11.3/11.4: TESTS FOR CONV./DIV. OF POSITIVE-TERM SERIES

all terms $a_n > 0$ "eventually" (i.e. $\forall n \geq$ some N)
(There's a point of no return after which the terms are always "+".)

11.3

Ⓐ $\int$ Test

Why are we starting w/ 2 instead of 1? $\frac{1}{0}$

Ex (#8) Does $\displaystyle\sum_{n=2}^{\infty} \underbrace{\frac{1}{n(\ln n)^2}}_{a_n}$ conv. or div.?

$$\frac{1}{2(\ln 2)^2} + \frac{1}{3(\ln 3)^2} + \ldots$$

Does this have a sum?

Conv. to smooth[?] thing. Interpolating func. Domain = cont. set of real #s. We'll formalize later (term deletion)

① Let $f(x) = \frac{1}{x(\ln x)^2}$.

② For $x \geq 2$, verify f is...

Swok.[?] requires $f(x) > 0$ for $x \geq 2$ (proof assumes this). $\int$ Test can be easily modified: $f(x) > 0$ $\forall x \geq M$. Same for "↓"

(If any of these fails, you can't apply the $\int$ Test.) hypotheses

ⓐ "+" valued [eventually.]

$$\frac{1}{\underset{+}{x}\underset{+}{(\ln x)^2}} > 0 \checkmark$$

Don't have to explain. See L11-10f

ⓑ cont. ✓

We're going to find $f'(x)$ and show it's what? NO

ⓒ decreasing (↘) [eventually]

Quot. or Recip. Rule. $\ln(1) = 0$

$$f'(x) = -\frac{\overbrace{D_x[x(\ln x)^2]}^{\text{Use Product Rule}}}{[x(\ln x)^2]^2}$$

Quotient Rule or Reciprocal Rule (p.114) $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$

$$= -\frac{(1)(\ln x)^2 + (x)[2\ln x \cdot \frac{1}{x}]}{x^2(\ln x)^4}$$

(Maybe easier to analyze)

Factor out $\ln x$ in num., den.; cancel.

$$= -\frac{\overset{+}{\ln x + 2}}{\underset{+}{x^2}\underset{+}{(\ln x)^3}}$$

$\ln(\text{what}) = 0$

$\left(\begin{smallmatrix}\ln 1 = 0 \\ \ln x \nearrow\end{smallmatrix}\right)$ $\ln x > 0$ for $x \geq 2$)

< 0 ✓

Rule:

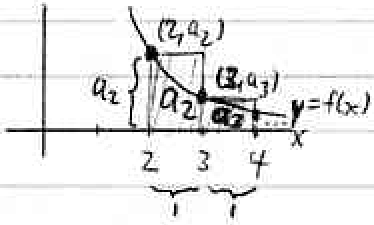
(3) If $\int_2^\infty f(x)\,dx$ conv. $\Rightarrow \sum_{n=2}^{\infty} a_n$ conv.

$\int \Rightarrow \infty$?

div. $\Rightarrow$ div.

Idea

Circumscribed rects.



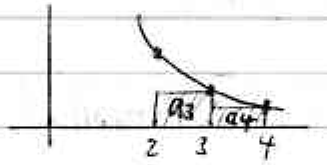
(like a Left-Hand Riemann Sum)

If little bro can't conv. then doorway, neither can big bro.

$\int$ div. $\Rightarrow \sum$ div. (Areas of rects. have no sum.)

little bro div. $\Rightarrow$ big bro div.

Inscribed rects.



(like a Right-Hand Riemann Sum)

$\int$ conv. $\Rightarrow \sum_{n=3}^{\infty} a_n$ conv. $\Rightarrow \sum_{n=2}^{\infty} a_n$ conv.

big bro conv. $\Rightarrow$ little bro conv.

throw in a_2 (doesn't change conv. vs. div.)

Does $\int_2^\infty \frac{1}{x(\ln x)^2}\,dx$ conv. or div.?

