## 11.1: SEQUENCES

Taylor polys

lists of #s (or funcs.)

### (A) $\{a_n\}$

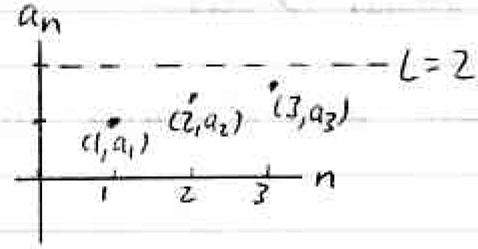
**Ex 1** $\{\underbrace{2-\tfrac{1}{n}}_{a_n\text{, the general } n^{th} \text{ term}}\}$

like $f(n)$, where domain: $n = 1, 2, 3, \ldots$ $a_1 = f(1)$, etc.

$$\underbrace{a_1, a_2, a_3}_{\text{Terms}}, \ldots \to ?$$

$$a_1 = 2 - \tfrac{1}{1} = 1$$
$$a_2 = 2 - \tfrac{1}{2} = \tfrac{3}{2} \text{ or } 1.5$$
$$a_3 = 2 - \tfrac{1}{3} = \tfrac{5}{3} \text{ or } 1.\overline{6}$$

Terms are func values corresp. to y-coords




$f$ interpolates $\{a_n\}$

Discretizing Calc 1 (Ch. 1?) limits limit prob.

People know how to $\int$ but not $\Sigma$.

### (B) $\lim\limits_{n\to\infty} a_n$

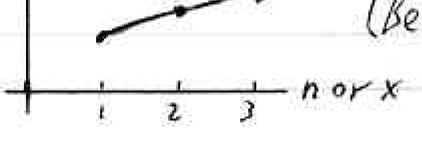
**Ex 1** Since $\lim\limits_{x\to\infty} \underbrace{\left(2 - \tfrac{1}{x}\right)}_{f(x),\ \text{defined on } [1,\infty)} = 2$ $\quad$ ($\tfrac{1}{x} \to 0$)

"Continuous" $f$ is a cont. interpolating func. on $[1,\infty)$

$\Rightarrow \lim\limits_{n\to\infty} a_n = 2$ "Discrete"

↑ a real # $\Leftrightarrow \{a_n\}$ conv. (else, div.)

Terms corresp. to beads on a string. If string $\to 2$ $\Rightarrow$ beads $\to 2$




(Beads on a string) $\{a_n\}$ $f(x)$

If the string goes off to $\infty$ $\Rightarrow$ the beads do too

More intuitive, less math honest: Can you think of a func where $\lim_{x\to\infty}$ DNE? Let's draw s-thing like a transformed version of $\sin x$, except maybe not perfectly periodic (maybe tweaked). Can we place beads so $y \to \#$?

or

Can you think of a string defined on $(1,\infty)$ that passes thru these pts, but $\lim_{x\to\infty}$ DNE?

lim(sum) = sum(lims) if exist

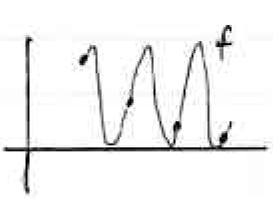
see (E)

Ex

$$\lim_{x\to\infty} f(x) = \infty \ (-\infty) \Rightarrow \lim_{n\to\infty} a_n = \infty \ (-\infty)$$

WARNING $\lim_{x\to\infty} f(x)$ DNE $\underset{\text{(does not imply)}}{\not\Rightarrow}$ $\lim_{n\to\infty} a_n$ DNE (i.e., if $\lim_{x\to\infty} f(x)$ DNE, then $\lim_{n\to\infty} a_n$ may or may not be DNE)

Counterex to $\Rightarrow$




$\lim_{x\to\infty} f(x)$ DNE, but

$a_n \to 0$

Thm 2.8 (p.60) extends to seqs.

limit of sum = sum of limits (if exist), etc.

## (C) Geometric Seqs.

Like K-mart used to do
A dress costs \$1
Marge

Ex $\{(0.9)^n\}$

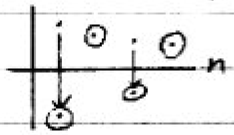
$0.9, 0.81, 0.729, \ldots \to 0$

(10% discounts)

What effect do you think "$(-1)^n$" has?
Factor out $-1$ in here; take $(-1)^n$ out of the whole thing.
Start w/ $(0.9)^n$

Ex $\{(-0.9)^n\}$ or $\{(-1)^n(0.9)^n\}$

$-0.9, 0.81, -0.729, \ldots \to 0$




if base is in this narrow band

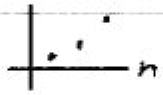
Edwards 535 pf.
$|r^n| = ||r|^n|$
$0<r<1$
$\frac{1}{r^n} = (1+a)^n$
$> 1+na$
$0 < r^n < \frac{1}{1+na}$
$\downarrow$
$0$

$$\boxed{\lim_{n\to\infty} r^n = 0 \text{ if } -1<r<1 \quad (|r|<1)}$$

How can you say ↗ using $| \, |$?
Get used to this!
More compact; books do it.

