11.8: MACLAURIN (M) and TAYLOR (T) SERIES

Stewart 762
Colin Maclaurin popularized M. series in 1742 calc book
766
M series for e^x, $\sin x$, $\cos x$ disc. by Newton using diff. methods
Did Indian astronomers know $\sin x$, $\cos x$ Σ, >1 century before Newton?
History

(A) M Series for $f(x)$

Power Σ rep. of $f(x)$ centered at 0

(11.7) (We found M. Series for 6 funcs.)

If f has a power Σ rep.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad (R \neq 0)$$

(If $R = 0$, just say something like $f(0) = 10$ (or whatever a_0 is), and be done with it. Why bother w/ Σ? Even a small $I \neq \{0\}$ is more interesting.)

Then

seed at $x=0$
sprinkle deriv. info
⇒ grow into better, higher-degree poly approxs. of func.

These all exist:

$$f^{(0)}(0) \overset{\text{which means}}{=} f(0)$$
$$f^{(1)}(0) = f'(0)$$
$$f^{(2)}(0) = f''(0)$$
$$\vdots$$

(We hope there's a nice pattern for our f.)

and

How can we use this deriv. info at 0 to construct our M/power Σ?
Maybe 0!, 1!

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \text{☆ (Compact!!)}$$

$$= \underbrace{f(0)}_{\text{const.}} + \underbrace{f'(0)}_{\text{const.}} x + \underbrace{\frac{f''(0)}{2!}}_{\text{const.}} x^2 + \underbrace{\frac{f'''(0)}{3!}}_{\text{const.}} x^3 + \dots + \underbrace{\frac{f^{(n)}(0)}{n!}}_{\text{const.}} x^n + \dots$$

<u>Ex</u> <u>If</u> e^x can be rep. by $\sum_{n=0}^{\infty} a_n x^n$ $(R \neq 0)$, what must it be?

i.e., Find the M series for e^x, assuming it exists.

$$f(x) = e^x$$

$$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = e^0 = 1 \quad (n = 0, 1, 2, \ldots)$$

$$e^x = \sum_{n=0}^{\infty} \frac{\overbrace{f^{(n)}(0)}^{\text{always } 1}}{n!} x^n$$

$$\Rightarrow \boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots}$$

for all real x

(Now you can believe me!! Wait, how about a proof of the template!)

Why not "Σ"? We're now talking about reps. of funcs, not just power Σ.

<u>Why?</u> (Proof of (A))

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

(We assume a M. series rep. exists.)

Show: $a_n = \frac{f^{(n)}(0)}{n!}$ $(n = 0, 1, 2, \ldots)$

(formula for coeffs of M power series)

Dems.

$$f^{(n)}(x) = D_x^{(n)}\Big(\underbrace{(a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}}_{n^{th} \text{ deriv} = 0}$$

Ex The 3rd deriv. of a 2nd-degree poly is 0.

Green

This is the term we really care about!

$$+ \underbrace{a_n x^n}_{n^{th} \text{ deriv.} = n(n-1)(n-2)\cdots(1)a_n x^0 = n! a_n}$$

Apply Power Rule n times $\Rightarrow$ -n from exponent

Reps.

$$+ a_{n+1} x^{n+1} + \ldots \Big)$$

some # times x

$$= n! a_n + \underbrace{c_1 x + c_2 x^2 + \ldots}_{\text{(Turns out we don't care what these are.)}} \quad (\text{"}c_i\text{"s real \#s})$$

$$f^{(n)}(0) = n!\,a_n$$ Solve for

$$\boxed{a_n = \frac{f^{(n)}(0)}{n!}}$$

Ex $f(x) = \sin x$ has a M series. Find it.