$\int \frac{1}{x(\ln x)^2}\,dx = \ldots = -\frac{1}{\ln x} + C$

$u = \ln x$
$du = \frac{1}{x}\,dx$

If see $\ln n$, $\frac{1}{n} \Rightarrow \int$ Test best!

Are we going to have to split the $\int$? Form still important!

$$\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^2}\,dx$$

cont. on $[2,\infty)$ (we vet this, already.)

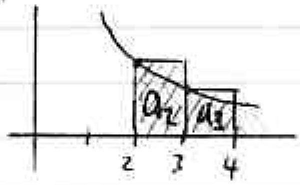
$$\overset{FTC}{=} \lim_{t\to\infty} \left[-\frac{1}{\ln x}\right]_2^t$$

$$= \lim_{t\to\infty} \left(-\underbrace{\frac{1}{\ln t}}_{\to 0\ (\ln t \to \infty)} - \left[-\frac{1}{\ln 2}\right]\right)$$

≈ 1.4427 $\qquad = \frac{1}{\ln 2}$ <u>NOT</u> Σ!!

(actually an underestimate)

$\Rightarrow \int$ conv.



≈ 2.1097 $\qquad \Rightarrow \boxed{\sum_{n=2}^\infty a_n \text{ conv.}}$

$\sum_{n=2}^\infty a_n$ = sum of areas of rects. $> \frac{1}{\ln 2}$

Up to 11 →

Also 11.2 pf

Construct little $\int$ to Σ that div.

<u>Ex 1 (p. 546)</u> Harmonic series div.

<u>Ex</u> $\sum_{n=1}^\infty \frac{1}{n-2.5}$ ← a_n

$f(x) = \frac{1}{x-2.5}$ (There's a problem w/ this interpolating func.)

OK to have even "cont. eventually" provided you restrict your analysis to $\int_3^\infty$, throw in other terms later. If f discont at 2, $f(x) = \frac{1}{x-2}$, a_2 undef.

Up to 11

f "+", cont., ↘ on $[3,\infty)$ $\Rightarrow$ Can use $\int$ Test there

Can show $\sum_{n=3}^\infty a_n$ div.   (Better: BCT from 11.3 ©)

$\Rightarrow \sum_{n=1}^\infty a_n$ div.

throw in a_1, a_2

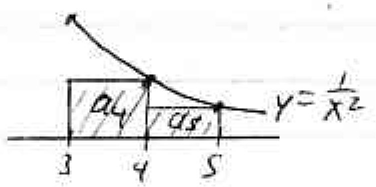
What's S_3?

Ex Approx. $\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\substack{\text{conv.} \\ \text{by } \int \text{ Test} \\ \text{(later: p-series Test)}}}$ by S_3, peform error analysis

What are the 1st 3 terms?

$$S = \underbrace{1 + \frac{1}{4} + \frac{1}{9}}_{S_3 \approx 1.3611} + \underbrace{\frac{1}{16} + \frac{1}{25} + \dots}_{\text{error} = \sum_{n=4}^{\infty} \frac{1}{n^2} \ (>0)}$$

Like Right hand Riemann



Start $\int$ at ③

<, not ≤;
f ↘ func.
Always gaps

$$\text{error} < \underbrace{\int_3^{\infty} \frac{1}{x^2}\,dx}_{= \frac{1}{3}}$$

upper bound on error

Better (thinner/ shorter) interval if $S_{10}, S_{100}, \dots$

$S_k \nearrow$ but $\int_k^{\infty} \searrow$

Eventually get 4-dec place accuracy

Range game up to 11

$$\Rightarrow 1.3611 < S < 1.3611 + \frac{1}{3}$$

$$\Rightarrow \underbrace{1.3611}_{\text{lower bound for } S} < S < \underbrace{1.6944}_{\text{upper bound for } S}$$

Turns out $S = \frac{\pi^2}{6} \approx 1.6449$ Euler (WOW!)

in

Ahlfors 190, Ross 69

Fourier S, Complex Analysis

$\Rightarrow \sum_1^{\infty} \frac{1}{n^p}$ if even (not for odd)

$\sum \frac{1}{n^4} = \frac{\pi^4}{90}$

Euler

We really subt. from 1.6944

Note (What if we use S_4 instead of S_3?

