10.2: LIMITS II: MORE INDET. FORMS

(A) Review "Limit Forms"

Books don't say indet.
DNE $\frac{}{0} \leftarrow \frac{\pm}{0}$

Ex $\lim_{x\to\infty} \frac{3}{\frac{\sin x}{x}}$
$\frac{\sin x}{x} \to 0$ but doesn't $\to 0^+$ or $\to 0^-$

	Limit	
$\frac{0}{3}$	0	
$\frac{0}{\pm\infty}$	0	
$\frac{3}{0}$	need more info	$\frac{\infty}{0}$ similar
$\frac{3}{0^+}$	∞	
$\frac{3}{0^-}$	$-\infty$	
$\frac{-3}{0^-}$	∞	
$\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$	indet. (L'H?)	

Note $\left(\lim_{x\to\infty} \frac{3}{\frac{\sin x}{x}} \text{ DNE} \right.$
$\frac{\sin x}{x} \to 0$ by Squeeze Thm.
but doesn't $\to 0^+$ or $\to 0^-$

(B) $0 \cdot \infty$ Indet. Form

(∞ or $-\infty$)

(Why Indet.?)

Exs As $x \to 0^+$, $\overset{\to 0}{x} \cdot \overset{\to\infty}{\frac{1}{x}} = 1 \to 1$
$x \cdot \frac{1}{x^2} = \frac{1}{x} \to \infty$ (Different results)
$x^2 \cdot \frac{1}{x} = x \to 0$

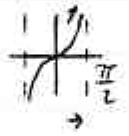
$\lim_{x\to 0} (\sin x \cdot \frac{1}{x^2})$ DNE:
$\lim_{x\to 0^+}$ is ∞
$\lim_{x\to 0^-}$ is $-\infty$

Trick Rewrite into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.
Use L'H?

I put in ().

Ex (#2) $\lim\limits_{x \to \frac{\pi}{2}^-} (\tan x)(\ln(\sin x))$ $\quad (\infty \cdot 0)$

$\tan x \downarrow \infty$ $\qquad \ln(\sin x) \downarrow \ln(\sin \frac{\pi}{2}) = \ln 1 = 0$ $\quad$ ☆



Choice: $\frac{\infty}{\infty}$ or $\frac{0}{0}$

$D_x\left(\frac{1}{\ln \sin x}\right) = \frac{\frac{1}{\sin x} \cdot \cos x}{(\ln(\sin x))^2} = \frac{\cot x}{(\ln(\sin x))^2}$

Rewrite $\tan x \ln(\sin x)$ as

$$\frac{\tan x}{\frac{1}{\ln(\sin x)}} \quad \text{or} \quad \frac{\ln(\sin x)}{\frac{1}{\tan x}}$$

Easier

$$= \frac{\ln(\sin x)}{\cot x}$$

You've already analyzed the limit.

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\ln(\sin x)}{\cot x} \quad \begin{matrix} \to 0 \text{ by ☆} \\ \to 0 \end{matrix} \quad \left(\frac{0}{0}\right)$$

If $\tan x \to \infty$, then $\cot x \to 0$

$$\overset{L'H}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{1}{\sin x} \cdot \overbrace{D_x(\sin x)}^{= \cos x}}{-\csc^2 x} \quad \leftarrow \text{Rewrite}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{\cos x}{\sin x}}{-\frac{1}{\sin^2 x}} \cdot \frac{\sin^2 x}{\sin^2 x} \quad \text{or} \quad \lim_{x \to \frac{\pi}{2}^-} \left(\frac{\cos x}{\sin x}\right)(-\sin^2 x)$$

$$= \lim_{x \to \frac{\pi}{2}^-} (-\cos x \sin x)$$

You know one's 0, other's 1

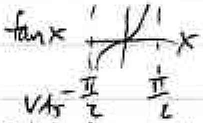
$$= -\underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1}$$

Up to 11

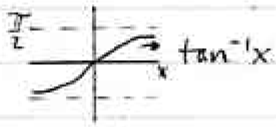
$$= \boxed{0}$$

$\underline{Ex}$ (#8) $\lim_{x\to\infty} x\left(\frac{\pi}{2}-\tan^{-1}x\right)$ $(\infty\cdot 0)$

$x \to \infty$; $\left(\frac{\pi}{2}-\tan^{-1}x\right) \to \frac{\pi}{2}-\frac{\pi}{2} = 0$



$\Rightarrow$ HAs for $\tan^{-1}x$



Could do $\frac{\infty}{\infty}$ form

$$= \lim_{x\to\infty} \frac{\frac{\pi}{2}-\tan^{-1}x}{\frac{1}{x}} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} \quad \leftarrow D_x(x^{-1}) = -x^{-2}$$

÷ by a frac ⇒ · by its recip.

$$= \lim_{x\to\infty} \left(-\frac{1}{1+x^2}\right)(-x^2)$$

$$= \lim_{x\to\infty} \frac{x^2}{1+x^2} \quad \left(\frac{\infty}{\infty}\right)$$

Could ÷ N, D by x^2 (MISO)

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{2x}{2x}$$

Just alg. simplify ($\forall x \neq 0$)

$$= \boxed{1}$$

Ⓒ 0^0, ∞^0, 1^∞ Indet. Forms

Yay!

Alg. I $\left.\begin{array}{l} x^0 = 1 \\ 0^x = 0 \end{array}\right\}$ if $x \neq 0$

0^0 ?

