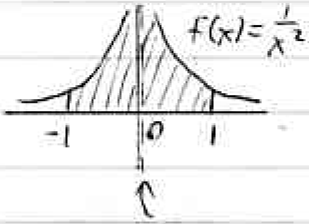
More on FOX

10.4: IMPROPER $\int$s II: WHEN INTEGRANDS EXPLODE

(A) Intro Ex

Edwards 523

$$\int_{-1}^{1} \frac{1}{x^2}\,dx \overset{\text{WRONG!}}{=\!=\!=} \left[-\frac{1}{x}\right]_{-1}^{1} \quad \text{(FTC)}$$
$$= -1 - \left(-\frac{1}{-1}\right)$$
$$= -1 - 1$$
$$= \cancel{-2}$$



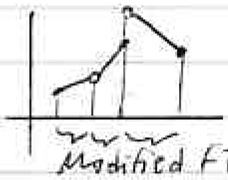
"-2" nonsense!
(<0)

f has an <u>interior discontinuity</u> at $x=0$.

It's an <u>infinite discont'y</u>, so FTC does <u>not</u> apply!

If you have a finite # of rem, jump disconts, you may be OK

(Note



Modified FTC may work.)

Correct

$$\int_{-1}^{1} \frac{1}{x^2}\,dx \quad (A)$$

discont. at $x=0$
cont. on $[-1,0) \cup (0,1]$

$$= \overbrace{\int_{-1}^{0} \frac{1}{x^2}\,dx}^{\text{I}} + \overbrace{\int_{0}^{1} \frac{1}{x^2}\,dx}^{\text{II}}$$

(*if* both ∫s converge)

If one or both diverge, then $\underbrace{\text{the original } \int}_{A}$ diverges.

← Write this out, even if (I) diverges. (Actually, you may want to do (II) first, since 1 may be easier to deal with than -1.)

$$\text{(I)} \quad \int_{-1}^{0} \frac{1}{x^2}\,dx = \lim_{t \to 0^-} \int_{-1}^{t} \frac{1}{x^2}\,dx \quad \text{(if exists)}$$

(You run towards the problem)
(Helps to draw arrow to see if 0^- or 0^+)

$$\overset{\text{FTC}}{=} \lim_{t \to 0^-} \left[-\frac{1}{x} \right]_{-1}^{t}$$

$$= \lim_{t \to 0^-} \left(-\frac{1}{t} - \left[-\frac{1}{-1} \right] \right)$$

$$= \lim_{t \to 0^-} \left(-\underbrace{\frac{1}{t}}_{-\infty} - 1 \right)$$

$$= \infty$$

$\Rightarrow$ (I) diverges

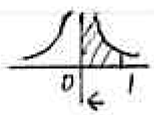
$\Rightarrow$ (A) $\boxed{\text{diverges}}$

<u>Note: (II)</u>

t ok

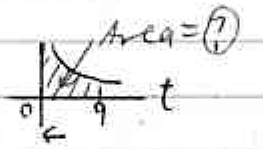
Do you think this diverges? By sym - even

$$\int_{0}^{1} \frac{1}{x^2}\,dx = \lim_{w \to 0^+} \int_{w}^{1} \frac{1}{x^2}\,dx \quad \text{(if exists)}$$



(B) Ex (#2)

$$\int_0^9 \frac{1}{\sqrt{x}}\,dx$$



Cont. on $(0,9]$
undef. at 0

$$= \lim_{t\to 0^+} \int_t^9 \frac{1}{\sqrt{x}}\,dx$$

$$\int x^{-\frac{1}{2}}\,dx = \frac{x^{1/2}}{1/2} + C$$
$$= 2\sqrt{x} + C$$

$$= \lim_{t\to 0^+} \left[2\sqrt{x}\right]_t^9$$

$$= \lim_{t\to 0^+} \left(\underbrace{2\sqrt{9}}_{=6} - \underbrace{2\sqrt{t}}_{\to 0}\right)$$