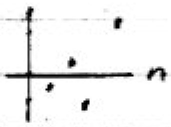
Ex $\{2^n\}$

$2, 4, 8, \ldots \to \infty$




Ex $\{(-2)^n\}$

$-2, 4, -8, 16, \ldots \to$ (DNE)




Geom. bec. really $\{2^n\}$

Ex $\{\underbrace{|(-2)^n|}_{\infty}\}$ same as $\{2^n\}$

$2^n, 3^n, \ldots$

$(-2)^n, (-3)^n, \ldots$

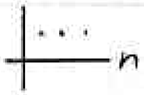
$$\boxed{\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r>1 \\ \text{DNE} & \text{if } r<-1 \end{cases}}$$

$$\boxed{\lim_{n\to\infty} |r^n| = \infty \text{ if } |r|>1}$$

Makes all terms "+"

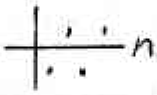
$\underline{r=1}$ $\{1^n\}$

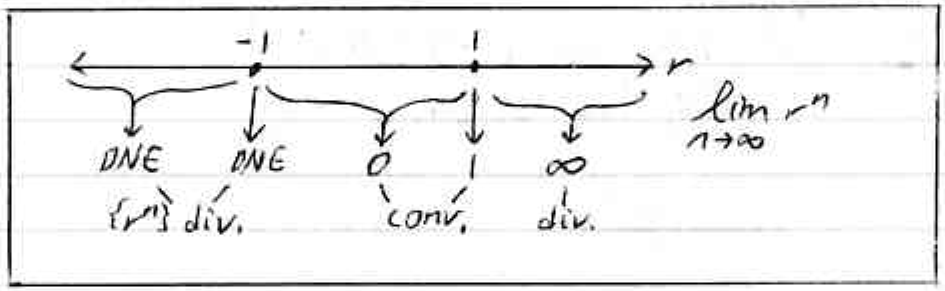
$1, 1, 1, \ldots \to 1$




$\underline{r=-1}$ $\{(-1)^n\}$

$-1, 1, -1, 1, \ldots \to$ (DNE)




2, 6, 18, what next? IQ test

## Ex (Geom. seq.)

$2, 6, 18, 54$

$\cdot 3 \quad \cdot 3$

$2\cdot 3^0 \quad 2\cdot 3^1 \quad 2\cdot 3^2$

$a_1 = 2$ (or a)

$r = 3$ (common ratio)

if we want to write gen. nth term... $\{2(3)^n\}$ we have to modify it starting w/ n=1

$$\{\underbrace{2\underbrace{(3)^{n-1}}_{\infty}}_{\infty}\}$$ as $n \to \infty$

$-2(\infty) = -\infty$

Tursy excl. r=0, a=0 ⇒ then Σ conv for |r|≥1

$\boxed{\text{General Form: } \{a_1 r^{n-1}\}}$ $(a_1 \neq 0)$

You do get \$40 in 1st yr.

Important Idea: $\sqrt{(1.04)^n}$ 1.04

## Ex Principal = \$1000

4% ann. comp. interest (annually compounded)

$$\{\underbrace{1000(1.04)^n}_{\$ \text{ after } n \text{ yrs.}}\} \text{ or } \{1040(1.04)^{n-1}\}$$

(following the form strictly)

What's r? sleazy bank r = 0.4 taking 96% of your \$.

$1040, 1081.6, \ldots \to \infty$

Manhattan Islands Indians - got restitution but not paid yet (by 2083?) Contract was illegal

Books never discuss

## (D) Sign Alternators

| $n =$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $(-1)^n$ | − | + | − | + |
| $(-1)^{n\pm 1}$ | + | − | + | − |

What if I want + −
How do I adjust?
Change parity
$n \pm 3, 5, \ldots$

$\underline{Ex}$ $3, -6, 12, -24, \ldots$

What's the first term?
common ratio?

$$\{3(-2)^{n-1}\} \quad \text{or} \quad \{(-1)^{n-1} 3 (2)^{n-1}\}$$

Terms $\to 0$
$\Leftrightarrow$ their $|\,|$s $\to 0$

## (E) $\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} |a_n| = 0$

($\Rightarrow$ Clear: limit def'n uses $|\,|$, anyway
$\Leftarrow$ Proven using Squeeze Thm,
$-|a_n| \le a_n \le |a_n|$)

