rightarrow \infty} \left[\frac{x^2}{2} \right]_{-t}^t \quad \left. \begin{array}{l} \text{or "f odd"} \\ \text{or "f even"} \end{array} \right\}$$

$$= \lim_{t \rightarrow \infty} \left(\frac{t^2}{2} - \frac{(-t)^2}{2} \right) \quad \left. \begin{array}{l} \text{-t} \\ \text{+t} \end{array} \right\}$$

= ~~WRONG!~~

$$\text{Ex } \int_0^\infty \sin x \, dx = \lim_{\substack{\text{cont. on} \\ [0, \infty)}} \int_0^t \sin x \, dx \text{ (if exists)}$$

$$= \lim_{t \rightarrow \infty} [-\cos x]_0^t$$

$$= \lim_{t \rightarrow \infty} (-\cos t - [-\cos 0]) \stackrel{=1}{=} 1$$

$$= \lim_{t \rightarrow \infty} (1 - \cos t)$$

$\lim_{t \rightarrow \infty} \cos t \text{ DNE}$ (and there are no ops. nor simplifications

that will prevent the whole thing from being DNE)

(Examples where

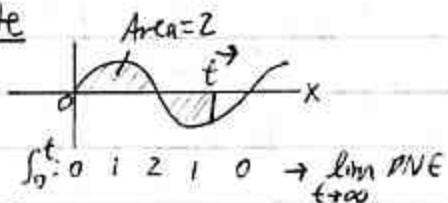
DNE is "prevented":

$$\lim_{t \rightarrow \infty} \frac{\sin t}{t} = 0 \quad (\text{squeeze thm.})$$

$$\lim_{t \rightarrow \infty} \frac{z + \cos t}{z + \cos t} = 1$$

Edwards:
by oscillations
Me: what if
then
 $\therefore \int_0^\infty \sin x \, dx$ diverges

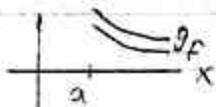
Note



$$\int_0^t: 0 \vdash z \vdash 1 \vdash 0 \rightarrow \lim_{t \rightarrow \infty} \text{DNE}$$

③ Comparison Tests

If f, g are cont., $\left. \begin{array}{l} \text{and } 0 \leq f(x) \leq g(x), \\ \text{on } [a, \infty) \end{array} \right\}$



If $\int_a^\infty g(x) \, dx$ converges $\Rightarrow \int_a^\infty f(x) \, dx$ converges.
(big bro fits \Rightarrow so do you)

If $\int_a^\infty f(x) \, dx$ diverges $\Rightarrow \int_a^\infty g(x) \, dx$ diverges.
(little bro can't fit \Rightarrow neither can you)

You diverge - you
have to
break up to
get them down.

(Remember Normal?)

Does this look
somewhat
familiar?
(Normal?)
Take a guess...
 $e^{-\frac{x^2}{2}} \geq e^{-x^2}$
e.g., $e^{-5} \geq e^{-10}$

Ex Show $\int_1^\infty e^{-x^2} dx$ converges.
(#28)

cont. ≥ 0
on $[1, \infty)$

① Show $e^{-x^2} \leq e^{-x}$ on $[1, \infty)$
"big bro"
(Can S)

Not true on $(0, 1)$
"unlight zone"
 $(\frac{1}{2})^2 < \frac{1}{2}$
MISO: Area bounded by x, x^2

Order \uparrow
preserved
Reciprocals,
 \downarrow don't preserve
order.

