11.2: SERIES

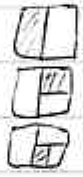
What did you have a hard time believing? You had a hard time believing this!

Ⓐ Intro Ex

From 10.3 — Standard Normal 
$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = 1$

① $\{(\frac{1}{2})^n\}$, the original sequence $\{a_n\}$

$a_1 = (\frac{1}{2})^1 = \frac{1}{2}$
$a_2 = (\frac{1}{2})^2 = \frac{1}{4}$
$a_3 = (\frac{1}{2})^3 = \frac{1}{8}$
$\vdots$



② $\{S_n\}$, the [associated] sequence of partial sums

$S_1 = a_1 = \frac{1}{2}$
$S_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$
$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

(Do 1st)
S_k = the k^{th} partial (cumulative) sum
$= a_1 + a_2 + \dots + a_k$
$= \sum_{n=1}^{k} a_n$ ↗ successively replace n with $1, 2, \dots, k$ and add the results

③ S, the sum of the infinite series $\sum_{n=1}^{\infty} a_n$

too sloppy for Diff. Eqs.!

(Do 1st)
$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \underbrace{\underbrace{\underbrace{\frac{1}{2}}_{S_1=\frac{1}{2}} + \frac{1}{4}}_{S_2=\frac{3}{4}} + \frac{1}{8}}_{S_3=\frac{7}{8}} + \dots$$
$\to S = 1$ 

We approach filling in the entire box

$\approx$ (10.3) $= 1$

$\overbrace{\sum_{n=1}^{\infty}}^{\text{Book says } \sum a_n \text{ (a bit sloppy)}} a_n = \underbrace{a_1 + a_2 + a_3 + \dots}$

has a sum (real #)
⇔ series converges
⇔ $\{S_n\}$ conv.
⇔ $\lim_{n\to\infty} S_n = S$ (real #)
else series diverges (has no sum)

Recap $\{a_n\}$ $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

$\{S_n\}$ $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots \to S = 1$

$\sum_{n=1}^{\infty} a_n$ $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$

The series has sum 1. Is $\sum a_n$ a sum? well.. does it have a sum?

(B) Telescoping Series

Ex (#2) $$\sum_{n=1}^{\infty} \frac{5}{(5n+2)(5n+7)}$$

(a) Find S_k (k^{th} partial sum)

PFD $$\frac{5}{(5n+2)(5n+7)} = \frac{A}{5n+2} + \frac{B}{5n+7}$$

what makes these = 0? we didn't see these in Ch 4 - could have.

$$5 = A(5n+7) + B(5n+2)$$

$n = -\frac{7}{5}$: $$5 = \cancel{A(0)} + B[5(-\tfrac{7}{5})+2]$$
$$5 = -5B$$
$$B = -1$$

$n = -\frac{2}{5}$: $$5 = A[5(-\tfrac{2}{5})+7] + \cancel{B(0)}$$
$$5 = 5A$$
$$A = 1$$

Stewart 714

for conv. series and for finite series, $\sum(a_n - b_n) = \sum a_n - \sum b_n$

Don't need abs conv. of partial sums $u_k = s_k + t_k$, $s_k \to s$, $t_k \to t$

Can't find nice S_k for many seqs.!

$$S_k = \sum_{n=1}^{k} \frac{5}{(5n+2)(5n+7)} = \sum_{n=1}^{k} \left(\frac{1}{5n+2} - \frac{1}{5n+7}\right)$$

$$= \underbrace{\left(\frac{1}{7} - \cancel{\frac{1}{12}}\right)}_{n=1} + \underbrace{\left(\cancel{\frac{1}{12}} - \cancel{\frac{1}{17}}\right)}_{n=2} + \underbrace{\left(\cancel{\frac{1}{17}} - \cancel{\frac{1}{22}}\right)}_{n=3} + \ldots + \underbrace{\left(\cancel{\frac{1}{5k+2}} - \frac{1}{5k+7}\right)}_{n=k}$$

Collapsing telescope

$$= \frac{1}{7} - \frac{1}{5k+7}$$

What do I do to find S, if ∃?

ⓑ Find S (sum of the series)

1st see if lim ∃ before say "S"

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(\frac{1}{7} - \underbrace{\frac{1}{5k+7}}_{\to 0}\right)$$

$$= \frac{1}{7}$$

S or S_∞

$$\Rightarrow S\left(= \sum_{n=1}^{\infty} \frac{5}{(5n+2)(5n+7)}\right)$$

Up to S

$$= \boxed{\frac{1}{7}}$$

ⓒ Geometric Series

Geom. seq.: $\{\underbrace{ar^{n-1}}_{a_n}\}$, $a \neq 0$

Don't say $a_n \to 0$ it looks too generic. When did this seq. conv. to 0?

$$(a, ar, ar^2, ar^3, \ldots \to 0) \overset{\text{if and}}{\underset{\text{only if}}{\Leftrightarrow}} -1 < r < 1 \quad (|r| < 1)$$

