Berresford/Rockett, Applied Calculus, 2nd ed., pp. 786-7 (Houghton Mifflin)

10.3 Taylor Series

APPLICATION PREVIEW

Calculators, Computers, and Guided Missiles

Throughout this book we have used calculators to evaluate and graph functions such as e^x, $\ln x$, and $\sin x$. How do calculators and computers evaluate such functions? Most calculators use a procedure called CORDIC (for COordinate Rotation DIgital Computer)

that is particularly suited to the limited circuitry of a calculator.* Computers, on the other hand, calculate values of these functions using series like the "Taylor series" shown below, which is discussed in this section.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

By taking enough terms, values of e^x can be found to any degree of accuracy using only the basic operations of addition, subtraction, multiplication, and division.

Actually, most computers modify this procedure in two ways. The first modification is called *range reduction*, which reduces the size of the number that is substituted into such formulas. The essential idea is that a number like $e^{1.36}$ can be expressed as $e^1 e^{0.36}$ (by adding exponents), so the formula is evaluated at $x = 0.36$ and the result multiplied by e to obtain $e^{1.36}$. In this way, the series is used only for x-values between -1 and 1. The second modification is to alter the coefficients of the series slightly, giving what is called an *economized series*, so that the error is more uniform over this interval.

In spite of these procedures, occasional errors slip through. For example, one popular personal computer evaluates ln 1.001 as 0.0009994461 (using software called BASICA) or as 0.000999547 (using QBASIC), both of which are wrong in the last few deciamal places, the correct value being 0.0009995003. Even such small errors can have major consequences. In 1991 when Iraq invaded Kuwait, 28 American soldiers were killed by a Scud missile, in part because an incorrect computer program generated an error in the sixth decimal place, allowing the Scud to slip through the Patriot missile defense system.†

Introduction

In this section we return to infinite series, but with the terms of the series now being *functions*. We begin by discussing *power series*, then specializing to *Taylor series*, which may be thought of as "infinitely long" Taylor polynomials. Taylor series lead to remarkably simple in-

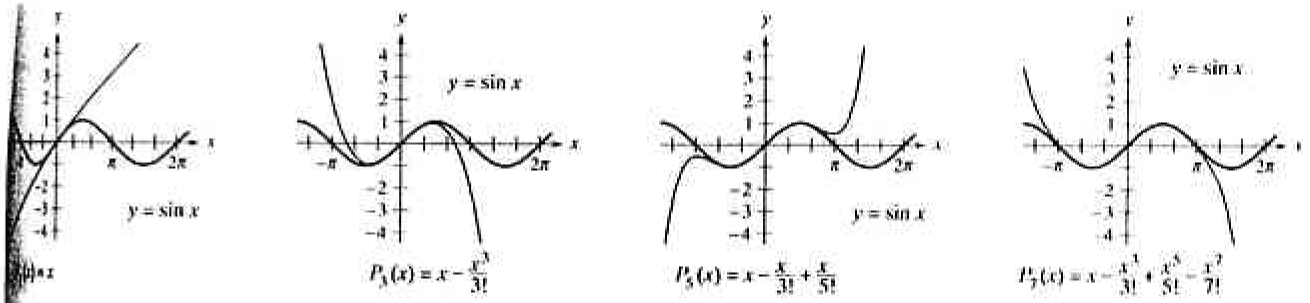
*For further details, see Richard J. Pulskamp and James A. Delaney, "Computer and Calculator Computation of Elementary Functions," UMAP module 708, COMAP, Arlington, Mass., 1991.

†See Robert Skeel, "Roundoff Error and the Patriot Missile," *SIAM News*, July 1992, p. 11, published by the Society for Industrial and Applied Mathematics, Philadelphia.

MACLAURIN and TAYLOR POLYNOMIALS

Maclaurin polynomials for $f(x) = \sin x$ (centered at $c = 0$).

Source: Calculus: Sixth Edition by Larson, Hostetler, Edwards, p.625; Houghton Mifflin, 1998.