$\underline{Ex}$ $\left\{\frac{(-1)^n}{n}\right\}$

$$-1, \tfrac{1}{2}, -\tfrac{1}{3}, \tfrac{1}{4}, \ldots$$

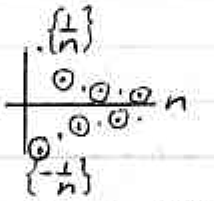
$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n}$$

$$= 0$$

OR $\underset{\downarrow\ 0}{-\frac{1}{n}} \le \underset{\downarrow\ \text{so, } 0}{\frac{(-1)^n}{n}} \le \underset{\downarrow\ 0}{\frac{1}{n}}$ $(n \ge 1)$ as $n \to \infty$

$|\,|$s $\to 0$, so
terms themselves
$\to 0$

$$\Rightarrow \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

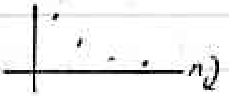



Circled: $\frac{(-1)^n}{n}$

$\pm 1 \ \pm\frac{1}{2} \ \pm\frac{1}{3} \ldots \to 0$
Regardless of
signs you pick

In fact, $1 \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \ldots \to 0$

<u>if flip any signs</u>




Can do HW
(all)

1, 1, 2, 3, 5, 8, 13, ...
what is this

## (F) Recursively Defined Seqs.

### Ex (#50)

Dictionary of Math: $a_1 = 2$, $a_2 = 1$
same recursion
$\Rightarrow$ Lucas #s
$\frac{l_n}{F_n} \to \sqrt{5}$ $r_k \to \tau$ also ) Conway 113

People have trouble! $\to$

Fibonacci seq.

$$\begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_{k+1} = a_k + a_{k-1} \quad (k \geq 2) \end{cases}$$

+

$$1,\ 1,\ 2,\ 3,\ 5,\ 8, \ldots$$

Conway
Fib. Quarterly!
Dictionary of Math: Fib.
# theorist, algebraist

Note:

Fib. used to model rabbit progeny
a pair of rabbits of one gen. produces a pair for next 2 gens., then die

c. 1175-1250 (Leonardo of Pisa) introduced Arabic # system to Europe!

In: almost all leaf arr., pineapples, cacti, sunflowers, pine cones, seashells

Closed form for $a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

GREAT BOOK: *The Book of Numbers* by Conway and Guy

Conway 202-3
Lucas:
$l_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$

New seq. $\{r_k\} = \left\{\frac{a_{k+1}}{a_k}\right\}$ ("ratio of successive terms)

$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \ldots \to \tau$ ("tau")
(assume exists)

Note (May see in Math 245: Discrete Math)

Find $\tau$

really confused by this...

$\div a_k$ $\qquad \dfrac{a_{k+1}}{a_k} = \dfrac{a_k}{a_k} + \dfrac{a_{k-1}}{a_k}$

Use notation $\qquad r_k = 1 + \dfrac{1}{\underbrace{\frac{a_k}{a_{k-1}}}_{= r_{k-1}}}$

Take limits $\qquad \lim\limits_{k\to\infty} r_k = \lim\limits_{k\to\infty} \left(1 + \dfrac{1}{r_{k-1}}\right)$

$r_k \to \tau$ $\qquad \tau = 1 + \dfrac{1}{\tau}$

$\cdot\tau$ $\qquad \tau^2 = \tau + 1$

$$\tau^2 - \tau - 1 = 0$$

Use QF: Quadratic formula

$$\tau = \frac{1 \pm \sqrt{5}}{2} \quad (\approx 1.618)$$

Golden ratio

Note 


Show $\tau - 1$ solves: $\dfrac{1+x}{1} = \dfrac{1}{x}$

$x^2 + x - 1 = 0$

$x = \dfrac{-1 \pm \sqrt{5}}{2}$

$= \tau - 1$

Conway 112
Knuth

Note "$r_k \to \tau$" Disc. by Kepler

Ancient Gr. considered it most pleasant ratio in art

Conway 184

$\frac{\text{diag}}{\text{side}}$ of reg. pentagon 

Cards! 

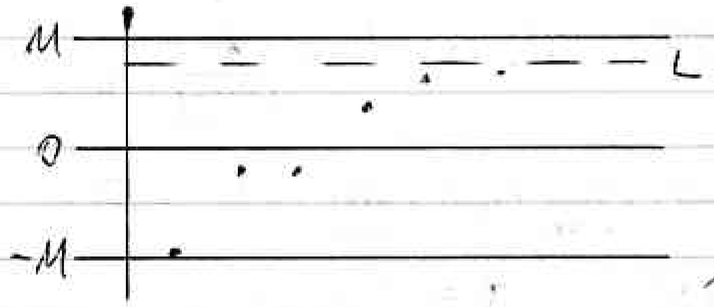
Discuss (Important in proofs, higher math)

## ⑥ A bounded, monotonic seq. converges

$|a_n| \le M$ (real #) for $n=1,2,3,\dots$

$\{a_n\}$ never $\searrow$ or never $\nearrow$

$a_{n+1} \ge a_n$ (or $\le$) for $n=1,2,3,\dots$




Bounded | Monotonic | Converges


These don't have to be tight bounds