S_3: $1.3611 < S < 1.6944$

both $+ \boxed{a_4}\ \frac{1}{16}$

BUT $-$ $\frac{1}{12}$ since error bound $\int_3^{\infty} \to \int_4^{\infty}$

3 4

NET relative to 1.6944

$\left.\left. \begin{matrix} \boxed{a_4} \end{matrix} \right\} + \frac{1}{16} \right\} - \frac{1}{12} = -\frac{1}{48}$

3 4

S_4: $1.4236 < S < 1.6736$

contained w/in S_3-based interval $1.3611 < S < 1.6944$)

(B) p-Series Test

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \quad (p>0) \qquad \begin{array}{l} \text{conv.} \Leftrightarrow p>1 \\ \text{div.} \Leftrightarrow p \le 1 \end{array}}$$

<u>Proof</u> by $\int$ Test

Not proven by $\int$ Test $\rightarrow$ $f(x) \nearrow$

<u>Note</u> $\sum_{n=1}^{\infty} \frac{1}{n^{-2}} = \sum_{n=1}^{\infty} \underset{\downarrow \infty}{n^2}$ div. by n^{th}-Term Test

<u>Exs</u> $p=\frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots$ div

(The harmonic series is the "smallest" p-Σ that div.)

$p=1$: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$ div

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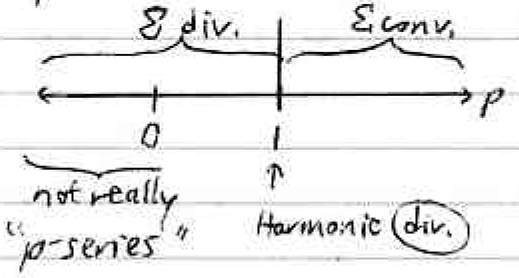
$p=1+\epsilon \Rightarrow$ conv. $(\epsilon>0)$

$p=1.1$: $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}} = 1 + \frac{1}{2^{1.1}} + \frac{1}{3^{1.1}} + \ldots$ conv.

$p=2$: $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \ldots$ conv.

Line of p-series:



Ⓒ Basic Comparison Tests (BCT)

Like when you were young Makes noises.

$\sum b_n$ "big brother" dominates

$\sum l_n$ "little brother"

Assume for all $n \geq$ some N,
(eventually, past some pt. of no return)

Otherwise, in magnitude, l_n dominant

$l_n \leq b_n$
$l_n, b_n > 0$

ⓐ If $\sum b_n$ conv. $\Rightarrow \sum l_n$ conv.
(big bro fits $\Rightarrow$ so does little bro)

ⓑ If $\sum l_n$ div. $\Rightarrow \sum b_n$ div.
(little bro too big $\Rightarrow$ so is big bro)

Ex Does $\displaystyle\sum_{n=3}^{\infty} \frac{4}{\ln n + 7^n}$ conv. or div.?

$\cancel{\int_3^{\infty} \frac{4}{\ln x + 7^x}\,dx}$ YUK!

Try to compare this to a geom. series or a p-series (easy to analyze).

We know $\displaystyle\sum_{n=3}^{\infty} \frac{4}{7^n} = \underbrace{\sum_{n=3}^{\infty} 4\left(\tfrac{1}{7}\right)^n}_{\substack{\text{Geom. series} \\ |r| = \frac{1}{7} < 1}}$ conv.

incl error analysis Stewart 734 11.4 end

You could compare to a $\sum$ where you need $\int$ test, but work!

What kind of $\sum$ is $\sum \frac{4}{7^n}$? Not p series here

$\sum \frac{4}{\ln n}$ div.

$\frac{4}{\ln n} > \frac{4}{n}$ div.

but $\frac{4}{\ln n} > \frac{4}{\ln n + 7^n}$
div.
wrong way!