	$f^{(n)}(0)$	for $n=$	
D_x $\sin x$	0	0	4
$\cos x$	1	1	5 ...
$-\sin x$	0	2	6
$-\cos x$	-1	3	7

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^{n}$$

$n=0 \Rightarrow 0$ term
$n=2 \Rightarrow 0$ term
⋮

$$= \underset{(n=1)}{\frac{1}{1!}x^1} + \underset{(n=3)}{\frac{(-1)}{3!}x^3} + \underset{(n=5)}{\frac{1}{5!}x^5} + \underset{(n=7)}{\frac{(-1)}{7!}x^7} + \ldots$$

Careful!
← The $n=7$ term here
← is the $n=3$ term here!!

$$\boxed{\begin{aligned} \Rightarrow \sin x &= \underset{(n=0)}{x} - \underset{(n=1)}{\frac{x^3}{3!}} + \underset{(n=2)}{\frac{x^5}{5!}} - \underset{(n=3)}{\frac{x^7}{7!}} + \ldots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ &\text{for all real } x \text{ (RAD)} \end{aligned}}$$

ODD

The graph is sym about origin if you replace x w/ $(-x)$
Same Σ, diff. form
We automatically skip over 0 terms
we're numbering the terms differently
Use k's.
More efficient/compact encoding of Σ.
Calc must be in RAD.

12: $\cos(x^2)$

Ex
$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$$

<u>Why not use M series template</u>
Repeatedly differentiating for $\sin(x^2)$, itself, would have been messy. Works in principle $\Rightarrow$ same series.

Can $\int \sin(x^2)\,dx$. Can approx. $\int_a^b \sin(x^2)\,dx$ for any real a, b.

Ex $f(x) = \cos x$

$$\cos x = D_x(\sin x)$$

$$= D_x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)$$

$$\Rightarrow \boxed{\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\ &\text{for all real } x \end{aligned}}$$

EVEN

Also, $D_x\Big(\underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}_{\sin x}\Big) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

Do **not** start w/ $n=1$. The $n=0$ term is **not** a constant.

sinh x cosh x

odd even

Exs $\sinh x$, $\cosh x$ (11.7)

Another way:

$$\sinh x \xleftrightarrow{D_x} \cosh x$$

$\sinh 0 = 0$ $\qquad$ $\cosh 0 = 1$ $\qquad$ ← Think: $\sin 0 = 0$, $\cos 0 = 1$

what if you forget defns? $\sinh x = \frac{e^x - e^{-x}}{2}$

If $f(x) = \sinh x \Rightarrow f^{(\text{even})}(0) = 0$ and $f^{(\text{odd})}(0) = 1$.

$$\sinh x = \frac{1}{1!}x^1 + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \overset{D_x}{\Rightarrow} \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

ODD $\qquad$ EVEN

same as (M series) for $\sin x$, $\cos x$, except all "+", no sign flipper.

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

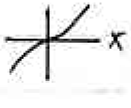
for all real x

Don't say pos-term or ||; what if $x<0$?

No sign flipper / alternator. Never alt. for any x.

The sign flips ⇒ periodicity Weird! Who'd suspect $\sin(10) = \sin(10+2\pi)$??

Note $\sinh x$ is ↗; not periodic like $\sin x$.



(Who'd suspect periodicity from Σ??)

11.7 method easier!

Ex $f(x) = \ln(1+x)$ | $f(0) = \ln 1 = 0$ (Won't follow the pattern, but he is 0...)

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$ | $f'(0) = 1$

$f''(x) = -(1+x)^{-2}$ | $f''(0) = -1$

$f'''(x) = 2(1+x)^{-3}$ | $f'''(0) = 2$

$f^{(4)}(x) = -3\cdot 2\,(1+x)^{-4}$ | $f^{(4)}(0) = -3!$

$\vdots$ | $\vdots$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$ | $f^{(n)}(0) = (-1)^{n-1}(n-1)!$ $(n=1,2,3,\dots)$ what I want!!

f''' + $f^{(4)}$ − if it's n: − + tweak

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f^{(0)}(0) = f(0) = 0$. Just drop the $n=0$ term.

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cancel{(n-1)!}}{\cancel{n!}_{\,n}} x^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} \quad \text{or} \quad \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$$

$k = n-1$

equals to our 11.7 result. Explains why n! in den. There's ! in num. up to 1.5

Ex $f(x) = \tan^{-1} x$
$f^{(n)}(x)$ hard!!
Use 11.7

$f'(x) = \frac{1}{1+x^2}$
$f''(x) = -\frac{2x}{(1+x^2)^2}$
$f'''(x) = \frac{6x^2-2}{(1+x^2)^3}$
$f^{(4)}(x) = -\frac{24x(x^2-1)}{(1+x^2)^4}$

YUK!!

Ex $\tan x = \frac{\sin x}{\cos x}$ > Expand, Long ÷ (theory messier)
There is a pattern, but it's complicated!!

