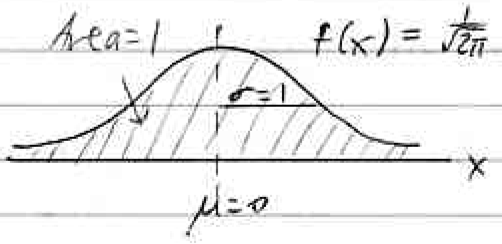
10.3: IMPROPER $\int$s I: $\int_{\pm\infty}^{\pm\infty} f(x)\,dx$ or

Ⓐ Intro Exs

Standard Normal Density Func. (Stats)

Tails fall rapidly. If a gallon of paint covers 1 ft.² (for a given thickness) Area = 1 ft² You won't run out of paint.



$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

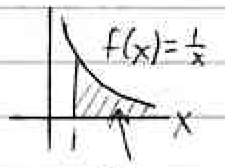
Casella 104-5
Calc III

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

a real # $\Leftrightarrow$ the $\int$ converges
Otherwise, it diverges (maybe to ∞ or $-\infty$).

Graph doesn't fall fast enough.

Ex



$\int_1^{\infty} f(x)\,dx$ diverges to ∞

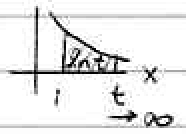
Swok 505: not definite
Harper Pitt: is

SKIP; Proof is Key.

Why? $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \underbrace{\int_1^t \frac{1}{x}\,dx}_{\text{has a value (for fixed } t \geq 1)}$

cont. on $[1,\infty)$ $\Longrightarrow$ has a value (for fixed $t \geq 1$)

Remem the defn.



$= \lim_{t\to\infty} \ln t$ by defn of $\ln t$ (p.382)

$= \infty$

Proof: pp. 389-90 · Key!

Idea of Proof:
$\ln 4 > \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$
$\ln 4 > 1$
$M \ln 4 > M$, $M \in \mathbb{Q}^+$
$\ln 4^M > M$
$\ln 4$ = area under curve
$\ln x$ can be made arbit. large

(B) Exs

Ex (#22) Does $\int_0^{\infty} xe^{-x}\,dx$ converge?
If so, find its value.

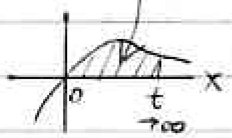
First, find $\int xe^{-x}\,dx$

$\int$ by Parts: $u = x \quad dv = e^{-x}dx$
$du = dx \quad v = -e^{-x}$

$$\int xe^{-x}dx = -xe^{-x} - \int -e^{-x}dx$$
$$= -xe^{-x} + \int e^{-x}dx$$
$$= -xe^{-x} - e^{-x} + C$$

$\int_0^{\infty}$ ticks us off

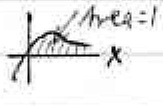
$$\underbrace{\int_0^{\infty} xe^{-x}dx}_{\text{cont. on } [0,\infty)} = \lim_{t\to\infty} \underbrace{\int_0^{t} xe^{-x}dx}_{\text{Definite }\int} \quad \text{(if it exists)}$$



Apply FTC. (Fund. Thm. of Calc.)

$$= \lim_{t\to\infty}\left[-xe^{-x} - e^{-x}\right]_0^t$$
$$= \lim_{t\to\infty}\left(\left[-te^{-t} - e^{-t}\right] - \left[0 - \underbrace{e^{0}}_{1}\right]\right)$$
$$= \lim_{t\to\infty}\left(-\underbrace{\frac{t}{e^t}}_{\to 0} - \underbrace{\frac{1}{e^t}}_{\to 0} + 1\right)$$
(★)
$$= 1$$

$\therefore$ $\boxed{\int_0^{\infty} xe^{-x}dx \text{ converges, and its value is } 1.}$



(next)

HW 10.1.31
(n int. > 0)
Obvious if n ≤ 0

(A) $\lim_{t\to\infty} \frac{t^{n}}{e^t} = 0$ (n ← real)

$\Rightarrow e^t$ "overwhelms" any poly. func. of t in the long run.

Big-O: papers over messy details

Analysis of algorithms
time, memory requirements as input size $\to\infty$
poly.-time vs. exp'l-time algorithms ("BAD!")