Everything's "+"

$$\frac{4}{\ln n + 7^n} \le \frac{4}{7^n} \quad (\text{for } n \ge 3)$$

$\ln n > 0$

Big bro. Σ conv. $\Rightarrow$

So does little bro. Σ

$\overset{BCT}{\Rightarrow} \sum_{n=3}^{\infty} \frac{4}{\ln n + 7^n}$ $\boxed{\text{conv.}}$

What if
Up to 17

__WARNING__ If big bro. Σ div. $\not\Rightarrow$
If little bro. Σ conv. $\not\Rightarrow$ can't use BCT

Stewart
732

see $\ln n, \frac{1}{n} \Rightarrow$
$\ln(\text{what}) = 1$

__Ex__ $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ ← a_n

$\int$ Test OK: $\int_3^{\infty} \frac{\ln x}{x}\, dx$ ← $f(x)$

Hypotheses are satisfied for $x \ge 3$:

ⓐ $f(x) > 0$ for $x > 1$ $\left(\begin{smallmatrix}\ln 1 = 0\\ \ln \nearrow\end{smallmatrix}\right)$

ⓑ f cont. for $x > 0$

ⓒ $f'(x) = \frac{1 - \ln x}{x^2}$ $\left(\begin{smallmatrix}\ln e = 1\\ \ln \nearrow\end{smallmatrix}\right)$
< 0 for $x > e$ (≈ 2.7)

$\Rightarrow f \downarrow$ there

Turns out $\int_3^{\infty} f(x)\, dx$ div.

$\overset{\int \text{Test}}{\Rightarrow} \sum_{n=3}^{\infty} a_n$ div.

$\Rightarrow \sum_{n=1}^{\infty} a_n$ $\boxed{\text{div.}}$

Up to 17

BCT quicker:

Everything's "+" for $n > 1$, or $n \ge 2$ $\left(\begin{smallmatrix}\ln 1 = 0\\ \ln \nearrow\end{smallmatrix}\right)$

$$\frac{1}{n} \le \frac{\ln n}{n}$$ holds for $n \ge 3$ $\left(\begin{smallmatrix}\ln e = 1\\ \ln \nearrow\end{smallmatrix}\right)$

little bro $\overset{BCT}{\Rightarrow}$ big bro
$\Sigma \frac{1}{n}$ div. $\quad \Sigma \frac{\ln n}{n}$ $\boxed{\text{div.}}$

Σ can be sloppy
ignore

factorial (See Ch.10 Notes on Gamma Func.)

Review $n!$

$0! = 1$ ← (by definition; convenient for series, $\binom{n}{k}$ combinations, Binomial Thm, etc.) Consistent w/gamma func.

$1! = 1$

$2! = (2)(1) = 2$

$3! = (3)(2)(1) = 6$

$4! = 4(3!) = 24$ $2\cdot4$

$5! = 120$ $2\cdot5$

$\vdots$

$n! = n(n-1)(n-2)\cdots(1)$, n is a natural #

Ex (#20) $\sum_{n=1}^{\infty} \frac{1}{n!}$ (You'd use the Ratio Test: 11.4)

What's your intuition? Denom. blowing up

We guess it conv.
Compare to an easy conv. Σ, say $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$\approx \frac{\text{Dominoes}}{5}$ know about induction. If you know mathematical induction, BCT much more powerful! Otherwise, pick a better test!

$$\sum_{n=1}^{\infty} \frac{1}{n!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots$$

Can ignore 1st few terms

for $n \geq 4$, $\frac{1}{n!} \leq \frac{1}{n^2}$

Big bro. $\sum \frac{1}{n^2}$ conv.

$\Rightarrow$ Little bro $\sum \frac{1}{n!}$ $\boxed{\text{conv.}}$

Ask me if you can assume! For something like this, convince me this is true! Otherwise, your use of BCT is suspect!
Here, you can use mathematical induction. Possible to do HW, tests w/out.