Stewart Long ÷ results

Calculator: $\tan 1 = \frac{\sin 1}{\cos 1}$??

Larson 598:
Brook Taylor
published early
comprehensive
work on poly
approx of
transc. funcs.
1715

Ⓑ T Series for $f(x)$ at c $(x=c)$

If f has a power Σ rep. centered at c

$$f(x)=\sum_{n=0}^{\infty} a_n (x-c)^n \quad (R\neq 0)$$

Then

$f(c), f'(c), f''(c), \ldots$ all exist

and

$$f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maybe $0!, 1!$

$$= f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3+\ldots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\ldots$$

M series: $c=0$

Ex (#20) Find the T series for $f(x)=e^x$ at $c=-3$.

$$f(x)=e^x$$
$$f^{(n)}(x)=e^x \quad f^{(n)}(-3)=e^{-3}=\frac{1}{e^3} \quad (n=0,1,2,\ldots)$$

$$e^x=\sum_{n=0}^{\infty}\frac{\overbrace{f^{(n)}(-3)}^{\frac{1}{e^3}}}{n!}(x-(-3))^n$$

$$=\boxed{\sum_{n=0}^{\infty}\frac{1}{e^3 n!}(x+3)^n}$$

$$\left(=\frac{1}{e^3}+\frac{1}{e^3}(x+3)+\frac{1}{e^3 \cdot 2!}(x+3)^2+\ldots\right)$$

Why? Later.

expanding in powers of $(x+3)$

Ex (#24) Find the 1st 3 terms of the T series for $f(x)=\tan x$ at $c=\frac{\pi}{4}$.

$f(x)=\tan x$	$f(\frac{\pi}{4})=\tan\frac{\pi}{4}=1$
$f'(x)=\sec^2 x = (\sec x)^2$	$f'(\frac{\pi}{4})=(\sec\frac{\pi}{4})^2=(\sqrt{2})^2=2$
$f''(x)=2\sec x\cdot\sec x\tan x = 2\sec^2 x\tan x$ Don't need a general pattern!	$f''(\frac{\pi}{4})=2(\sec\frac{\pi}{4})^2(\tan\frac{\pi}{4}) = 2(2)(1) = 4$

$$\tan x=\frac{f(\frac{\pi}{4})}{\underbrace{0!}_{=1}}+\frac{f'(\frac{\pi}{4})}{\underbrace{1!}_{=1}}(x-\tfrac{\pi}{4})+\underbrace{\frac{f''(\frac{\pi}{4})}{2!}}_{\frac{4}{2}=2}(x-\tfrac{\pi}{4})^2+\ldots$$

(may help) →

$$=\underline{1+2(x-\tfrac{\pi}{4})+2(x-\tfrac{\pi}{4})^2+\ldots}$$

Can do HW

Why Bother? (Why not just use M series?)

Ex Approx. $e^{-2.9}$.

M series $(c=0)$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots$$

Wait: $e^0=1$. Don't even have int. part right, let alone any dec places. I'm offended by $\approx$

$$e^{-2.9} = \underbrace{1 + (-2.9) + \frac{(-2.9)^2}{2!} + \dots}$$] STRUGGLE! ($n!$ eventually "really wins out")

$$e^{-2.9} \not\approx 2.30500$$

T series $(c=-3)$

We can approx. e, and we could, in principle, cube the approx. by hand, or w/ cheap calculator, and take reciprocal. Why not center Σ at -2.9? Then, you'd have to know $e^{-2.9}$. The whole problem we're attacking is the integration of e^x between integer values of x.

$$e^x = \frac{1}{e^3} + \frac{1}{e^3}(x+3) + \frac{1}{e^3 2!}(x+3)^2 + \dots$$ (A cheap calculator can approx. e^3.)

$$= \frac{1}{e^3}\left[1 + (x+3) + \frac{(x+3)^2}{2!} + \dots\right]$$

$$e^{-2.9} = \frac{1}{e^3}\left[1 + (-2.9+3) + \frac{(-2.9+3)^2}{2!} + \dots\right]$$

$$= \underbrace{\frac{1}{e^3}\left[1 + (0.1) + \frac{(0.1)^2}{2!} + \dots\right]}$$] NO STRUGGLE!

$$\approx 0.05501$$

Terms → 0 faster
Faster convergence

Like Price is Right. The envelope please. No contest

Turns out $e^{-2.9} \approx 0.05502$

M series: struggle bet. powers of -2.9 and $n!$

T series: powers of .1 → 0 anyway. $n!$ just speeds it up. Fast conv.