#12: 1, ∞

Ex $\int_{-\infty}^{3} \frac{x}{(1+x^2)^2}\,dx$

cont. on $(-\infty, 3]$

$$\int \frac{x}{(1+x^2)^2}\,dx = \ldots = -\frac{1}{2(1+x^2)} + C$$

Strategy?

Let $u = 1+x^2$
$du = 2x\,dx$

$$\int_{-\infty}^{3} \frac{x}{(1+x^2)^2}\,dx = \lim_{t\to-\infty} \int_t^3 \frac{x}{(1+x^2)^2}\,dx \quad \left(\text{if exists}\right)$$

$$= \lim_{t\to-\infty} \left[-\frac{1}{2(1+x^2)}\right]_t^3$$

$$= \lim_{t\to-\infty} \left(-\frac{1}{20} - \underbrace{\left[-\frac{1}{2(1+t^2)}\right]}_{\to 0}\right)$$

Up to 13

$$= \boxed{-\frac{1}{20}, \int \text{converges}}$$

Not in books

Note

Method 2 (no one does this, though)

$$\int_{-\infty}^{3} \frac{x}{(1+x^2)^2}\,dx$$

Let $u = 1+x^2$
$du = 2x\,dx$
$\Rightarrow \frac{1}{2}du = x\,dx$

$$= \lim_{t\to-\infty} \int_{x=t}^{x=3} \frac{x}{(1+x^2)^2}\,dx \quad \left(\text{if exists}\right)$$

$x = t \Rightarrow u = 1+t^2$
$x = 3 \Rightarrow u = 10$

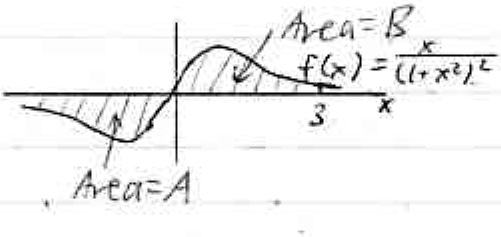
$$= \lim_{t\to-\infty} \int_{u=1+t^2}^{u=10} \frac{\frac{1}{2}du}{u^2}$$

$$= \lim_{t\to-\infty} \left[-\frac{1}{2u}\right]_{u=1+t^2}^{u=10}$$

$$= \lim_{t\to-\infty} \left(-\frac{1}{20} - \underbrace{\left[-\frac{1}{2(1+t^2)}\right]}_{\to 0}\right)$$

$$= \boxed{-\tfrac{1}{20},\ \int \text{converges}}$$

In terms of A,B what is the value of $\int$?

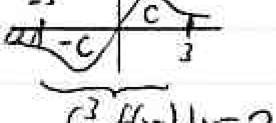


$$\int_{-\infty}^{3} f(x)\,dx = -A + B$$
$$= -\tfrac{1}{20}$$

$= \overbrace{\int_{-\infty}^{-3}}^{=-\frac{1}{20}\text{ also}} + \underset{\int_{-3}^{3}}{0}$ odd func. I'll mention later

Note $\Big($ f odd $\Rightarrow \int_{-\infty}^{-3} f(x)\,dx = -\frac{1}{20}$, also.

$\int_{3}^{\infty} f(x)\,dx = \frac{1}{20}$ $\Big)$



$\int_{-3}^{3} f(x)\,dx = 0$

Same integrand as previous Ex.

Ex $$\int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2}\,dx \quad ☆$$

cont. on $(-\infty,\infty)$

$$= \overset{\text{I}}{\int_{-\infty}^{0} \frac{x}{(1+x^2)^2}\,dx} + \overset{\text{II}}{\int_{0}^{\infty} \frac{x}{(1+x^2)^2}\,dx}$$

(if both $\int$s converge)

Note 1 If one or both diverge, then the original $\int$ ☆ diverges.

