11.6: POWER Σ

(A) Ex

Like a poly that goes on and on in ↗ powers. Not all geom Σ are power Σ ($1+\frac{1}{2}+\frac{1}{4}\ldots$ has no x (variable)). If x is some # w/abs. value <1, what do we know about this Σ?

Ex Poly: $1 + x + x^2$

Ex Power Σ: $1 + x + x^2 + \ldots + x^n + \ldots$

This also happens to be a geom. Σ with $a=1, r=x$. (Not all power Σ are.)

Case 1: If $|x|<1 \Rightarrow \Sigma$ conv. w/sum $S = \frac{a}{1-r} = \frac{1}{1-x}$

Ex If $x = \frac{1}{10}$,

$$1 + \tfrac{1}{10} + \left(\tfrac{1}{10}\right)^2 + \ldots + \left(\tfrac{1}{10}\right)^n + \ldots$$

i.e., $1 + .1 + .01 + \ldots$

converges w/sum

$$S = \frac{1}{1-\frac{1}{10}} \left(= \frac{1}{\frac{9}{10}} \right) = \frac{10}{9} \quad (= 1.\overline{1})$$

Case 2: If $|x| \geq 1 \Rightarrow \Sigma$ div., no sum

Ex If $x = 5$,

$$1 + 5 + (5)^2 + \ldots \quad \text{div}$$

Ⓑ Power Σ in x

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

coeffs charac'ize the power Σ

Notes
ⓐ Know $\{a_n \text{ coeffs}\} \Rightarrow$ Know Σ
ⓑ $x^0 = 1$, even if $x = 0$. (Normally, 0^0 gives us pause.)
ⓒ Maybe not geom. Σ
ⓓ Coeffs can be 0.

If you have a 4th-deg poly, do you have to have an x^2 term? No. Book: r

Ⓒ R, the Radius of Convergence, and I, the Interval of Conv.

... for which $\Sigma a_n x^n$ yields a real # output - like the ... of f

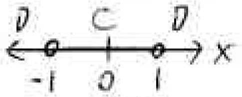
$$I = \{\text{all real values of } x \text{ for which } \overbrace{\Sigma a_n x^n}^{f(x)} \text{ conv.}\}$$
$= \text{Domain of } f$

<u>Ex</u> $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$ (all $a_n = 1$)

This conv.

$I = (-1, 1)$ symmetry about 0 not a coincidence!

$\Leftrightarrow x$ is in $\underbrace{(-1, 1)}_{I}$.



$\boxed{R = 1}$ = half-width of I

White shirt...

What can I look like?

3 possibs:

① $I = \{0\}$ $R = 0$

is this an "interval"? - Don't worry.

② $I = (-\infty, \infty)$ $R = \infty$

all real #s

AC on $(-d, d)$
AC/CC/D at $-d, d$
D outside
AC vs. CC
Not a big deal:
we'll do Σ alt harm.
$= \ln 2$ $\underbrace{}_{CC}$

③ $I = [(-d, d)]$ $R = d$
or or

for some $d > 0$

44: $\sum a_n$ AC $\Rightarrow \sum a_n x^n$ is AC $\forall x$ in $[-1,1]$
45: $I = [-r, r]$ i'c
46: AC at one endpt. $\Rightarrow$ AC at other

① Find I

Ex (#8) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n\sqrt{n}} x^n$

u_n depends on n, x

Consider Ratio Test (or Root Test)
1st, for what x is $L < 1$?

What guarantees that $\sum$ conv.? If L...

$$L = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$$

(Go ahead and write "L" even though "$L = \infty$" is inappropriate to write if $\lim\limits_{n\to\infty} \cdots = \infty$)

$$= \lim_{n\to\infty} \left| \frac{\frac{1}{4^{n+1}\sqrt{n+1}} x^{n+1}}{\frac{1}{4^n\sqrt{n}} x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{4^n\sqrt{n}}{4^{n+1}\sqrt{n+1}} \cdot \frac{x^{n+1}}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1}{4} \underbrace{\frac{\sqrt{n}}{\sqrt{n+1}}}_{\to 1} x \right|$$

$$= \frac{1}{4}|x|$$

This L limit value depends on what x is. (Sometimes, it doesn't matter: $R = \infty$)

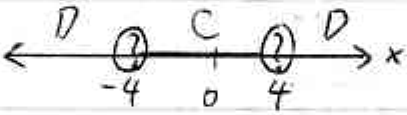
$\sum$ Ⓒ if
$\frac{1}{4}|x| < 1$
$|x| < 4$
$R = 4$

($\sum$ Ⓓ if
$\frac{1}{4}|x| > 1$
$|x| > 4$)

Don't have to write.