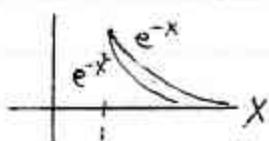
$$\begin{aligned} &x^2 \geq x \text{ on } [1, \infty) \\ \Rightarrow &-x^2 \leq -x \\ \Rightarrow &e^{-x^2} \leq e^{-x} \quad , \quad e^{\star} \text{ is } \nearrow \text{ in } \mathbb{R} \\ &\left(\text{because } e^{\star} \text{ is an increasing func of } \mathbb{R} \right) \end{aligned}$$

② Show $\int_1^\infty e^{-x} dx$ converges.

$$\begin{aligned} \int_1^\infty \underbrace{e^{-x}}_{\substack{\text{cont. on} \\ [1, \infty)}} dx &= \lim_{t \rightarrow \infty} \int_1^t e^{-x} dx \quad (\text{if exists}) \\ &= \lim_{t \rightarrow \infty} [-e^{-x}]_1^t \\ &= \lim_{t \rightarrow \infty} (-e^{-t} - [-e^{-1}]) \\ &= \lim_{t \rightarrow \infty} \left(-\frac{1}{e^t} + \frac{1}{e} \right) \\ &= \frac{1}{e}, \quad \text{converges} \end{aligned}$$

can skip if no time

③ $\therefore \int_1^\infty e^{-x^2} dx$ converges.



Show $\int_0^\infty \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{2}} dx$
conv.
Key: Show
 $\int_1^\infty e^{-\frac{x^2}{2}} dx$ conv.
 $\leq e^{-\frac{x^2}{2}}$

More on PDX 10.4: IMPROPER Js II: WHEN INTEGRANDS EXPLODEA) Intro Ex

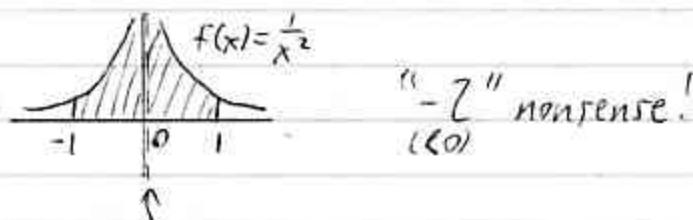
Edwards 523

$$\int_{-1}^1 \frac{1}{x^2} dx \stackrel{\text{(WRONG!)}}{=} \left[-\frac{1}{x} \right]_{-1}^1 \quad (\text{FTC})$$

$$= -1 - \left(-\frac{1}{-1} \right)$$

$$= -1 - 1$$

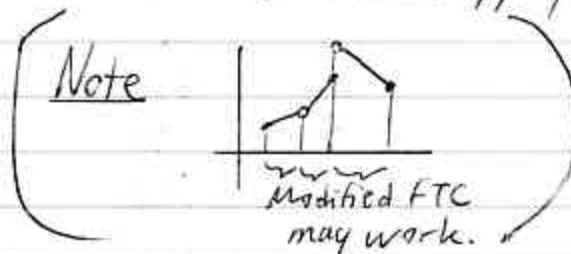
$$= \cancel{-2}$$



f has an interior discontinuity at $x=0$.

It's an infinite discontinuity, so
FTC does not apply!

If you have
a finite #
at rem, jump
discont., you
may be ok.

Correct

$$\int_{-1}^1 \left(\frac{1}{x^2} \right) dx \quad \text{(A)}$$

discont. at $x=0$
cont. on $[-1, 0) \cup (0, 1]$

$$= \underbrace{\int_{-1}^0 \frac{1}{x^2} dx}_{\text{(if both } \int \text{s converge)}} + \underbrace{\int_0^1 \frac{1}{x^2} dx}_{\text{(if both } \int \text{s converge)}}$$

← Write this out, even if \textcircled{I} diverges.
 (Actually, you may want to do \textcircled{II} first, since it may be easier to deal with than -1 .)

If one or both diverge, then the original \int diverges.