Note 2 $\int_{-\infty}^{1} + \int_{1}^{\infty}$ OK (Doesn't matter where split)
$\Rightarrow$ Same result

Why? $$\underbrace{\int_{-\infty}^{0} + \int_0^1}_{\int_{-\infty}^{1}} + \int_1^\infty \quad \text{with } \int_0^1 + \int_1^\infty = \int_0^\infty$$

$\int_{-\infty}^{\infty} = \int_{-\infty}^{0} + \int_0^1 + \int_1^\infty$ if converge

t OK but confusing in (2) lim expr self-contained

$$\text{I} = \lim_{t\to-\infty} \int_t^0 \frac{x}{(1+x^2)^2}\,dx \quad \text{(if exists)}$$

$$= \lim_{t\to-\infty} \Bigg[\overbrace{-\frac{1}{2(1+x^2)}}^{\text{from before}}\Bigg]_t^0$$

$$= \lim_{t\to-\infty} \left(-\frac{1}{2} - \left[-\underbrace{\frac{1}{2(1+t^2)}}_{\to 0}\right]\right)$$

$$= -\frac{1}{2}$$

$$\text{II} = \lim_{w\to\infty} \int_0^w \frac{x}{(1+x^2)^2}\,dx \quad \text{(if exists)}$$

$$= \lim_{w\to\infty} \left[-\frac{1}{2(1+x^2)}\right]_0^w$$

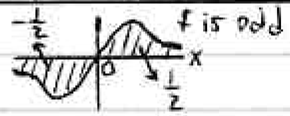
$$= \lim_{w\to\infty} \left(-\underbrace{\frac{1}{2(1+w^2)}}_{\to 0} - \left[-\frac{1}{2}\right]\right)$$

$$= \frac{1}{2}$$

Why should this not be surprising? Up to 23

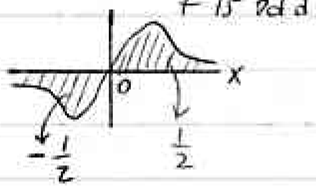
$$\text{I} + \text{II} = -\frac{1}{2} + \frac{1}{2}$$

$$= \boxed{0, \int \text{converges}}$$



Begs the ? Is that true of all odd funcs? NO!

Can go up to 23 we'll talk more.



Ex $\int_{-\infty}^{\infty} x\,dx$ (A)

$= \underbrace{\int_{-\infty}^{0} x\,dx}_{\mathrm{I}} + \underbrace{\int_{0}^{\infty} x\,dx}_{\mathrm{II}}$ (if $\int$s converge)

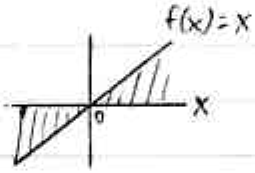
Wait to see if (I) diverges! (Then (A) div.)
Write this anyway, though! If (I) div, you don't have to work it out.

You may want to do this before (I), since ∞ may be easier to deal with than $-\infty$.

$\mathrm{I} = \lim_{t\to-\infty} \int_{t}^{0} x\,dx$

$= \lim_{t\to-\infty} \left[\frac{x^2}{2}\right]_t^0$

$= \underbrace{\lim_{t\to-\infty} \left(0 - \frac{t^2}{2}\right)}_{= -\infty}$



$\Rightarrow$ (I) diverges

$\Rightarrow \underbrace{\int_{-\infty}^{\infty} x\,dx}_{A}$ $\boxed{\text{diverges}}$ (x is odd, but answer is <u>not</u> 0!)

$\nexists$ may be inapp. "div." like $x = \frac{1}{0}$

WARNING $\int_{-\infty}^{\infty} f(x)\,dx$ does not necessarily equal $\lim_{t\to\infty} \int_{-t}^{t} f(x)\,dx$.

Dumb Plan

Edwards 528 $\int_{-\infty}^{\infty} \frac{1+x\,dx}{1+x^2}$ div. but $\lim_{t\to\infty}\int_{-t}^{t} = \pi$

Me: If $\int_{-\infty}^{\infty}$ converges $\Rightarrow \lim_{t\to\infty}\int_{-t}^{t} = ?$

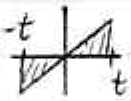
$\infty - \infty$, but we know: 0 | $\int_{-1M}^{1M} = 0$

$\int_{-\infty}^{\infty} x\,dx \overset{\text{WRONG}}{=\!=} \lim_{t\to\infty} \int_{-t}^{t} x\,dx$

$= \lim_{t\to\infty} \left[\frac{x^2}{2}\right]_{-t}^{t}$

$= \lim_{t\to\infty} \left(\frac{t^2}{2} - \frac{t^2}{2}\right)$

or "f odd"



$= \cancel{0}$ WRONG!

