11.5: WHEN Σ CAN BE < 0

11.2: n^{th}-Term Test
Now, for Cv.
11.5: Tests for Conv.

Ⓐ Alternating Series Test (AST) for Conv.

"eventually" too risky - need $a_n > 0$ $\forall n \geq 1$ for my Exs.

Assume all $a_n > 0$ (delete finite # of terms as necessary)

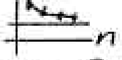
$$\text{Alt. } \Sigma \begin{cases} \displaystyle\sum_{n=1}^{\infty} \overbrace{\underbrace{(-1)^n}_{\text{sign}} \underbrace{a_n}_{\text{magnitude}}}^{b_n = \text{general } n^{th} \text{ term}} = -a_1 + a_2 - a_3 + a_4 \ldots & a_n = |b_n| \\ \displaystyle\sum_{n=1}^{\infty} (-1)^{n \pm 1} a_n = a_1 - a_2 + a_3 - a_4 \ldots & (\leftarrow \text{I tend to use this in my Exs.}) \end{cases}$$

(⓪ Have alt. Σ like above.)

If $\begin{cases} \text{① } a_n \searrow \text{ (nonincreasing)} \\ \text{② } a_n \to 0 \end{cases}$

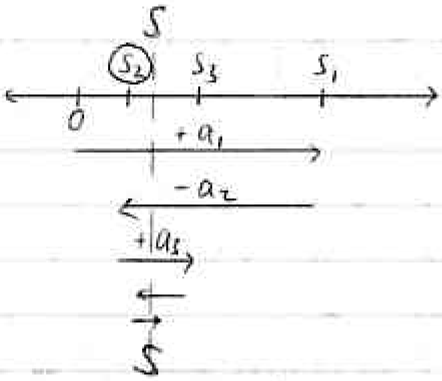
$\Rightarrow$ each Σ conv.

Note ①✓, ② fails:



Note ① fails ②✓

Idea



What do we do for $-a_2$?

Finger wagging

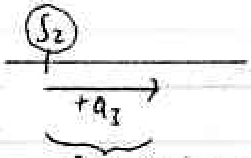
(Alt. Σ: Arrows keep reversing direction.
$a_n \searrow$: Arrows never grow.
$a_n \to 0$: Arrow lengths $\to 0$.)

Ex Approx. S by S_2

S = sum of whole Σ; $S_2 = a_1 - a_2$

S is w/in a_3 of S_2.

i.e., $\underbrace{|S - S_2|}_{\text{distance bet. } S, S_2} \leq a_3$



Note Need $\leq a_3$

$a_4, a_5, \ldots$ assumed non-0, BUT could have:

In general Approx. S by S_N

$$|\text{error}| \leq |1^{st} \text{ neglected term}|$$
$$|S - S_N| \leq a_{N+1}$$

(100!)
Add 1st N terms. Peek at next term to see how far off you can be from S.

Harmonic div.
This puppy conv.

Ex (Alternating Harmonic Σ)

$$\sum_{n=1}^{\infty} (-1)^{n-1} \underbrace{\left(\frac{1}{n}\right)}_{a_n > 0} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots$$

If there's any work, it's here

Verify conditions of AST

⓪ alt. Σ ✓

① $\frac{1}{n}$ ...

ⓐ $a_{n+1} \le a_n$?

$\frac{1}{n+1} \le \frac{1}{n} \quad (n \ge 1)$ ✓

Hint at Ratio Test

✓ what?

or ⓑ $\frac{a_{n+1}}{a_n} \le 1$?

$$\frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \le 1 \quad (n \ge 1) ✓$$

or ⓒ $f(x) = \frac{1}{x}$

$f'(x) = -\frac{1}{x^2} < 0 \quad (x \ge 1)$ ✓

For $\frac{1}{n}$, you don't have to show, but I'm showing you techniques for verifying harder cases.

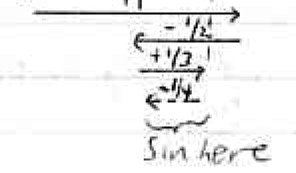
② $\frac{1}{n} \to 0$ ✓ (I believe you! Show as appropriate.) $\lim_{n \to \infty} a_n = 0$

$\therefore$ (Therefore) Σ $\boxed{\text{conv.}}$

Approx. S by S_3 (say)

$S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

$|\text{error}| < \frac{1}{4}$



We last added, so upper bound on S.

$$\underbrace{\frac{5}{6} - \frac{1}{4}}_{\frac{7}{12}} \le S \le \frac{5}{6}$$

$0.5833 \le S \le 0.8333$

Turns out: $S = \ln 2 \approx 0.6931$ wow!! (We'll see in 11.7.)

