11.4: RATIO, ROOT TESTS

Not comparison tests

(Deals w/ratios of successive terms)

LCT ($a_1 + a_2 + a_3 + \dots$
(Deals w/ratios of corresponding terms) $b_1 + b_2 + b_3 + \dots$

Geom. Series (Core of proofs here)

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \underbrace{\left(\frac{1}{2}\right)^n}_{a_n} + \dots$$

What's $\frac{1}{4}$ over $\frac{1}{2}$... Actually, you'd just use Geom. Series Test

① $\frac{a_{n+1}}{a_n} = r = \frac{1}{2}$
$\left|\frac{1}{2}\right| < 1$, so Σ conv. by Ratio Test

② $\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{1}{2}\right)^n} = \frac{1}{2}$
$\left|\frac{1}{2}\right| < 1$, so Σ conv. by Root Test

Generalize!

Assume all $a_n \neq 0$ (eventually)

① Proof
If $L < 1$
Let r: $0 \le L < r < 1$
$\forall n \ge$ some N
$\frac{a_{n+1}}{a_n} < r$
$a_{N+1} + a_{N+2} + \dots$
$< a_N r + a_N r^2 + \dots$
conv.
If $L > 1$ $\forall n \ge$ some N
$\frac{a_{n+1}}{a_n} > r > 1$
$a_{n+1} > a_n$
n^{th} Term test
② $a_n < r^n$ $(L < 1)$

① Ratio Test

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ or ∞

Don't need if all $a_n > 0$

② Root Test

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$ or ∞

If $L < 1 \Rightarrow \Sigma a_n$ conv.
$L > 1$ or "∞" $\Rightarrow$ div.
$L = 1$ ("or ONE") $\Rightarrow$? Need more info! Inconclusive!

Already excluded by first "IF" → Like n^{th} term Test when $a_n \to 0$

same rules, though L for Root Test may differ from L for Ratio Test

Note ⓐ

Maybe $L=.8$ for ①
$L=.9$ for ②
Never have 1 test ⇒ conv. Other ⇒ div.

Notes

ⓐ L for ①,② may differ.

ⓑ If $L=1$, the tests "fail" (are inconclusive).
Ex For ① and ②, $L=1$ for $\sum \frac{1}{n}$, $\sum \frac{1}{n^2}$
div. conv.

$L=1$ if $\sum$ algebraic (rational w/ but maybe roots), incl. p-series

Ex $\frac{n^3+n^{5/2}}{2n+\sqrt{n}+1}$ $n^{5/2} = \sqrt{n^5}$

Stewart: alg. ⇒ $L=1$ for ratio

Stewart: Me: ratio/root
If ① fails ⇒ ② but Ross p 74 has counterex.
Power Rule more important later

*ⓒ If ② fails ⇒ ① fails. If ① fails ⇒ ② often fails. (They tend to fail together in 151.)
Root Ratio Ratio Root

ⓓ If a_n has $n!$, 2^n, products, ..., try ① Ratio.
$(\ )^n$, 2^n, ..., try ② Root.
(including)

If $(\ )^n$ not so nice ⇒ root usu. better than ratio?

ⓔ Proofs use BCT to compare $\sum a_n$ w/a geom. series. (for conv.) and n^{th}-term Test (for div.)
$(L<1)$ $(L>1)$

$1+2+4+8+\ldots$ div.
Ratio: $L=2$

If you're curious!

*Note on Note (Ross, Elementary Analysis, p. 74)

Root works, Ratio fails for: $\sum 2^{(-1)^n-n} = \begin{cases} 2^{1-n}, & n \text{ even} \\ 2^{-1-n}, & n \text{ odd} \end{cases}$ >0

Conv. by BCT : $2^{(-1)^n-n} \le 2^{1-n} = \frac{1}{2^{n-1}}$
("Easy to see")

$\sum$ conv. (Geom.)

$\Rightarrow \sum$ conv.