\textcircled{A}

$$\textcircled{I} \quad \int_{-1}^0 \frac{1}{x^2} dx = \lim_{t \rightarrow 0^-} \int_{-1}^t \frac{1}{x^2} dx \quad (\text{if exists})$$

$\overbrace{-1 \rightarrow 0^-}^{\text{You run towards the problem}}$
 (Helps to draw arrow to see if 0^- or 0^+)

$$\stackrel{\text{etc}}{=} \lim_{t \rightarrow 0^-} \left[-\frac{1}{x} \right]_{-1}^t$$

$$= \lim_{t \rightarrow 0^-} \left(-\frac{1}{t} - \left[-\frac{1}{-1} \right] \right)$$

$$= \lim_{t \rightarrow 0^-} \left(-\left(\frac{1}{t}\right) - 1 \right) \underset{\rightarrow \infty}{\sim}$$

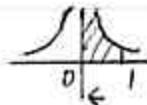
$$= \infty$$

\textcircled{I} diverges

\Rightarrow \textcircled{A} diverges

Note: \textcircled{II}

$$\int_0^1 \frac{1}{x^2} dx = \lim_{w \rightarrow 0^+} \int_w^1 \frac{1}{x^2} dx \quad (\text{if exists})$$



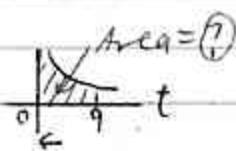
+ OK

Do you think
this diverges?
By symmetry?

B) Ex (#7)

$$\int_0^9 \frac{1}{\sqrt{x}} dx$$

cont. on $(0, 9]$
undef. at 0



$$= \lim_{t \rightarrow 0^+} \int_t^9 \frac{1}{\sqrt{x}} dx$$

$$\int x^{-\frac{1}{2}} dx = \frac{x^{1/2}}{1/2} + C \\ = 2\sqrt{x} + C$$

$$= \lim_{t \rightarrow 0^+} [2\sqrt{x}]_t^9$$

$$= \lim_{t \rightarrow 0^+} (2\sqrt{9} - 2\sqrt{t})$$

$$= [6, \text{ } \int \text{ converges}]$$

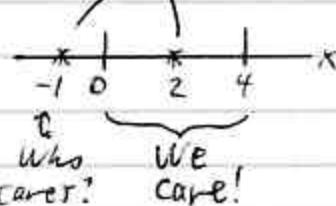
Curve does
approach
 y -axis
fast enough!

C) Ex (#16)

$$\textcircled{A} \int_0^4 \frac{1}{x^2 - x - 2} dx = \int_0^4 \frac{1}{(x-2)(x+1)} dx$$

cont.

except at



$$\text{I} \quad \text{II}$$

$$= \int_0^2 \frac{1}{(x-2)(x+1)} dx + \int_2^4 \frac{1}{(x-2)(x+1)} dx \quad (\text{if converge})$$

First, find $\int \frac{1}{(x-2)(x+1)} dx$

Partial fractions!

I''' steps

$$\frac{1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

$$1 = A(x+1) + B(x-2)$$

$$x = -1 \Rightarrow 1 = -3B \Rightarrow B = -\frac{1}{3}$$

$$x = 2 \Rightarrow 1 = 3A \Rightarrow A = \frac{1}{3}$$

$$\begin{aligned} \int \frac{1}{(x-2)(x+1)} dx &= \int \frac{\frac{1}{3}}{x-2} dx - \int \frac{\frac{1}{3}}{x+1} dx \\ &= \frac{1}{3} \ln|x-2| - \frac{1}{3} \ln|x+1| + C \\ &= \frac{1}{3} \ln \left| \frac{x-2}{x+1} \right| + C \end{aligned}$$

(It turns out to be helpful to transform differences into quotients.)

$$\textcircled{I} \int_0^2 \frac{1}{(x-2)(x+1)} dx = \lim_{t \rightarrow 2^-} \int_0^t \frac{1}{(x-2)(x+1)} dx \quad (\text{if exists})$$

$\frac{1}{x}$ \rightarrow * $\frac{1}{4}$ \quad (Again, you run towards the problem.)