In HW sheet (To approx. to "4 dec. places," ensure $|\text{error}| < 0.00005$.)

Ⓑ CC vs. AC Σ

$|b_n| = a_n$ in alt. $\Sigma(-1)^{n-1}a_n$

Σb_n is <u>conditionally convergent (CC)</u> $\Leftrightarrow$
Σb_n conv., <u>BUT</u> $\underbrace{\Sigma|b_n|}_{\text{corresp. absolute value series, all "+" terms (maybe 0)}}$ div.

<u>Ex</u> (Alt. Harmonic Σ)

$$\sum_{n=1}^{\infty} \underbrace{(-1)^{n-1}\frac{1}{n}}_{b_n} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots \text{ is CC, because}$$

Σb_n conv. by AST (we saw this in Ⓐ)
<u>BUT</u> $\Sigma|b_n| = \Sigma\frac{1}{n}$ div.

(<u>Note</u> In 11.7, we'll see that $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = \ln 2$ ← Sum)

Σb_n is <u>absolutely convergent (AC)</u> $\Leftrightarrow$
Σb_n, $\Sigma|b_n|$ both conv.

$\Sigma\frac{1}{n}$ div, but what conv.?

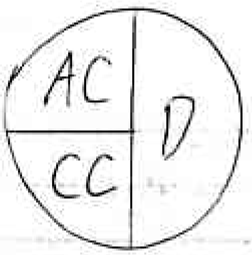
<u>Ex</u> $$\sum_{n=1}^{\infty} \underbrace{(-1)^{n-1}\frac{1}{n^2}}_{b_n} = 1 - \frac{1}{4} + \frac{1}{9} - \ldots \text{ is AC, because}$$

Σb_n conv. by AST
(alt. Σ, $a_n = \frac{1}{n^2}$ ↘ and $\to 0$)
<u>AND</u> $\Sigma|b_n| = \Sigma\frac{1}{n^2}$ conv. (p-series, $p = 2 > 1$)

In fact, Σb_n conv. $\Uparrow$ implies by ACT in Ⓒ $\Sigma|b_n|$ conv.

(<u>Note</u> Mathematica says
$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} = \frac{\pi^2}{12}$$
$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^4} = \frac{7\pi^4}{720}$$)

Σ



"+" term Σ can't be what?

"+" term Σ can't be CC

$|b_n| = b_n$

Larson 8.5.53

Note

Alt. p-series

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$ conv. if $p > 0$ (by AST)

In particular, $\begin{cases} AC, & \text{if } p > 1 \\ CC, & \text{if } 0 < p \leq 1 \end{cases}$

Alt. Geom. Series

are geom. Σ, anyway! We saw in 11.2.

Ex $\sum_{n=1}^{\infty} (-1)^n (\frac{1}{2})^n = \sum_{n=1}^{\infty} (-\frac{1}{2})^n$

Geom. Σ is $\begin{cases} AC, & \text{if } |r| < 1 \\ D, & \text{otherwise} \end{cases}$
(never CC)

© Absolute Convergence Test (ACT) (for conv. only)

If $\sum |b_n|$ conv. $\Rightarrow \sum b_n$ conv. (in fact, AC)

Ex "+" term $\sum |b_n| = \underbrace{1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots}$ conv. $\left(\sum_{n=1}^{\infty} \frac{1}{n^2}, \text{ p-series } p=2>1\right)$

flip any signs (any sign configuration)
$\Rightarrow$ resulting $\sum b_n$ conv. by ACT

I don't know sum; Mathematica failed me! Even for approx.

for instance, $1 + \frac{1}{4} - \frac{1}{9} - \frac{1}{16} + \ldots$
also conv. In fact, AC.

(Note An "alt." $\sum$ can't go $++--++--\ldots$
Can't use AST.

Note This is $\sum_{n=1}^{\infty} (-1)^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{n^2}$, where
$\lfloor\ \rfloor$ is the floor or greatest integer operator/function.
$\lfloor 4.2 \rfloor = 4$.)

Some tests were OK for mixed-sign $\sum$, already:
(Telescoping $\sum$s)
Geom.
n^{th}-Term
modified LCT
11.5 versions of Ratio, Root

ACT allows us to use the 11.2-11.4 tests for "+" term $\sum$ to possibly verify that a mixed-sign $\sum$ conv. and, in particular, AC.

In high school!