If you care about $x \approx c$, expand T series at c.

Sometimes, we can't use M series to find a func. value. Re-center!

Ex Approx. $\ln 3$. Want $\ln(1+x)$ where $x=2$. M series for $\ln(1+x)$: $I=(-1,1]$ No 2!!

1st term in expansion: $\ln c$. Is there a # whose ln we know besides 1? If we picked 2, circular

Try T series for $\ln x$ at e. $\ln x = \underbrace{\ln e}_{=1} + \frac{1}{e}(x-e) - \dots$ (R=e)

I need a center c so that f(c) is "known."

Ⓒ M, T Polys.

If it exists, M. series for $f(x)$

$$= \underbrace{f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n}_{\substack{P_n(x)\text{, the } n^{th}\text{-degree} \\ \text{M. poly. for } f(x)}} + \ldots$$

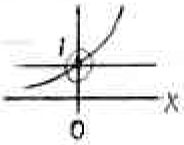
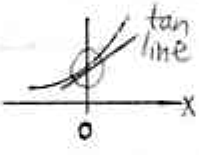
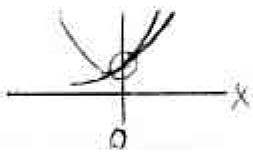
Best cubic approx. given by $P_3(x)$, at least near $x=0$.

best n^{th}-deg. poly. approx. for $f(x)$ near $x=0$.

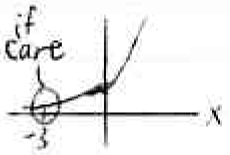
I'm Mr. Colin Mac. Give me info at 0.

Ex (p.591) $f(x) = e^x = 1 + x + \frac{1}{2}x^2 + \ldots$

Gardening
If you sprinkle deriv. info on seed, how does the poly approx. grow?
A.I.

Use	Poly. Approx.		
$f(0)=1$	$P_0(x) = 1$		(Best constant approx. to f $y=f(0)$)
and $f'(0)=1$	$P_1(x) = 1+x$		$\left(\begin{array}{l} y = b + mx \\ y = f(0) + f'(0)x \end{array}\right)$
and $f''(0)=1$	$P_2(x) = 1 + x + \frac{1}{2}x^2$		
⋮			

What's the slope at?
At least it's inc.
At least concave up
"concavity"



Use T poly. at -3.
(Calcs. use T polys modified so that error is spread more evenly through an interval.)

Stewart 780

Handout

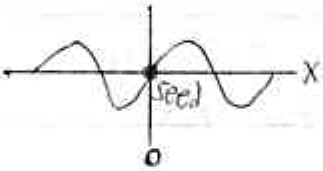
Where is valid?

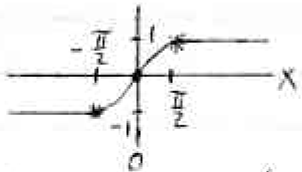
① For what x is $f(x)$ rep. by its M series?

g from Larson (24

Ex

$$f(x) = \sin x \qquad g(x) = \begin{cases} -1 & , x < -\frac{\pi}{2} \\ \sin x & , |x| \le \frac{\pi}{2} \\ 1 & , x > \frac{\pi}{2} \end{cases}$$





$f(0) = g(0)$
$f'(0) = g'(0)$

(All bets are off.) ← g'' DNE → (All bets are off.)
(Brick walls here)

Note:
All M series are power series.
They have derivs. of all orders on $(-R, R)$, $R > 0$
or $(-\infty, \infty)$, $R = \infty$.
If $f^{(n)}$ falls apart, the M series can't rep. it there.

$$f^{(n)}(0) = g^{(n)}(0) \quad (n = 0, 1, 2, \dots)$$ (All orders of derivs. at 0 match up.)

Based on these values, we construct the M. series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Where?
OK: $\sin x$ somewhere
1 out here
-1
$D_x^2(\sin x) = -\sin x$

$\vec{0}$ $\overset{c}{\uparrow}$ (CU)
$\overset{-}{\uparrow}$ $\frac{c}{0}$
(CD)
Stopped dead in our tracks?