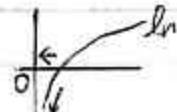
$$= \lim_{t \rightarrow 2^-} \left[\frac{1}{3} \ln \left| \frac{x-2}{x+1} \right| \right]_0^t$$

Note:
 $\frac{t-2}{t+1} \rightarrow 0^-$

$$= \lim_{t \rightarrow 2^-} \left(\underbrace{\frac{1}{3} \ln \left| \frac{t-2}{t+1} \right|}_{\substack{\rightarrow 0^+ \\ \rightarrow -\infty}} - \frac{1}{3} \ln 2 \right) \quad (\text{not } 0^- \text{ because of } 1/1)$$

$\ln \cancel{x}$
 careful:
 used \cancel{x}

$$= -\infty$$



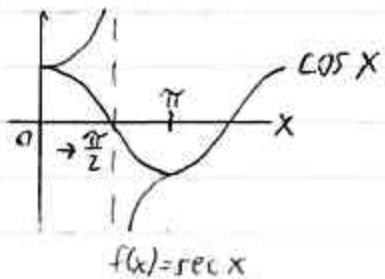
$\Rightarrow \textcircled{I}$ diverges

$\Rightarrow \textcircled{A}$ diverges

① Ex (#24)

$$\int_0^\pi \sec x \, dx \quad \textcircled{A}$$

$$\begin{aligned}\sec x &= \frac{1}{\cos x} \\ \text{second.} &\Leftrightarrow \cos = 0 \\ \sec \pi &\Leftrightarrow \cos \pi \\ \sec 1 &\Leftrightarrow \cos 1\end{aligned}$$



$$= \underbrace{\int_0^{\pi/2} \sec x \, dx}_{\textcircled{I}} + \underbrace{\int_{\pi/2}^\pi \sec x \, dx}_{\textcircled{II}} \quad (\text{if converge})$$

$$\textcircled{I} = \lim_{t \rightarrow \frac{\pi}{2}^-} \int_0^t \sec x \, dx$$

$$= \lim_{t \rightarrow \frac{\pi}{2}^-} [\ln |\sec x + \tan x|]_0^t$$

$$= \lim_{t \rightarrow \frac{\pi}{2}^-} (\underbrace{\ln |\sec t + \tan t|}_{\substack{\rightarrow \infty \\ \text{graph} \\ \uparrow \uparrow \uparrow \uparrow}} - \underbrace{\ln |\sec 0 + \tan 0|}_{=0})$$

$\cancel{\text{graph}}$

$$= \infty \quad \text{↑ slope}$$

\Rightarrow (I) diverges

\Rightarrow (A) diverges

(E) Ex (#28)

$$\int_{-1}^3 \frac{x}{\sqrt[3]{x^2 - 1}} dx \quad \textcircled{A}$$

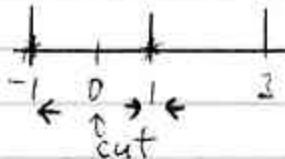
cont. everywhere except at $x = \pm 1$



If you have a problem at both ends \Rightarrow cut

$$\text{Idea: } = \int_{-1}^1 + \int_1^3$$

discontinuity at both $\#s \Rightarrow$ Need a cut!



$$= \underbrace{\int_{-1}^0}_{\textcircled{I}} + \underbrace{\int_0^1}_{\textcircled{II}} + \underbrace{\int_1^3}_{\textcircled{III}} \text{ (if converge)}$$

$$= \lim_{t \rightarrow -1^+} \int_t^0 + \lim_{w \rightarrow 1^-} \int_0^w + \lim_{m \rightarrow 1^+} \int_m^3 \text{ (if exist)}$$

$$= -\frac{3}{4} + \left(-\frac{3}{4}\right) + 3$$

$$= 3, \boxed{\text{converges}}$$

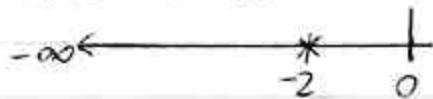
You probably
have to find
the value of
 \textcircled{I} , anyway.