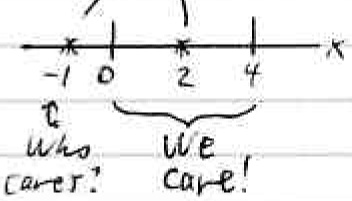
Curve does approach y-axis fast enough!

$$= \boxed{6, \int \text{converges}}$$

(C) Ex (#16)

$$\int_0^4 \frac{1}{x^2-x-2}\,dx = \int_0^4 \frac{1}{(x-2)(x+1)}\,dx$$

cont.
except at



$$= \underbrace{\int_0^2 \frac{1}{(x-2)(x+1)}\,dx}_{\text{I}} + \underbrace{\int_2^4 \frac{1}{(x-2)(x+1)}\,dx}_{\text{II}} \quad \left(\text{if converge}\right)$$

<u>First, find $\int \frac{1}{(x-2)(x+1)}\,dx$</u>

Partial fractions!

I'll skip

$$\left(\begin{aligned} \frac{1}{(x-2)(x+1)} &= \frac{A}{x-2} + \frac{B}{x+1} \\ 1 &= A(x+1) + B(x-2) \\ x=-1 \Rightarrow 1 &= -3B \\ \Rightarrow B &= -\tfrac{1}{3} \\ x=2 \Rightarrow 1 &= 3A \\ \Rightarrow A &= \tfrac{1}{3} \end{aligned} \right)$$

$$\int \frac{1}{(x-2)(x+1)}\,dx = \int \frac{\frac{1}{3}}{x-2}\,dx - \int \frac{\frac{1}{3}}{x+1}\,dx$$

$$= \tfrac{1}{3}\ln|x-2| - \tfrac{1}{3}\ln|x+1| + C$$

$$= \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + C$$

(It turns out to be helpful to transform differences into quotients.)

(I) $\displaystyle\int_0^2 \frac{1}{(x-2)(x+1)}\,dx = \lim_{t\to 2^-} \int_0^t \frac{1}{(x-2)(x+1)}\,dx$ (if exists)

(Again, you run towards the problem.)

$$= \lim_{t\to 2^-} \left[\frac{1}{3} \ln \left| \frac{x-2}{x+1} \right| \right]_0^t$$

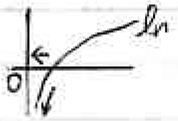
Note: $\frac{t-2}{t+1} \to 0^-$

$$= \lim_{t\to 2^-} \left(\underbrace{\frac{1}{3} \ln \underbrace{\left| \frac{t-2}{t+1} \right|}_{\to 0^+}}_{\to -\infty} - \frac{1}{3} \ln 2 \right)$$

(not 0^- because of $|\,|$)

$\ln \nexists$ careful: I used $\nexists$

$$= -\infty$$



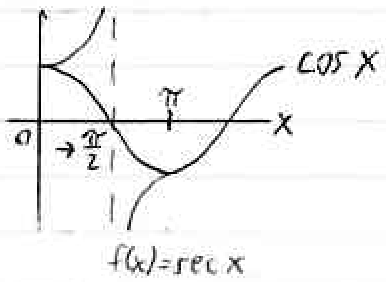
$\Rightarrow$ (I) diverges

$\Rightarrow$ ($\nexists$) $\boxed{\text{diverges}}$

① Ex (#24)

$$\int_0^{\pi} \sec x \, dx \quad Ⓐ$$

$\sec x = \frac{1}{\cos x}$
sec und. $\Leftrightarrow$ $\cos = 0$
sec $\nearrow$ $\Leftrightarrow$ cos $\searrow$
$\sec = 1 \Leftrightarrow \cos = 1$



$$= \underbrace{\int_0^{\pi/2} \sec x \, dx}_{Ⓘ} + \underbrace{\int_{\pi/2}^{\pi} \sec x \, dx}_{ⒾⒾ} \quad \text{(if converge)}$$