Ex $7, 7(\frac{1}{2}), 7(\frac{1}{2})^2, \ldots$ $(a = 7, r = \frac{1}{2})$
$7, \frac{7}{2}, \frac{7}{4}, \ldots \to 0$

Geom. Series Test for Conv. or Div.:

Geom. series: $\sum_{n=1}^{\infty} a_n = a + ar + ar^2 + \ldots + \underbrace{ar^{n-1}}_{n^{th}\text{ term}} + \ldots$

conv. $\Leftrightarrow -1 < r < 1$ $(|r| < 1)$

Then, the sum, $S = \frac{a}{1-r}$ (Pf W/last Words I-p.120)

(Note: $S_k = \frac{a - ar^k}{1-r}$, if $r \neq 1$. If $|r| < 1 \Rightarrow (ar^k \to 0, S_k \to S)$)

else diverges

Ex $7 + \frac{7}{2} + \frac{7}{4} + \ldots + 7(\frac{1}{2})^{n-1} + \ldots$

$a = 7$

$r = \frac{1}{2} \Rightarrow$ series $\boxed{\text{conv.}}$ w/sum

$S = \frac{a}{1-r} = \frac{7}{1-\frac{1}{2}} = \frac{7}{1/2} = \boxed{14}$

Note: Except for $0 + 0 + \ldots$, arithmetic series never have a sum. Ex $3 + 7 + 11 + 15 + \ldots$ diverges

What kind of a # is this?
⇐ Grade school

Ex $0.1\overline{987} \Rightarrow$ nice fraction
(rational)

$0.1987987987\ldots$

Odd man out

$= 0.1 \quad \} \ \frac{1}{10}$
$+0.0987$
$+0.0000987$
$\vdots$
Geom series!
$a = 0.0987$
$r = \frac{1}{1000}$, or 0.001

When you move a dec place 3 places to left, you're ÷ by what?

$\Rightarrow S = \frac{a}{1-r}$

$= \frac{0.0987}{1-0.001}$
$= \frac{0.0987}{0.999}$ ⇐ · by ⇐ 10,000
$= \frac{987}{9990}$ ⇐ ÷ by ⇐ 3
$= \frac{329}{3330}$

$= \frac{329}{3330}$

Don't forget!

$= \frac{1}{10} + \frac{329}{3330}$
$= \boxed{\frac{331}{1665}}$

Duh! Now, you can finish grade school....

Ex Find the sum of $\sum_{n=1}^{\infty} 2^{n+3}5^{-n}$

$$= \sum_{n=1}^{\infty} \frac{2^{n+3}}{5^n}$$

(Mostly "rewriting")

How can I break up 2^{n+3}?

$$= \sum_{n=1}^{\infty} \frac{2^n \cdot 2^3}{5^n} \quad (2^3 = 8)$$

Massaged into form that looks like geom.

$$= \sum_{n=1}^{\infty} 8\left(\frac{2}{5}\right)^n$$

r pretty clearly $\frac{2}{5}$
r not the base if $_n$ en, though.

$$\sum_{n=1}^{\infty} 8\left(\frac{2}{5}\right)^n \quad \text{or} \quad \sum_{n=1}^{\infty} 8\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)^{n-1}$$

$= \underbrace{8\left(\frac{2}{5}\right)}_{a=\frac{16}{5}} + 8\left(\frac{2}{5}\right)^2 + 8\left(\frac{2}{5}\right)^3 + \ldots$ $\quad r = \frac{2}{5}$

$= \sum_{n=1}^{\infty} \underbrace{\frac{16}{5}\left(\frac{2}{5}\right)^{n-1}}_{ar^{n-1}}$ $\left(a = \frac{16}{5}, r = \frac{2}{5}\right)$ "fitting the form"

$|r| < 1 \Rightarrow$ sum exists

$$S = \frac{a}{1-r}$$

$$= \frac{16/5}{1 - 2/5}$$

$$= \frac{16/\cancel{5}}{3/\cancel{5}}$$

Up to 15

$$= \boxed{\frac{16}{3} \text{ or } 5.\overline{3}}$$

① <u>n^{th}-Term Test</u> (for Div.)

For a <u>geom.</u> series,

$$\underbrace{\sum_{n=1}^{\infty} a_n}_{\text{“the series”}} \text{ conv} \Leftrightarrow (\overset{\text{“the terms”}}{a_n} \to 0 \underbrace{\text{ as } n \to \infty}_{\text{implied}})$$

i.e., $|r| < 1$

BUT, in general,

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \underset{\text{(does not imply)}}{\not\Leftarrow} (a_n \to 0)$$

As a minimal requirement Like eligibility for Prez: 35 but you also need $\$60$M

"$a_n \to 0$" is a <u>necessary</u> but <u>not sufficient</u> condition for $\sum_{n=1} a_n$ to converge.

(i.e., we need "$a_n \to 0$," but it's not enough!)

Larson 570

Ex (Harmonic Series)

(General Harmonic Series: $\sum \frac{1}{an+b}$
In music, strings w/ same diameter, tension, material whose lengths form a harmonic $\sum$ produce harmonic tones.)