Basis Step ($n=4$)
$\frac{1}{4!} \leq \frac{1}{4^2}$
$\frac{1}{24} \leq \frac{1}{16}$ ✓

Inductive Step
Let $k \geq 4$ (k int.)
Assume $\frac{1}{k!} \leq \frac{1}{k^2}$ (*)
(i.e., $k! \geq k^2$)
Show $\frac{1}{(k+1)!} \leq \frac{1}{(k+1)^2}$
(i.e., $(k+1)! \geq (k+1)^2$)

$(k+1)! = (k+1)\cdot k!$
$\geq (k+1)\cdot k^2$ by (*)
$\geq (k+1)(k+1)$
$= (k+1)^2$
bec. $k^2 \geq k+1$ for $k \geq 4$
I believe this; can use.

We'll see in 11.7

Up to 19

True for $k \geq 2$, but $\frac{1}{2!} > \frac{1}{2^2}$

Note $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots = \boxed{e}$

$n = 0$ not 1

We'll show in 11.7

ⓓ Limit Comparison Test (LCT)

Why?

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ Try BCT:

$\frac{1}{\sqrt{n}+1} \leq \frac{1}{\sqrt{n}}$ (Everything's "+", same num., smaller denom. ⇒ this is bigger, overall.)

Is larger or smaller?

↑ Big $\sum \frac{1}{\sqrt{n}}$ div. (p-series, $p = \frac{1}{2} \leq 1$)

? (BCT fails.)

Frustrating! So close!

How can we easily compare $\sum \frac{1}{\sqrt{n}+1}$, $\sum \frac{1}{\sqrt{n}}$?

LCT Idea

Every term matters for sum! BCT wouldn't work, but...

Num. of a_n: $7 - (\frac{1}{10})^n$

In the long run, what's happening?

compare to:

$\sum a_n = \frac{6.9}{2} + \frac{6.99}{4} + \frac{6.999}{8} + \ldots$ conv. w/sum = ?

$\sum b_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ conv. w/sum S $(=1)$

Note: $\frac{7}{2} + \frac{7}{4} + \frac{7}{8} + \ldots$ conv. w/sum $7S$ $(=7)$

a_n terms → $7 \cdot b_n$ terms

$\frac{a_n}{b_n} \to 7$

LCT

Me: Don't need $a_n, b_n > 0$ (makes proof easier) or $c > 0$. Div. Test Contrapos.; $\sum a_n$ C/D $\Leftrightarrow \sum(-a_n)$

If lim = 0, situation more complicated

Look at S1, S2

lim = 0

ΣD C ⇒ ΣN C

lim = ∞

ΣD D ⇒ ΣN D

Pf Idea: (S2)

$\exists M$: If $k > M$, $\frac{a_k}{b_k} > 1$

$\Rightarrow a_k > b_k$

(S1): $\frac{a_k}{b_k} < 1$

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ ← non-0 real # (like 7, say)

(Book assumes "+" term series, so it says "$c > 0$." implies non-0, real #.)

$\Rightarrow \sum a_n, \sum b_n$ both conv. or both div.

Given $\sum a_n$, how pick $\sum b_n$?

$a_n =$ ______ } Take dominant terms, Make constant factors = 1

$\Rightarrow b_n$

(old)

Ex $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$

Why cares if the limit 7 or $\frac{1}{7}$? Well, tell me the lim, anyway, though. 0, ∞ messed up, anyway. Order matters if you exploit #51, 52

Let $a_n = \frac{1}{\sqrt{n}+1}$, $b_n = \frac{1}{\sqrt{n}}$ } can switch

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} \begin{matrix} \leftarrow \div\sqrt{n} \\ \leftarrow \div\sqrt{n} \end{matrix}$$

(How to Ace p. 40 Typos! $\sqrt{n+1}$)

$$= \lim_{n\to\infty} \frac{1}{1+\underbrace{\frac{1}{\sqrt{n}}}_{\to 0}}$$

$$= 1$$

> 0, real ✓

(a_n terms → b_n terms)

We know $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ div. (p-series, $p = \frac{1}{2} \le 1$) $\sqrt{n} = n^{\frac{1}{2}}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1} \boxed{\text{div.}}$$

