Edwards 525

GAMMA FUNCTION, Γ

Euler used this to interpolate $n!$ $\quad (\Gamma(n+1) = n!)$

pron. "oiler"

One of the greatest! (1707-1783)

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\,dx \qquad (t>0)$$

Γ is a continuous function of t.

You can show:

ⓐ $\Gamma(1) = 1 \qquad$ (i.e., $0! = 1$)

$$\Gamma(1) = \int_0^\infty \underbrace{x^{1-1}}_{\substack{=x^0\\=1}} e^{-x}\,dx$$

$$= \int_0^\infty e^{-x}\,dx$$

$$= \lim_{t\to\infty} \int_0^t e^{-x}\,dx$$

$$= \lim_{t\to\infty} \left[-e^{-x}\right]_0^t$$

$$= \lim_{t\to\infty} \left(-\underbrace{e^{-t}}_{\to 0} - (-\underbrace{e^0}_{=1})\right)$$

$$= 1$$

(b) If n is a natural # $(1,2,3,\dots)$,
$$\underbrace{\Gamma(n+1) = n\Gamma(n).}$$

i.e., $n! = n\cdot(n-1)!$ (Recursive relation for $n!$)

$$\begin{aligned}
\Gamma(n+1) &= \int_0^\infty x^{(n+1)-1}e^{-x}\,dx \\
&= \int_0^\infty x^n e^{-x}\,dx \\
&= \lim_{t\to\infty}\int_0^t x^n e^{-x}\,dx
\end{aligned}$$

$\int$ by Parts: $u = x^n \quad dv = e^{-x}dx$
$du = nx^{n-1}dx \quad v = -e^{-x}$

$$\begin{aligned}
&= \lim_{t\to\infty}\left(\left[-x^n e^{-x}\right]_0^t - \int_0^t(-nx^{n-1}e^{-x})\,dx\right) \\
&= \lim_{t\to\infty}\Big(\underbrace{-t^n e^{-t}}_{= -\frac{t^n}{e^t} \to 0} + 0 + n\underbrace{\int_0^t x^{n-1}e^{-x}\,dx}_{\Gamma(n),\ \text{a } \#}\Big) \\
&= n\Gamma(n)
\end{aligned}$$

(c) By using mathematical induction,

$$\left.\begin{aligned} &\text{(a) } \Gamma(1) = 1 \\ &\text{(b) } \Gamma(n+1) = n\Gamma(n),\ n\text{: natural } \# \end{aligned}\right\} \Rightarrow \Gamma(n+1) = n!$$

Idea:
$$\begin{aligned}
\Gamma(1) &= 1 && \leftarrow 0! \\
\Gamma(2) &= 1\cdot\Gamma(1) = 1 && \leftarrow 1! \\
\Gamma(3) &= 2\cdot\Gamma(2) = 2\cdot 1 && \leftarrow 2! \\
\Gamma(4) &= 3\cdot\Gamma(3) = 3\cdot 2\cdot 1 && \leftarrow 3! \\
&\ \vdots
\end{aligned}$$

Note: You could think of $\Gamma(3.5)$ as $2.5!$.