Root works: n even: $\sqrt[n]{2^{1-n}} = \sqrt[n]{\frac{2}{2^n}} = \frac{\sqrt[n]{2}}{2} \to \frac{1}{2}$ (Sweep over only even n)

$\sqrt[n]{2} = 2^{\left(\frac{1}{n}\right)}$, $\frac{1}{n} \to 0$, $\sqrt[n]{2} \to 1$

n odd: $\sqrt[n]{2^{-1-n}} = \sqrt[n]{\frac{1}{2}\cdot\frac{1}{2^n}} = \sqrt[n]{\frac{1}{2}}\cdot\frac{1}{2} \to \frac{1}{2}$ (" odd n)

$\sqrt[n]{2^{-1-n}} = \sqrt[n]{\frac{1}{2^{n+1}}}$, $\sqrt[n]{\frac{1}{2}} \to 1$

$\Rightarrow L=\frac{1}{2}$ Conv.

Ratio fails: n even: $\frac{a_{n+1}}{a_n}$ ← odd $= \frac{2^{-1-(n+1)}}{2^{1-n}} = \frac{2^{-2-n}}{2^{1-n}} = \frac{2^{-2}}{2^1}\cdot\frac{2^{-n}}{2^{-n}} = \frac{1}{8}$

n odd: $\frac{a_{n+1}}{a_n}$ ← even $= \frac{2^{1-(n+1)}}{2^{-1-n}} = \frac{2^{-n}}{2^{-1-n}} = \frac{1}{2^{-1}}\cdot\frac{2^{-n}}{2^{-n}} = 2$

$\Rightarrow \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ DNE inconclusive

Ex (#10) $\sum_{n=1}^{\infty} \frac{n!}{(n+1)^5}$

$n! \Rightarrow$ Use Ratio Test?

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{(n+1)!}{[(n+1)+1]^5}}{\frac{n!}{(n+1)^5}}$$

all $a_n > 0$

$$= \lim_{n\to\infty} \frac{(n+1)!}{[n+2]^5} \cdot \frac{(n+1)^5}{n!}$$

Orange shirt
Green pants

$$= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{(n+1)^5}{(n+2)^5}$$ (Exchanging parts)

why?

Ex $\frac{3!}{2!} = \frac{3\cdot\cancel{2}\cdot\cancel{1}}{\cancel{2}\cdot\cancel{1}} = 3$

Ex $\frac{(n+1)!}{n!} = \frac{(n+1)\cdot\cancel{n!}}{\cancel{n!}} = (n+1)$

$$= \lim_{n\to\infty} (n+1) \cdot \frac{(n+1)^5}{(n+2)^5}$$

$$= \lim_{n\to\infty} \frac{(n+1)^6}{(n+2)^5}$$ ← polys in n

$\Downarrow$

$$= \lim_{n\to\infty} \frac{n^6+\ldots}{n^5+\ldots}$$ ← Only consider dominant terms
(Could use L'H)

$$= \infty$$

LCT: ∞ useless
(except for 11.3 \#52)
Up to 9

$\Rightarrow \sum$ $\boxed{\text{div.}}$

$\underline{\text{Ex}}$ $\sum_{n=1}^{\infty} \frac{n^4}{2^{1+3n}}$

Ratio Test works.
Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n^4}{2^{1+3n}}}$$

all $a_n \geq 0$

We know laws of exps. better than laws of radicals.

$$= \lim_{n\to\infty} \left(\frac{n^4}{2^{1+3n}}\right)^{1/n}$$

Mult. exps. by $\frac{1}{n}$

$$= \lim_{n\to\infty} \frac{n^{4/n}}{2^{\frac{1+3n}{n}}}$$

$$= \lim_{n\to\infty} \frac{(n^{1/n})^4}{2^{\frac{1}{n}+3}}$$

$n^{1/n} \to 1$, $\frac{1}{n} \to 0$

Can use $n^{\frac{1}{n}} \to 1$ on HW, tests.
$(\sqrt[n]{n})$ ln trick
$(\sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \sqrt[5]{5}, \ldots \to 1)$

$$= \frac{1}{2^3}$$

$$= \frac{1}{8}$$

Ratio Test may have diff. L.

$$L = \frac{1}{8} < 1 \Rightarrow \sum \boxed{\text{conv.}}$$