High school has him and all other Σ where you just flip signs. Also most likely to diverge. From queen: all "+"'s Dennis Miller Σ.

If you're sarcastic, you may say- duh, of course sum ↓, but how do we know we have a sum? Why not oscillatory behavior for S_k like $\sin x$?

ACT

Idea $\Sigma|b_n|$ is "most likely to div." (all "+"'s)

If conv. $\Rightarrow$ any Σ where you just flip signs also conv. (AC)

If you're curious!

ACT

Proof Idea

Assume $\Sigma|b_n|$ conv.

b_n	+	$\lvert b_n\rvert$	$= 0$ or $2\lvert b_n\rvert$
1		1	2
$\frac{1}{4}$		$\frac{1}{4}$	$\frac{1}{2}$
$-\frac{1}{9}$		$\frac{1}{9}$	0
$-\frac{1}{16}$		$\frac{1}{16}$	0
$\vdots$		$\vdots$	$\vdots$

$\underset{\text{Sum}=T}{\Sigma \text{ conv.}} \Rightarrow \underset{\text{Sum}=U}{\Sigma \text{ conv.}}$

So, Σ conv. Sum $= U - T$

Idea Σ conv. $\Leftrightarrow$ has a sum

Sum of Σ

T = Sum of $\Sigma|b_n|$ (all "+" terms)

0

Flip "+"'s to "−"'s.
It's a nontrivial observation that $S_k \to$ some # for each of these new Σ's. (Partial sums don't oscillate ~~wildly~~ forever.)

$-T$ (all "−" terms) = Sum of $\Sigma(-|b_n|) = -\Sigma|b_n|$

(D) Ratio, Root Tests

$\left|\frac{a_{n+1}}{a_n}\right|$ OK for Ratio w/ alt. Σ.

If $\lim\limits_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = L$ or ∞ ... see 11.4 $\begin{cases} \text{if } L<1 \Rightarrow \Sigma b_n \text{ conv. (AC)} \\ \text{if } L>1 \text{ or "}\infty\text{"} \Rightarrow \Sigma b_n \text{ div.} \\ \text{if } L=1 \text{ (or "DNE")} \Rightarrow \text{?} \end{cases}$

" , $\sqrt[n]{|b_n|}$ " , "

Like in 11.4, but use "| |"
For either (Ratio/Root), if $L<1 \Rightarrow \Sigma$ conv. (in fact, AC)

SURPRISE! We can use Ratio, Root tests to verify that a mixed-sign Σ diverges!! AST, ACT couldn't do that.

(Note A CC Σ will yield $L=1$ for both tests. (or DNE)
BUT $L=1$ for a test $\Rightarrow$? Could be AC, CC, or D!!)
inconclusive

Ⓔ Exs

Ex (#2) Does $\sum_{n=1}^{\infty} (-1)^{n-1} n 5^{-n}$ conv. or div.? C|D

$$= \sum_{n=1}^{\infty} \overbrace{(-1)^{n-1} \left(\frac{n}{5^n}\right)}^{b_n}$$

$a_n > 0$ (in our Alt-Σ form)

$$= \frac{1}{5} - \frac{2}{25} + \frac{3}{125} \ldots \quad (\leftarrow \text{Not required})$$

Method 1 (AST) ⓪ alt. Σ ✓

On HW, they're told to verify 1,2

Verify ① a_n ... ↘

$\frac{a_{n+1}}{a_n} \le 1$ (for $n \ge 1$?) **or** if $f(x) = \frac{x}{5^x}$
$\Rightarrow f'(x) < 0$ (for $x \ge 1$?)

$$\frac{\frac{n+1}{5^{n+1}}}{\frac{n}{5^n}} = \frac{n+1}{5^{n+1}} \cdot \frac{5^n}{n}$$

$$= \frac{n+1}{n} \cdot \frac{5^n}{5^{n+1}}$$

$$= \underbrace{\left(1 + \overbrace{\frac{1}{n}}^{\le 1 \text{ for } n \ge 1}\right)}_{\le 2 \text{ for } n \ge 1}\left(\frac{1}{5}\right)$$

$$\le \frac{2}{5}$$

$$\le 1 \quad (n \ge 1) \checkmark$$

$$f'(x) = \frac{5^x(1) - x(5^x \ln 5)}{(5^x)^2}$$

$$= \frac{5^x - 5^x x \ln 5}{5^{2x}}$$

$$= \frac{\overbrace{5^x}^{>0}(1 - \overbrace{x \ln 5}^{>1})}{\underbrace{5^{2x}}_{>0}}$$

$$\le 0 \quad (x \ge 1)$$

$\ln e = 1$
$\ln \nearrow$

② $a_n \to 0$

exp'ls beat up polys

$$\lim_{n\to\infty} \frac{n}{5^n} \left(= \lim_{x\to\infty} \frac{x}{5^x} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{1}{5^x \ln 5} \right) = 0$$

cont. interpolating func. $f(x)$ (unless DNE; $\infty, -\infty$ OK) $\left(\frac{\infty}{\infty}\right)$

I'll accept what work on test.

