CH. 10: LIMITS and IMPROPER $\int$s

10.1: LIMITS I: INDET(ERMINATE) FORMS $\frac{0}{0}$, $\frac{\infty}{\infty}$

need more work!

All derivatives involve $\frac{0}{0}$: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Pop #1, 2, 4, 14

Ⓐ L'Hôpital's Rule

If

$$\lim_{x \to c} \frac{f(x)}{g(x)} \quad \frac{\to 0}{\to 0} \text{ or } \frac{\to \pm\infty}{\to \pm\infty}$$

(or $x \to c^-, c^+, \infty, -\infty$)

Then

$$= \lim \frac{f'(x)}{g'(x)}$$

(not Quotient Rule)

Your 150 teacher would've slapped you

Provided,

1) This limit is not "DNE" ($\infty, -\infty$ OK). (If DNE, maybe analyze $\lim_{x \to c^-}$ vs. $\lim_{x \to c^+}$?)
2) f, g are differentiable, and $g' \neq 0$ "near where we care" $x \to c$, $x \to \infty$

   $\lim_{x \to c}$: "around" (maybe not at) c

   $\lim_{x \to \infty}$: "eventually"

Ignore in 1st

If you get DNE, you've wasted your time. L'H was inapp. to apply in 1st place

Repeat until

1) you can determine the limit, maybe after manipulation, or
2) L'Hôpital can't be applied

and worse

10.2 - manip so we can use L'H

(B) Exs

Ex (Pop #4)

Leading term of a poly determines its long-term behavior ($x \to \pm\infty$)

$$\lim_{x\to\infty} \frac{\overbrace{2x^3}^{\text{leading term}} - x + 3}{\underbrace{4x^2} + 1} \quad \begin{matrix}\leftarrow D_x \\ \leftarrow D_x\end{matrix} \quad \underset{\text{Write!}}{\left(\frac{\infty}{\infty}\right)}$$

Write!

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{6x^2-1}{8x} \quad \left(\frac{\infty}{\infty}\right)$$

$\frac{12}{0}$? No!

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{12x}{8} \quad \begin{matrix}\to \infty \\ \to 8\end{matrix} \qquad \cancel{L'H}$$

$$= \boxed{\infty}$$

$$\text{or} \quad \lim_{x\to\infty} \frac{6x^2-1}{8x} = \lim_{x\to\infty} \left(\overset{\to\infty}{\left(\frac{3}{4}x\right)} - \overset{\to 0}{\left(\frac{1}{8x}\right)} \right)$$

$$= \boxed{\infty}$$

(Discuss Pop #1)

Ex (Pop #14) $\displaystyle \lim_{x\to 0} \frac{\sin x}{x} \quad \left(\frac{0}{0}\right)$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\cos x}{1} \quad \begin{matrix}\to \cos 0 = 1 \\ \to 1\end{matrix}$$

uglier proof on A1.09-10

$$= \boxed{1}$$

If you forget what this is...

Circular reasoning! $D_x(\sin x) = \cos x$

Pop #12: $\lim_{x\to\infty} \sin x$ DNE

vs. #15 $\lim_{x\to\infty} \frac{\sin x}{x}$ $\frac{\to \text{DNE}}{\to \infty}$ $\cancel{L'H}$

Ex $\lim_{x\to 0^+} \frac{e^{3x}-1}{x}$ $\frac{\to e^0-1=1-1=0}{\to 0}$ $\left(\frac{0}{0}\right)$

$\stackrel{L'H}{=} \lim_{x\to 0^+} \frac{3e^{3x}}{1}$ $\frac{\to 3e^0=3}{\to 1}$

$= \boxed{3}$

Would splitting work? NO

$\lim_{x\to 0^+} \left(\frac{e^{3x}}{x} - \frac{1}{x}\right)$ $\quad \frac{e^{3x}}{x}$ $\frac{\to 1}{\to 0^+}$

$\infty \qquad \infty$

$\infty-\infty$ is indet.
($\infty+\infty$ is not. $\to\infty$)

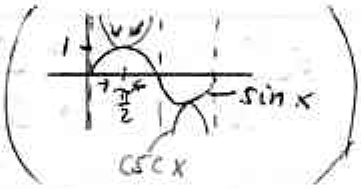
The remedy would be to recombine → single fraction
$\lim_{x\to 0^+} \frac{e^{3x}+1}{x}$ $\frac{\to 2}{\to 0^+}$
$=\infty$ (Don't use L'H)

If $\lim_{x\to 0}$, consider $\lim_{x\to 0^-}$, also.

Wherever trig funcs are defined we have 2-sided limits.

$\sin \frac{\pi}{2} = 1 \Rightarrow \csc \frac{\pi}{2} = 1$

Ex $$\lim_{x \to \frac{\pi}{2}} \frac{1 - \csc x}{\cos x} \quad \begin{matrix} \to 1 - 1 = 0 \\ \to \cos \frac{\pi}{2} = 0 \end{matrix} \quad \left(\frac{0}{0}\right)$$



$$\lim_{x \to \frac{\pi}{2}^-} \frac{\sec x}{\tan x} \overset{L'H}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{\sec x \tan x}{\sec^2 x} \quad \text{loop!}$$

Have to → $\lim_{x \to \frac{\pi}{2}^-} \frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}$

$$\overset{L'H}{=} \lim_{x \to \frac{\pi}{2}} \frac{\csc x \cot x}{-\sin x}$$

Rewrite

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\cancel{\sin x}} \cdot \frac{\cos x}{\cancel{\sin x}}}{-\sin x} \cdot \frac{\cancel{\sin^2 x}}{\sin^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{-\sin^3 x} \quad \begin{matrix} \to 0 \\ \to -[\underbrace{\sin(\frac{\pi}{2})}_{=1}]^3 = -1 \end{matrix}$$

$$= \boxed{0}$$

Tell me what you think.

Ex $$\lim_{x \to 0} \frac{\cos x}{x} \overset{\cancel{L'H}}{=} \lim_{x \to 0} \frac{-\sin x}{1} = 0$$

NO!

$$\frac{\cos x}{x} \quad \begin{matrix} \to 1 \\ \to 0 \end{matrix}$$

~~L'H~~ (does not apply!)

Correct: $\lim_{x \to 0^-} \frac{\cos x}{x} \begin{matrix} \to 1 \\ \to 0^- \end{matrix} = -\infty$ | $\lim_{x \to 0^+} \frac{\cos x}{x} \begin{matrix} \to 1 \\ \to 0^+ \end{matrix} = \infty$

$$\Rightarrow \lim_{x \to 0} \frac{\cos x}{x} \ \boxed{DNE}$$