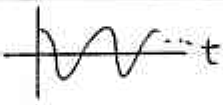
$\underline{Ex}$ $\int_0^\infty \underbrace{\sin x}_{\text{cont. on } [0,\infty)} dx = \lim_{t\to\infty} \int_0^t \sin x\, dx$ (if exists)

$= \lim_{t\to\infty} [-\cos x]_0^t$

$= \lim_{t\to\infty} (-\cos t - [\overbrace{-\cos 0}^{=1}])$

$= \lim_{t\to\infty} (1 - \underbrace{\cos t}_{})$

$\lim_{t\to\infty} \cos t$ DNE



(and there are no ops. nor simplifications that will prevent the whole thing from being DNE)

(Examples where DNE is "prevented":

$\lim_{t\to\infty} \frac{\sin t}{t} = 0$ ↑squeeze thm.

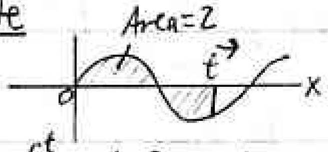
$\lim_{t\to\infty} \frac{2+\cos t}{2+\cos t} = 1$)

DNE

Edwards: D by oscillation
Me: What if

$\therefore \int_0^\infty \sin x\, dx$ $\boxed{\text{diverges}}$

$\underline{Note}$



$\int_0^t$: 0 1 2 1 0 → $\lim_{t\to\infty}$ DNE

© Comparison Tests

If f, g are cont., and $0 \le f(x) \le g(x)$, } on $[a, \infty)$

If $\int_a^\infty g(x)\, dx$ converges $\Rightarrow \int_a^\infty f(x)\, dx$ converges.
(big bro fits ⇒ so do you)

You diverge - you have to break up to get thru door.

If $\int_a^\infty f(x)\, dx$ diverges $\Rightarrow \int_a^\infty g(x)\, dx$ diverges.
(little bro can't fit ⇒ neither can you)

(Remember Normal?)

Does this look somewhat familiar? (Normal?) Take a guess... $e^{-\frac{x^2}{2}} \geq e^{-x^2}$ e.g., $e^{-5} \geq e^{-10}$

Ex (#28) Show $\int_1^\infty \underbrace{e^{-x^2}}_{\text{cont., } \geq 0 \text{ on } [1,\infty)} dx$ converges.

① Show $e^{-x^2} \leq \underbrace{e^{-x}}_{\substack{\text{"big bro"} \\ \text{(Can } \int)}}$ on $[1,\infty)$

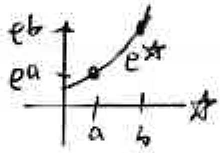
Not true on $(0,1)$ twilight zone $(\frac{1}{2})^2 < \frac{1}{2}$ MISO: Area bounded by x, x^2

$$x^2 \geq x \text{ on } [1,\infty)$$
$$\Rightarrow -x^2 \leq -x$$
$$\Rightarrow e^{-x^2} \leq e^{-x}$$

Order is preserved. Take reciprocals, $\cdot(-1)$ don't preserve order.

(because $e^{\star}$ is an increasing func. of $\star$)

$e^{\star}$ is $\nearrow$ in $\star$



② Show $\int_1^\infty e^{-x} dx$ converges.

$$\int_1^\infty \underbrace{e^{-x}}_{\text{cont. on } [1,\infty)} dx = \lim_{t\to\infty} \int_1^t e^{-x} dx \quad (\text{if exists})$$
$$= \lim_{t\to\infty} \left[-e^{-x}\right]_1^t$$
$$= \lim_{t\to\infty} \left(-e^{-t} - [-e^{-1}]\right)$$
$$= \lim_{t\to\infty} \left(\underbrace{-\frac{1}{e^t}}_{\to 0} + \frac{1}{e}\right)$$
$$= \frac{1}{e}, \int \text{ converges}$$

can skip if no time

③ $\therefore \int_1^\infty e^{-x^2} dx$ converges.

Show $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ conv.

Key: Show $\int_1^\infty \underbrace{e^{-\frac{x^2}{2}}}_{\leq e^{-\frac{x}{2}}} dx$ conv.