Can do 1,3

$\therefore \sum b_n$ $\boxed{\text{conv.}}$ by AST

Method 2 (Ratio Test)

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n \frac{n+1}{5^{n+1}}}{(-1)^{n-1}\frac{n}{5^n}}\right|$$

(| | kills sign alternators if everything else "+".)

$$= \lim_{n\to\infty}\frac{\frac{n+1}{5^{n+1}}}{\frac{n}{5^n}} = \lim_{n\to\infty}\frac{n+1}{5^{n+1}}\cdot\frac{5^n}{n} = \lim_{n\to\infty}\frac{n+1}{n}\cdot\frac{5^n}{5^{n+1}}$$

(exchange parts)

$$= \lim_{n\to\infty}\underbrace{\left(\frac{n+1}{n}\right)}_{\to 1}\cdot\frac{1}{5}$$

$$= \frac{1}{5}$$

$$< 1$$

$\overset{\text{Ratio}}{\Rightarrow} \sum$ $\boxed{\text{conv.}}$ (in fact, AC)

Method 3 (Root Test)

L for Root doesn't have to = L for Ratio

$$\lim_{n\to\infty}\sqrt[n]{|b_n|} = \lim_{n\to\infty}\sqrt[n]{\left|(-1)^{n-1}\frac{n}{5^n}\right|}$$

$$= \lim_{n\to\infty}\frac{\overset{\to 1}{n^{1/n}}}{5}$$

$$= \frac{1}{5}$$

$$< 1$$

$\overset{\text{Root}}{\Rightarrow} \sum$ $\boxed{\text{conv.}}$ (in fact, AC)

Ex $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{n+1}{n}\right)^n = \sum_{n=1}^{\infty}(-1)^n\underbrace{\left(1+\frac{1}{n}\right)^n}_{\to e\neq 0}$ $\boxed{\text{div}}$ by n^{th}-Term Test

Should be aware

In any event

DNE

No way $\to 0$

Ex (#8) Is $\sum_{n=1}^{\infty} \overbrace{(-1)^{n+1} \underbrace{\frac{n}{n^2+4}}_{a_n > 0 \text{ in alt. } \Sigma \text{ form}}}^{b_n}$ AC, CC, or div.? $\frac{AC}{CC}$ | D

(We think it conv. alt, $a_n \downarrow$, $\to 0$) If div, AST can't verify div. If AC, just test $\Sigma|b_n|$ if you can.

AST to verify conv. (Don't need if can show Σ AC using Ratio, Root, or otherwise showing $\Sigma|b_n|$ conv.)

⓪ Alt. Σ ✓

Verify ① a_n ...

$$f(x) = \frac{x}{x^2+4}$$

$$f'(x) = \frac{(x^2+4)(1)-(x)(2x)}{(x^2+4)^2}$$

$$= \frac{-x^2+4}{(x^2+4)^2} \begin{matrix} \leftarrow - \text{ if } x \geq 3 \\ \leftarrow + \end{matrix}$$

$$< 0 \quad (x \geq 3) \checkmark$$

② $a_n = \frac{n}{n^2+4} \to 0$ ✓

$\therefore \sum_{n=3}^{\infty} b_n$ conv. by AST

$\Rightarrow \sum_{n=1}^{\infty} b_n$ conv.

$\frac{AC}{CC}$ | ~~div.~~

Is it AC or CC?

$$\sum_{n=1}^{\infty} |b_n| \left(= \sum_{n=1}^{\infty} a_n\right)$$

$$= \sum_{n=1}^{\infty} \frac{n}{n^2+4}$$

c_n?

Compare w/ $\frac{n}{n^2} = \frac{1}{n}$

BCT fails: $\frac{n}{n^2+4} \leq \frac{n}{n^2}$ or $\frac{1}{n}$ Σ div. Σ ?