What are the only 2 values of x we don't know about yet?

all 4 combos CC is possible



$I = [(-4, 4)]$
or or

✓ $x = -4, 4$ (then, $L = \frac{1}{4}|x| = 1 \Rightarrow$ Ratio Test inconclusive)

<u>If $x = -4$</u>

$$\sum_{n=1}^{\infty} \frac{1}{4^n \sqrt{n}} (-4)^n = \sum_{n=1}^{\infty} \frac{1}{\cancel{4^n} \sqrt{n}} (-1)^n \cancel{4^n}$$

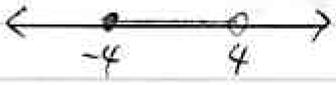
$$= \sum_{n=1}^{\infty} (-1)^n \underbrace{\left(\frac{1}{\sqrt{n}}\right)}_{\to 0} \quad \text{Ⓒ by AST}$$

alt. Σ,

<u>If $x = 4$</u>

$$\sum_{n=1}^{\infty} \frac{1}{\cancel{4^n} \sqrt{n}} \cancel{4^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad \text{Ⓓ by } p\text{-series } (p = \tfrac{1}{2} \le 1)$$

<u>Find I</u>



$$I = [-4, 4)$$

<u>What If...</u>

indep. of n, can pop out of $\lim_{n \to \infty}$

or $|x| \lim_{n\to\infty} \frac{1}{n}$

① $L = \ldots = \overbrace{\lim_{n \to \infty} \frac{|x|}{n}}^{} = 0 < 1$ for all real x

$R = \infty$, $I = (-\infty, \infty)$

Up to 33 (1, 2, (4 later))

② $L = \ldots = \lim_{n \to \infty} n|x| = \begin{cases} 0 & ; \text{ if } x = 0 \\ \infty & ; \text{ if } x \neq 0 \end{cases}$

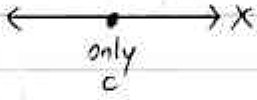
$R = 0$, Σ conv. only for $x = 0$ (I'm OK with $I = \{0\}$.)

Ⓔ Power Σ in $(x-c)$ (Before: $c=0$)

$$\sum_{n=0}^{\infty} a_n (x-c)^n \quad \text{"Power } \Sigma \text{ centered at } c\text{"}$$

Possible "I"s

What value of x guarantees conv., regardless of "a_n"s? Not really an interval here

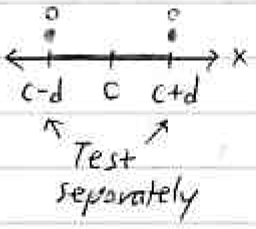
① 

$R=0$ $\quad I=\{c\}$

Is this an interval! Don't worry!

② 

$R=\infty$ $\quad I=(-\infty,\infty)$

③ 

$R=d$ $(d>0)$ $\quad I=[(c-d, c+d)]$

or $\quad$ or

Ex (#12) Find I for $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{1}{n(n+1)}(x-2)^n}_{=u_n}$

For what x is "Ratio Test" (or Root) $L<1$?

$$L=\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)(n+2)}(x-2)^{n+1}}{\frac{1}{n(n+1)}(x-2)^n}\right|$$

$$=\lim_{n\to\infty}\left|\frac{n\cancel{(n+1)}}{\cancel{(n+1)}(n+2)}(x-2)\right|$$

$$= \lim_{n\to\infty} \left| \underbrace{\frac{n}{n+2}}_{\to 1} (x-2) \right|$$

$$= |x-2|$$

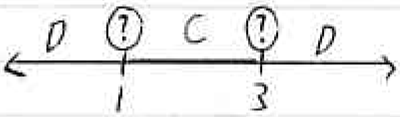
Σ Ⓒ if $|x-2| < 1$

How can we rewrite w/out $|\,|$?

$$|☆| < 1$$
$$-1 < ☆ < 1$$

$$-1 < x-2 < 1$$
$$+2 \quad\quad +2 \quad\quad +2$$

$$1 < x < 3$$



$I = [(1,3)]$
or or

<u>If $x=1$</u>

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} (1-2)^n = \sum_{n=1}^{\infty} (-1)^n \underbrace{\frac{1}{n(n+1)}}_{\to 0}$$

Alt. Σ

Ⓒ by AST

L11-24
11.6

If $x=3$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \underbrace{(3-2)^n}_{=(1)^n=1} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

I thought 1^∞ indet? Not if the base is just "1"!

$$= \sum_{n=1}^{\infty} \frac{1}{n^2+n} \quad >0$$

BCT

$$\frac{1}{n^2+n} \le \frac{1}{n^2} \quad (n \ge 1)$$

Big Σ conv.

$\Rightarrow$ Little Σ ©

I :

$$\boxed{[1,3]}$$