(Could be "2"!)

Note $\left(\lim_{x \to 0^+} \underset{\downarrow \, 0}{x}^{\frac{\ln 2}{1+\ln x} \to 0} = 2 \right)$

$\simeq x^{\frac{\ln 2}{\ln x}} = x^{\log_x 2}$
$= 2$
Larson 7.7.86 ↗

1^∞ Exs $\lim_{x \to \infty} \underset{\downarrow \, 1}{\left(1 + \frac{1}{x}\right)}^{x \to \infty} = \boxed{e}$ (in Cont. Comp. Interest)

but $\lim_{x \to \infty} 1^x = \boxed{1}$ — different!

$(0^+)^\infty \Rightarrow 0$ Think $\left(\frac{1}{10}\right)^{1000} \simeq 0$

Not indet.

Is this indet? no;
$x^{1/x}$ problem every
$(0^+)^{-\infty} = \infty$ ↗
Larson 7.7.83-4

Ex (#22) $\lim_{x \to 0^+} \underset{\downarrow \, 1}{(1+3x)}^{\csc x \to \infty}$ (1^∞)

$\sin x \to 0^+$
$\Rightarrow \csc x \to \infty$

Note $\csc x$

How do we bring this exponent down to Earth?

① Let $y = (1+3x)^{\csc x}$

② $\ln y = \ln (1+3x)^{\csc x}$
$\ln y = \csc x \cdot \ln \underbrace{(1+3x)}_{>0 \text{ as } x \to 0^+}$

Like Log Diff but only log a problem in Log Diff
Avoids | | issue, also
$\ln x^2 = 2 \ln |x|$

③ $\lim_{x \to 0^+} \ln y \overset{ⓔ}{=} \lim_{x \to 0^+} \underset{\downarrow \, \infty}{\csc x} \cdot \underset{\ln 1 = 0}{\ln (1+3x)}$ $(\infty \cdot 0)$

Reminder!

$\overset{ⓔ}{=} \lim_{x \to 0^+} \frac{\ln (1+3x)}{\sin x}$ $\left(\frac{0}{0}\right)$

In the end, what will we use to kill off ln? Remind yourself! I always forget at the end!
If ' ' for lim ln y, you forget, anyway

$$\overset{L'H}{\underset{ⓔ}{=}} \lim_{x\to 0^+} \frac{\frac{1}{1+3x}\cdot 3}{\cos x}$$

$$\underset{ⓔ}{=} \lim_{x\to 0^+} \frac{3}{(1+3x)\cos x} \quad \frac{\to 3}{\to (1)\cos 0 = 1}$$

$$\underset{ⓔ}{=} 3$$ NOT DONE!

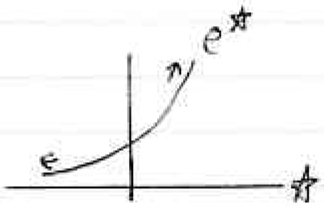
④ $\lim_{x\to 0^+} \ln y = 3$

Continuity of $e^{☆}$

$$\Rightarrow \lim_{x\to 0^+} e^{\ln y} = e^3$$

$$\Rightarrow \lim_{x\to 0^+} y = \boxed{e^3}$$

Note If $\ln y \to \infty$
$\Rightarrow y \to \infty$ ("$e^{\infty} = \infty$")



If $\ln y \to -\infty$
$\Rightarrow y \to 0$ ("$e^{-\infty} = 0$")

Up to 21

Ⓓ $\infty - \infty$ Indet. Form

As $x \to \infty$, $x^2 - x^2 \to 0$, $x^2 - x \to \infty$, $x - x^2 \to -\infty$ } different!

Ex (#24) $\lim_{x\to 1^+} \left(\frac{1}{\underbrace{x-1}_{\to 0^+}} - \frac{1}{\underbrace{\ln x}_{\to 0^+}} \right)$ $(\infty - \infty)$

Trick: Get 1 fraction

To + We need a ... LCD

In num.: who's missing?

$$= \lim_{x\to 1^+} \frac{\overbrace{\ln x}^{\text{missing from 1st frac.}} - (x-1)}{(x-1)\ln x}$$ (Simplify 1st, then identify limit form.)

$$= \lim_{x \to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x} \quad \begin{matrix} \to \overbrace{\ln 1}^{=0} - 1 + 1 = 0 \\ \to 0 \underbrace{\ln 1}_{=0} = 0 \end{matrix} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{x \to 1^+} \frac{\frac{1}{x} - 1}{\ln x + (x-1)(\frac{1}{x})} \cdot \frac{x}{x}$$

By Product Rule

$$= \lim_{x \to 1^+} \frac{1 - x}{\underbrace{x \ln x}_{\text{Use Product Rule}} + x - 1} \quad \begin{matrix} \to 0 \\ \to 1 \underbrace{\ln 1}_{=0} + 1 - 1 = 0 \end{matrix} \quad \left(\frac{0}{0}\right)$$

$$\overset{L'H}{=} \lim_{x \to 1^+} \frac{-1}{\ln x + \underbrace{x(\frac{1}{x})}_{=1} + 1}$$

$$= \lim_{x \to 1^+} \frac{-1}{\ln x + 2} \quad \begin{matrix} \to -1 \\ \to \underbrace{\ln 1}_{=0} + 2 = 2 \end{matrix}$$

$$= \boxed{-\frac{1}{2}}$$