LCT $\lim_{n\to\infty} \frac{\frac{n}{n^2+4}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+4}$

$= \underbrace{1}_{>0,\text{ real}}$

(if polys w/ same degree $\Rightarrow$ take ratio of leading coeffs.)

somersault

Carson 584 $\Sigma \frac{\sin n}{n^2}$

We know $\sum_{n=1}^{\infty} \frac{1}{n}$ div. $\Rightarrow \sum_{n=1}^{\infty} \underbrace{\frac{n}{n^2+4}}_{|b_n|}$ div.

$\therefore \sum_{n=1}^{\infty} b_n$ $\boxed{CC}$

(F) Rearrangements (AC vs. CC)

Larson
Stewart 722
Guido Ubaldus

Do we get a medal?

Ex $0 = 0 + 0 + 0 + \ldots$
$= (1-1) + (1-1) + (1-1) + \ldots$
$\neq 1 + (-1+1) + (-1+1) + \ldots$
$= 1$

NO! Regrouping can change sums. Assoc., Comm. laws of "+" break down.

(Guido Ubaldus thought this proved the existence of God: "something has been created out of nothing.")

① If $\sum b_n$ is AC with sum $= S$,
then any regrouping or reordering of terms
$\Rightarrow$ conv. $\sum$ with sum $= S$. (When you add the order of the terms doesn't matter)

If you consider all poss. reorderings, you can get all real sums.
Ref. Larson 586
Gelbaum 54
If div
Ex may have to $\to\infty$ ($-\infty$)

② If $\sum b_n$ is CC, and if r is any real #,
then there is a reordering $\Rightarrow \sum_{new}$ has sum $= r$. (" matter!)

Stewart 11.C #39

Do if have ~~time~~

Idea Consider $\sum$("+" b_n terms), $\sum$("−" b_n terms)

① If AC $\Rightarrow$ both conv.
② If CC $\Rightarrow$ div.

Let r be a real #. Say $r > 0$ (for our argument.)

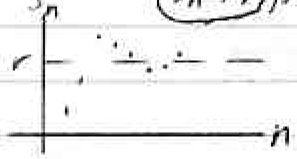
Stewart 11.C #40
Sols $|S_n - r| < |b_n|$
$\downarrow$
0

Add "+" terms until $S_n > r$
Then, "−" " $< r$
Repeat!

constructing a new $\sum$

$\boxed{S_n \to r}$, so sum of — $= r$

at least do graph



Remember, $\sum b_n$ conv.,
so $\boxed{b_n \to 0}$.

Larson 586 has reference
Gelbaum: "derangement"

Big Knopp 141, see 318
Rearrange Alt. Harm.
$\Rightarrow$ div.: $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^8} - 1 + \ldots + \frac{1}{2^{16}} - \frac{1}{3} \ldots$

Small Knopp 81 do
Big Knopp 140 same
Gelbaum 54

Riemann proved

(Note Why $S_n \to r$? Consider the "b_n"s that make S_n cross the r-line.
$|S_n - r| < |b_n| \Rightarrow |S_n - r| \to 0$
Distance bet. S_n, r $\;\downarrow 0$ $\;\Rightarrow S_n \to r$)

Can get div. $\sum$.
(My idea: Take a bunch of "+" terms until their sum $\geq$ |next neg. term| + 1
(Can make $\lim_{n\to\infty} S_n = \infty, -\infty$, or DNE.)

(G) Some Strategies for Alternating $\sum b_n = \sum \underbrace{(-1)^{n(\pm 1)} \underbrace{a_n}_{>0}}_{b_n}$

AC/CC | D ?

Is it really a geom. Σ? $\Rightarrow$ Geom. Σ Test

Do I think $\sum b_n$ ...

D ?	AC ?	CC ?
Try: n^{th}-Term Ratio Root	Try: Ratio, Root } Fail if a_n algebraic or Show $\sum \lvert b_n \rvert (= \sum a_n)$ C $\Rightarrow \sum b_n$ C, AC by ACT	Try: AST to show $\sum b_n$ C Then, use some test to show $\sum a_n$ D. WARNING: Ratio, Root Tests fail if $\sum b_n$ CC.

ACT often linked w/ another test.

Note on AST hypotheses:
(0) Alt. Σ
(1) $a_n \searrow$
} Failure does not imply D.

(2) $a_n \to 0$ } Failure implies that n^{th}-Term Test can be used to show D.

Confirmed by a book →

The failure of the hypoths. to hold does not negate the conclusion.

$\overbrace{\text{"If I get an A,}}^{\text{hypothesis}}$ then $\overbrace{\text{I pass the class."}}^{\text{conclusion}}$

Not true: "If I don't get an A, then I don't pass the class."
(What about B, C?)