Like 31 $\frac{1+2^n}{1+3^n}$ unassigned

Ex $\sum_{n=1}^{\infty} \underbrace{\frac{3^n+1}{n+2^n}}_{a_n}$

$b_n = \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n$

Orange shirt Green pants vs. Green orange Change pants! (mult. commut.)

exp'ls kick polyr' butt

$$\lim_{n\to\infty} \frac{\frac{3^n+1}{n+2^n}}{\frac{3^n}{2^n}} = \lim_{n\to\infty} \frac{3^n+1}{n+2^n} \cdot \frac{2^n}{3^n}$$

$$= \lim_{n\to\infty} \frac{3^n+1}{3^n} \cdot \frac{2^n}{n+2^n} \begin{matrix} \leftarrow \div 2^n \\ \leftarrow \div 2^n \end{matrix}$$

$$= \lim_{n\to\infty} \left(1+\underbrace{\frac{1}{3^n}}_{\to 0}\right)\left(\frac{1}{\underbrace{\frac{n}{2^n}}_{\text{L'H} \to 0}+1}\right)$$

$$= 1$$

> 0, real

We know $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ div. (Geom. series, $|r| = \frac{3}{2} \geq 1$)

$\Rightarrow \sum_{n=1}^{\infty} \frac{3^n+1}{n+2^n}$ $\boxed{\text{div.}}$

<u>Ex</u> $\sum_{n=1}^{\infty} \underbrace{\frac{7}{\sqrt[3]{(4n^2+1)(n^3+5)}}}_{= a_n \; (>0)} \leftarrow \sqrt[3]{\underbrace{4n^5}_{\text{dominant, ignore 4}} + \ldots}$

$\sqrt[3]{n^5} = n^{5/3}$

$b_n = \frac{1}{n^{5/3}} \quad (>0)$

$$\lim_{n\to\infty} \frac{\dfrac{7}{\sqrt[3]{(4n^2+1)(n^3+5)}}}{\dfrac{1}{n^{5/3}}} = \lim_{n\to\infty} \frac{7n^{5/3}}{\sqrt[3]{(4n^2+1)(n^3+5)}}$$

$$= \lim_{n\to\infty} \frac{7\left(\sqrt[3]{n^5}\right)}{\sqrt[3]{(4n^2+1)(n^3+5)}}$$

Doesn't work for BCT!

In 11.4, 11.5, they pull this → (lower order terms) OK for test

"Dominant term analysis":

$$= \lim_{n\to\infty} 7\left(\sqrt[3]{\frac{n^5}{4n^5 + \underbrace{\ldots}_{\text{lower-order terms}}}}\right)$$

$\frac{n^5}{4n^5+\ldots}$ ← rational, same degree

↓ as $n\to\infty$

ratio of leading coeffs $= \frac{1}{4}$

More precisely,

$$= \lim_{n\to\infty} 7\left(\sqrt[3]{\frac{n^5}{(4n^2+1)(n^3+5)}}\right) \begin{matrix} \leftarrow \div n^5 \\ \leftarrow \div n^5 \end{matrix}$$

$$= \lim_{n\to\infty} 7\left(\sqrt[3]{\frac{1}{\left(\frac{4n^2+1}{n^2}\right)\left(\frac{n^3+5}{n^3}\right)}}\right)$$

$$= \lim_{n\to\infty} 7\left(\sqrt[3]{\frac{1}{\left(4+\underbrace{\frac{1}{n^2}}_{\to 0}\right)\left(1+\underbrace{\frac{5}{n^3}}_{\to 0}\right)}}\right)$$

Don't worry about rationalizing

$$= 7\left(\sqrt[3]{\frac{1}{4}}\right)$$

>0, real

We know $\sum_{n=1}^{\infty} \frac{1}{n^{5/3}}$ conv. (p-series, $p = \frac{5}{3} > 1$)

$\Rightarrow \sum_{n=1}^{\infty} a_n$ $\boxed{\text{conv.}}$