This represents $f(x)$ for all real x
but $g(x)$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, only.
(g'' DNE)

Proof? Later!

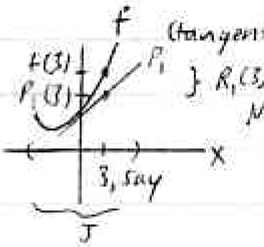
Redundant: $\exists f^{(n+1)} \Rightarrow \exists$ previous

Let J be some interval containing 0.

If $f, f', \ldots, f^{(n+1)}$ exist on J, (actually, if $f^{(n+1)}$ exists $\Rightarrow$ so do the previous ones)
then for any x in J,

Larson 602: MVT special case $f(x) = f(c) + f'(z)(x-c)$

$$f(x) = P_n(x) + \underbrace{R_n(x)}_{\text{remainder or error}}$$



(tangent line at $x=0$ y-int.)
$\}\ R_1(3)$
Need $R_n(3) \to 0$ as $n \to \infty$

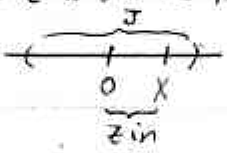
Reminds you of AST error analysis.

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$$

} Looks like 1st neglected term in M series, except $0 \to z$.

Edwards 514 "Lagrange form" for rem. 1st appeared 1797 book by L.

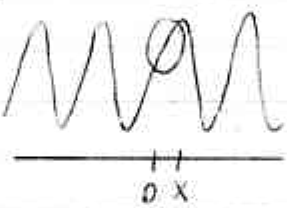
where z is some # bet. $0, x$.



11.9: Bound |error|, or $|R_n(x)|$

Idea

$f''=0$, $|f''|$ not high
Your poly approxs. might fly off pretty quickly.



If $|f^{(n+1)}|$ can be high
$\Rightarrow$ |error|!
(Your poly. approx may "fly off" pretty quickly.)

If an $f^{(n)}$ doesn't exist, see my note on ② L11-38b.

If $f, f', f'', \ldots$ exist on J,
and $R_n(x) \to 0$ for every x in J,
then $f(x)$ is rep. by its M. series on J.

$$\left(\underline{\text{must}} \text{ be } \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \right)$$

Ross 179-50
F, not always
True for many familiar funcs.

Can J be I? "Frequently"
T. series at c: similar.

$\underline{Ex}^2$ (p.584) Prove: $f(x) = \sin x$ is rep. by its M. series for all real x

$$|R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right|$$

$$f^{(n+1)}(z) = \pm \cos z$$
or $\pm \sin z$

$$\Rightarrow |f^{(n+1)}(z)| \le 1 \quad \text{(for any real } z)$$

Books: Stewart?
$e^x = \sum \frac{x^n}{n!}$ c $\forall x$
$\Rightarrow \left(\frac{x^n}{n!} \to 0\right)$

$$\Rightarrow \underbrace{|R_n(x)|}_{\text{So } \downarrow \, 0} \le \underbrace{\left| \frac{x^{n+1}}{(n+1)!} \right|}_{\downarrow \, 0 \text{ (for any real } x)}$$

$$\Rightarrow R_n(x) \to 0 \quad \text{(for any real } x)$$

sin odd $\Rightarrow 2n+1$
conv $\forall$ real $x \Rightarrow$!
periodic $\Rightarrow (-1)^n$

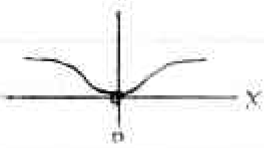
$\therefore$ M series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

Can plug in $x = 1$ M

represents $\sin x$ for all real x.

Ross 181: Such fancs vital to theory of dist^{n}s – related to adv. diff eqs., Fourier analysis

$\underline{Ex\ 6}$ (pp. 586-7) $f(x) = \begin{cases} e^{-1/x^2}, & x \ne 0 \\ 0, & x = 0 \end{cases}$



$$f^{(n)}(0) = 0 \quad (n = 0, 1, 2, \dots)$$

Since there's no interval containing 0, where valid we say M. series no good.

M series $= 0$, valid only for $x = 0$. $R = 0 \Rightarrow$ "No M Series"
$R_n(x) \not\to 0$ $\binom{\text{for all}}{x \ne 0}$