(To show that \textcircled{A} converges,
you must show that
 \textcircled{I} , \textcircled{II} , and \textcircled{III} all converge!)

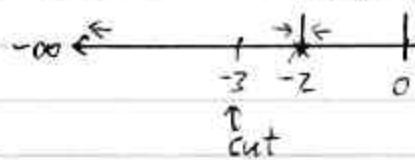
F) Ex (#30)

Caution:
possibly improper

$$\int_{-\infty}^0 \frac{1}{x+2} dx \quad \textcircled{A}$$



$$= \int_{-\infty}^{-2} \frac{1}{x+2} dx + \int_{-2}^0 \frac{1}{x+2} dx$$



$$= \underbrace{\int_{-\infty}^{-3} \frac{1}{x+2} dx}_{\textcircled{I}} + \int_{-3}^{-2} \frac{1}{x+2} dx + \int_{-2}^0 \frac{1}{x+2} dx$$

$$\textcircled{I} = \lim_{t \rightarrow -\infty} \int_t^{-3} \frac{1}{x+2} dx$$

$$= \lim_{t \rightarrow -\infty} [\ln|x+2|]_t^{-3}$$

$$= \lim_{t \rightarrow -\infty} (\underbrace{\ln 1}_{=0} - \underbrace{\ln|t+2|}_{\rightarrow \infty})$$

$$= -\infty$$

\Rightarrow (I) diverges

\Rightarrow (A) diverges

CH.10: REVIEW

(10.1/10.2) LIMITS

Review Limit forms
Trig, Inv trig graphs

L'Hôpital's Rule $\frac{0}{0}, \pm\infty$ (Write ^{indet.} forms!)

$\lim \frac{f}{g} = \lim \frac{f'}{g'}$ unless DNE
Simplify or Repeat (if you can!)

0·∞

$\lim f \cdot g = \lim \frac{f}{\frac{1}{g}}$ or $\lim \frac{g}{\frac{1}{f}}$
Use L'H

$0^{\circ}, \infty^{\circ}, 1^{\infty}$ (Indet. forms w/Exponent \Rightarrow Use ln)

$\lim f(x)$ (Write indet. forms!)

- ① $y = f(x)$
- ② $\ln y = \ln$ (Use Power/Smackdown Rule: $\ln M^p$)
- ③ Find $\lim \ln y$ ($= L$ if exists)
- ④ $\lim y = \lim e^{\ln y} = e^L$

Ideas $e^{\infty} = \infty$
 $e^{-\infty} = 0$

$\infty - \infty$

Get 1 fraction

(10.3/10.4) IMPROPER \int s

$\int = \text{a real } \#$
 $\Leftrightarrow \int \text{ converges (else diverges)}$

$$\textcircled{I} \quad \int_a^{\infty} \underbrace{f(x) dx}_{\substack{\text{cont. on} \\ [a, \infty)}} = \lim_{t \rightarrow \infty} \underbrace{\int_a^t}_{\substack{\text{Maybe find } \int \text{ 1st} \\ \text{Apply FTC later}}} f$$

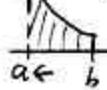
$$\int_{-\infty}^b \underbrace{f(x) dx}_{\substack{\text{cont. on} \\ (-\infty, b]}} = \lim_{t \rightarrow -\infty} \int_t^b f$$

$$\int_{-\infty}^{\infty} \underbrace{f(x) dx}_{\substack{\text{cont. everywhere}}} = \int_{-\infty}^{(c)} + \int_{(c)}^{\infty} \quad \xleftarrow{\substack{c \\ \text{cut}}} \quad \begin{matrix} \nearrow \\ \searrow \end{matrix}$$