Recips of nat. #s

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots$$ (Sum of reciprocals of the natural #s)

$$a_n = \frac{1}{n} \to 0$$

Pf in Ex. 3
$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots >$
$1 + (\frac{1}{2}) + \underbrace{(\frac{1}{4} + \frac{1}{4})}_{\frac{1}{2}} + \underbrace{(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8})}_{\frac{1}{2}} + \dots$
$S_{2k} > (k+1)(\frac{1}{2})$
$\int_1^{\infty} \frac{1}{x}\,dx$ div. ($\int$ Test)

BUT more work needed to see if $\sum_{n=1}^{\infty} \frac{1}{n}$ conv. or div.
turns out (Ex. 3 in your book)
(or $\int$ Test: 11.3)

(Handout: Proof w/out Words II 116)

Pf. p. 538. conv $\Rightarrow \exists S$
$a_n = S_n - S_{n-1}$
$\lim_{n \to \infty} a_n = S - S = 0$

$$\boxed{\sum_{n=1}^{\infty} a_n \text{ conv.} \Rightarrow (a_n \to 0)}$$

Contrapositive of "$A \Rightarrow B$" (Daddy, Man) is "not $B \Rightarrow$ not A" (not Man, not Daddy) } equiv.

Pict
No Springer stuff!
If you accept $A \Rightarrow B$, you must accept $\sim B \Rightarrow \sim A$

$$\boxed{(a_n \not\to 0) \Rightarrow \sum_{n=1}^{\infty} a_n \text{ div.} \quad (n^{th}\text{-Term Test (for Div.)})}$$ (can't use to prove $\sum$ conv.)

Ex $$\sum_{n=1}^{\infty} \underbrace{\sin\left(\pi n - \frac{\pi}{2}\right)}_{a_n} = \sin\frac{\pi}{2} + \sin\frac{3\pi}{2} + \dots$$
$$= 1 - 1 + 1 - 1 + \dots$$

Can't say sum $= 0$

$S_1 = 1$
$S_2 = 0$
$S_3 = 1$
$S_4 = 0$

$$\lim_{k \to \infty} S_k \text{ DNE} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ div.}$$

Shortcut $\{a_n\}$ $1, -1, 1, -1, \dots \not\to 0$
$\Rightarrow \sum_{n=1}^{\infty} a_n$ div. (by n^{th}-Term Test)

(E) Altering Series

Recips. of squares $\int_1^\infty \frac{1}{x}\,dx$ vs. $\int_1^\infty \frac{1}{x^2}\,dx$

Turns out: $1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} + \ldots$ conv.

Throwing in a finite # of terms does not change the situation w/ conv. vs. div.

$\Rightarrow 100 + 7 + 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} + \ldots$ conv.

$$\sum a_n \Longrightarrow \sum b_n$$

Insert, delete, or change a finite # of real terms

no $\frac{1}{0}$

Both conv. (sums may differ)
or Both div.

Up to 39

(F) Linearity Properties

Assume $\underbrace{\sum_{n=1}^{\infty} a_n}_{\text{Sum}=A}, \underbrace{\sum_{n=1}^{\infty} b_n}_{B}, \underbrace{\sum_{n=1}^{\infty} c_n}_{C}$ conv.

If I mult each term in the a_n series by 7, the resulting Σ will conv. w/ sum 7A.

Then, $\sum_{n=1}^{\infty} \underbrace{(7a_n + b_n - 9c_n)}_{\text{a linear combo of } a_n, b_n, c_n}$ conv.
w/ sum $7A + B - 9C$

($2x + 3y$ is a linear combo of x, y.)

Swoke wouldn't pull out 7 until he knew $\Sigma(\frac{1}{2})^n$ conv.

Ex $\sum_{n=1}^{\infty} 7\left(\frac{1}{2}\right)^n = 7 \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n}_{\text{conv.}} = 7(1) = \boxed{7}$

$$7\left(\tfrac{1}{2}\right) + 7\left(\tfrac{1}{4}\right) + \ldots$$
$$= 7\underbrace{\left(\tfrac{1}{2} + \tfrac{1}{4} + \ldots\right)}_{=1}$$
$$= 7$$

If div.guy has 0 coeff., we don't have to worry about him!

Ruins the party, but 2 nuts can knock each other off.

Linear combo weights: 7, 1

"Ice" X-Files

$$\boxed{\text{If } \underline{\text{exactly one}} \text{ of the series is div., then } \sum_{n=1}^{\infty} \underbrace{(7a_n + b_n - 9c_n)}_{\text{linear combo w/non-0 coeff. on "div.guy"}} \text{ is div.}}$$

Ex $\sum_{n=1}^{\infty} \frac{1}{n}$ div. $\qquad \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ conv.

$$\sum_{n=1}^{\infty} \left[\frac{7}{n} + \left(\frac{1}{2}\right)^n\right] \quad \boxed{\text{div.}}$$

(On HW, if 2 or more of the series diverge, then they may interact in an interesting way, and the resulting series may converge.)