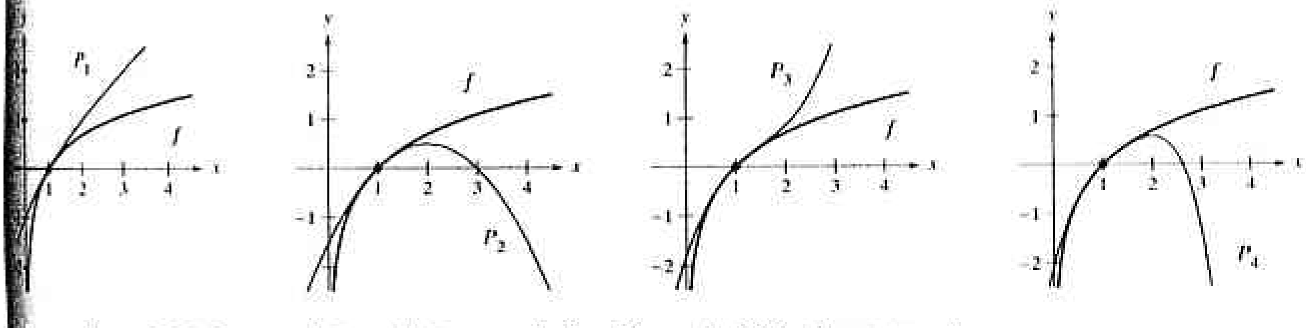


increases, the graph of P_n more closely resembles the sine function.
8.23

Taylor polynomials for $f(x) = \ln x$ (centered at $c = 1$).

Source: Calculus: Sixth Edition by Larson, Hostetler, Edwards, p.599; Houghton Mifflin, 1998.

Figure 8.14 compares the graphs of P_1, P_2, P_3, and P_4 with the graph of $f(x) = \ln x$. Note that near $x = 1$ the graphs are nearly indistinguishable. For instance, $P_4(0.9) \approx -0.105358$ and $\ln(0.9) \approx -0.105361$.



increases, the graph of P_n becomes a better and better approximation of the graph of $f(x) = \ln x$ near $x = 1$.
8.14

FUN WITH e^x

We begin with the Maclaurin series for e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

This holds for any complex value of x!! (z is often used instead of x.)

Now, plug in $(i\theta)$ for x, where θ represents an arbitrary real number, possibly representing an angle measure in radians.

$$
\begin{aligned}
e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \ldots \\
&= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \ldots
\end{aligned}
$$

Remember that the powers of i cycle between 1, i, -1, and $-i$.

$$
= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \ldots
$$

Now, separate the real terms and the terms in i.

$$
\begin{aligned}
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + \left(i\theta - i\frac{\theta^3}{3!} + i\frac{\theta^5}{5!} - \ldots\right) \\
&= \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right)}_{\text{This is } \cos\theta.} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)}_{\text{This is } \sin\theta.}
\end{aligned}
$$

We now have Euler's formula:

$$
e^{i\theta} = \cos\theta + i\sin\theta
$$

The right side is the polar (trig) form of a complex number with modulus 1 and argument (standard angle) θ in the complex plane.

Plug in $\theta = \pi$:

$$
\begin{aligned}
e^{i\pi} &= \underbrace{\cos\pi}_{=-1} + i\underbrace{\sin\pi}_{=0} \\
e^{i\pi} &= -1
\end{aligned}
$$

$$
e^{i\pi} + 1 = 0 \qquad (\text{Euler's Theorem})
$$

"[It] connects the five most important constants of mathematics (and also the three most important mathematical operations – addition, multiplication, and exponentiation). These five constants symbolize the four major branches of classical mathematics: arithmetic [0 and 1]; algebra [i]; geometry [π]; and analysis [e]." (Eli Maor, e: The Story of a Number, p.160)

"...perhaps the most compact and famous of all formulas.... It appeals equally to the mystic, the scientist, the philosopher, the mathematician." (Kasner and Newman, Mathematics and the Imagination, 1940).