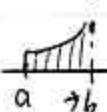
converges \Leftrightarrow both converge

Comparison Tests

$$\textcircled{II} \quad \int_a^b \underbrace{f(x) dx}_{\substack{\text{exists in } [a, b]}}$$



$$\int_a^b = \lim_{t \rightarrow a^+} \int_a^t f$$

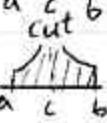


$$\int_a^b = \lim_{t \rightarrow b^-} \int_a^t f$$



$$\int_a^b = \int_a^{(c)} + \int_{(c)}^b$$

cut at discontin.



$$\int_a^b = \int_a^{(c)} + \int_c^{(d)} + \int_d^b$$

$\downarrow \text{conv.} \Leftrightarrow \text{both conv.}$

$$\textcircled{I+II} \quad \int_a^{\infty} \underbrace{f(x) dx}_{\substack{\text{cut}}} = \int_a^{(c)} + \int_c^{\infty}$$

POP QUIZ II

Edwardr S2S

GAMMA FUNCTION, Γ

Euler used this to interpolate $n!$ ($\Gamma(n+1) = n!$)
 pron. "oiler"
 One of the greatest! (1707-1783)

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx \quad (t > 0)$$

Γ is a continuous function of t .

You can show:

$$\textcircled{a} \quad \Gamma(1) = 1 \quad (\text{i.e., } 0! = 1)$$

$$\begin{aligned}\Gamma(1) &= \int_0^\infty \underbrace{x^{1-1}}_{=x^0} e^{-x} dx \\ &= 1\end{aligned}$$

$$\begin{aligned}&= \int_0^\infty e^{-x} dx \\ &= \lim_{t \rightarrow \infty} \int_0^t e^{-x} dx \\ &= \lim_{t \rightarrow \infty} [-e^{-x}]_0^t \\ &= \lim_{t \rightarrow \infty} (-e^{-t} - (-e^0)) \\ &= 1\end{aligned}$$

⑥ If n is a natural # (1, 2, 3, ...),
 $\underbrace{\Gamma(n+1)}_{\Gamma(n+1) = n\Gamma(n)}$

i.e., $n! = n \cdot (n-1)!$ (Recursive relation for $n!$)

$$\begin{aligned}\Gamma(n+1) &= \int_0^\infty x^{(n+1)-1} e^{-x} dx \\ &= \int_0^\infty x^n e^{-x} dx \\ &= \lim_{t \rightarrow \infty} \int_0^t x^n e^{-x} dx\end{aligned}$$

$$\text{Integrate by Parts: } u = x^n \quad dv = e^{-x} dx \\ du = nx^{n-1} dx \quad v = -e^{-x}$$

$$\begin{aligned}&= \lim_{t \rightarrow \infty} \left([-x^n e^{-x}]_0^t - \int_0^t (nx^{n-1} e^{-x}) dx \right) \\&= \lim_{t \rightarrow \infty} \left(\underbrace{-t^n e^{-t}}_{\rightarrow 0} + 0 + n \underbrace{\int_0^t x^{n-1} e^{-x} dx}_{\Gamma(n), \text{ a #}} \right) \\&= -\frac{t^n}{e^t} \\&= n\Gamma(n)\end{aligned}$$

⑦ By using mathematical induction,

$$\left. \begin{array}{l} \textcircled{a} \quad \Gamma(1) = 1 \\ \textcircled{b} \quad \Gamma(n+1) = n\Gamma(n), \quad n: \text{natural #} \end{array} \right\} \Rightarrow \Gamma(n+1) = n!$$

Idea:

$\Gamma(1) = 1$	$\leftarrow 0!$
$\Gamma(2) = 1 \cdot \Gamma(1) = 1$	$\leftarrow 1!$
$\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1$	$\leftarrow 2!$
$\Gamma(4) = 3 \cdot \Gamma(3) = 3 \cdot 2 \cdot 1$	$\leftarrow 3!$
⋮	

Note: You could think of $\Gamma(3.5)$ as $2.5!$.