$$Ⓘ = \lim_{t \to \frac{\pi}{2}^-} \int_0^t \sec x \, dx$$

$$= \lim_{t \to \frac{\pi}{2}^-} \left[\ln|\sec x + \tan x|\right]_0^t$$

$$= \lim_{t \to \frac{\pi}{2}^-} \left(\ln|\underbrace{\sec t}_{\to\infty} + \underbrace{\tan t}_{\to\infty}| - \underbrace{\ln|\underbrace{\sec 0}_{=1} + \underbrace{\tan 0}_{=0}|}_{=0}\right)$$

$$= \infty$$

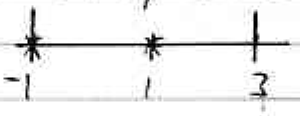
$\Rightarrow$ Ⓘ diverges

$\Rightarrow$ Ⓐ $\boxed{\text{diverges}}$

(E) Ex (#28)

$$\int_{-1}^{3} \frac{x}{\sqrt[3]{x^2-1}}\,dx \quad ⚝$$

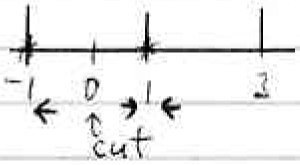
cont. everywhere except at $x=\pm 1$



If you have a problem at both ends ⇒ cut

Idea: $= \int_{-1}^{1} + \int_{1}^{3}$

discont. at both #s ⇒ Need a cut!



$$= \underbrace{\int_{-1}^{0} \cdots}_{\text{I}} + \underbrace{\int_{0}^{1} \cdots}_{\text{II}} + \underbrace{\int_{1}^{3} \cdots}_{\text{III}} \quad \text{(if converge)}$$

OK to just use t; there are self-contained lim exprs.

$$= \lim_{t \to -1^+} \int_{t}^{0} + \lim_{w \to 1^-} \int_{0}^{w} + \lim_{m \to 1^+} \int_{m}^{3} \quad \text{(if exist)}$$

$\vdots$

$$= \frac{3}{4} + \left(-\frac{3}{4}\right) + 3$$

$$= \boxed{3, \int \text{converges}}$$

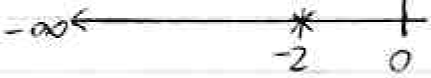
You probably have to find the value of ⚝, anyway.

(To show that ⚝ converges, you must show that I, II, and III all converge!)

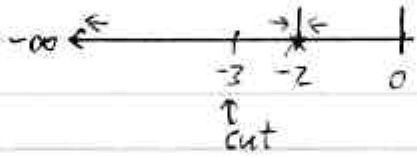
(F) Ex (#30)

Larson: doubly improper

$$\int_{-\infty}^{0} \frac{1}{x+2}\,dx \quad (\star)$$



$$= \int_{-\infty}^{-2} \frac{1}{x+2}\,dx + \int_{-2}^{0} \frac{1}{x+2}\,dx$$



$$= \underbrace{\int_{-\infty}^{-3} \frac{1}{x+2}\,dx}_{\text{(I)}} + \int_{-3}^{-2} \frac{1}{x+2}\,dx + \int_{-2}^{0} \frac{1}{x+2}\,dx$$

$$\text{(I)} = \lim_{t\to-\infty} \int_{t}^{-3} \frac{1}{x+2}\,dx$$

$$= \lim_{t\to-\infty} \Big[\ln|x+2|\Big]_{t}^{-3}$$

$$= \lim_{t\to-\infty} \Big(\underbrace{\ln 1}_{=0} - \underbrace{\ln \underbrace{|t+2|}_{\to\infty}}_{\to\infty}\Big)$$

$$= -\infty$$

$\Rightarrow$ (I) diverges

$\Rightarrow$ $(\star)$ $\boxed{\text{diverges}}$

p. 538 Larson
circumference
of unit circle