

Taylor polys

II.1: SEQUENCES

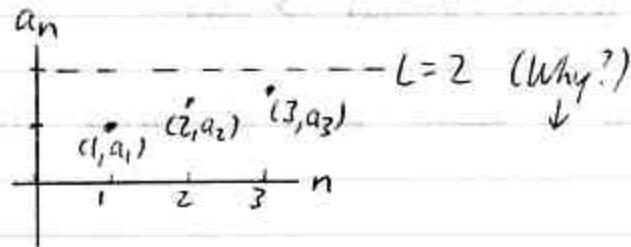
'lists of #'s (or funcs.)

(A)  $\{a_n\}$ Ex 1  $\{2 - \frac{1}{n}\}$  $a_n$ , the general  $n^{\text{th}}$  term  
like  $f(n)$ , where domain:  $n = 1, 2, 3, \dots$   $a_1 = f(1)$ , etc. $a_1, a_2, a_3, \dots \rightarrow ?$   
 $\begin{array}{c} \diagdown \\ 1 \\ \diagup \end{array}$   
Terms

$$a_1 = 2 - \frac{1}{1} = 1$$

$$a_2 = 2 - \frac{1}{2} = \frac{3}{2} \text{ or } 1.5$$

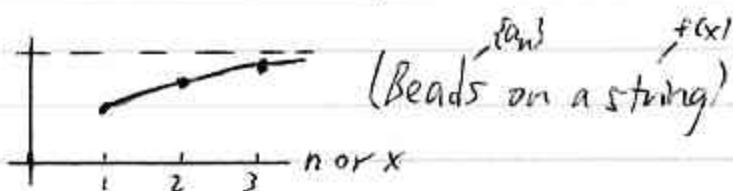
$$a_3 = 2 - \frac{1}{3} = \frac{5}{3} \text{ or } 1.6$$

Terms are  
func values  
converg. to y-coordsf interpolates  
 $\{a_n\}$ (B)  $\lim_{n \rightarrow \infty} a_n$ discretizing  
Ch. 0-limit  
func. para.people know how  
to f but not  
E.Ex 1 Since  $\lim_{x \rightarrow \infty} (2 - \underbrace{\frac{1}{x}}_{f(x)}) = 2$   
defined on  $(1, \infty)$ 

$$\Rightarrow \lim_{n \rightarrow \infty} a_n = 2$$

 $\Leftrightarrow$  a real #  
 $\{a_n\}$  conv.  
(crete, div.)

"Continuous"  
f is a cont. interpolating  
func. on  $[1, \infty)$   
"Discrete"

Terms  
converg. to  
beads on a  
string.  
If string  $\rightarrow 2$   
 $\rightarrow$  beads  $\rightarrow 2$ 

If the string  
goes off to  $\infty$   
& the beads do too

More intuitive, less  
math honest?

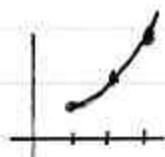
Can you think  
of a func  
where  $\lim \text{DNE}$ ?  
 $x \rightarrow \infty$

Let's draw it. It's  
like a transformed  
version of  $\sin x$ ,  
except maybe not  
perfectly periodic.  
(maybe tweaked,  
placeboids so  $y \neq \#$ ?  
or  
Can you think  
of a rhng  
defined on  
(1,  $\infty$ ) that  
passes thru  
these pts.  
but  $\lim \text{DNE}$ ?  
 $x \rightarrow \infty$

$\lim(\text{sum}) =$   
 $\text{sum}(\lim r)$   
if exist

see  $\textcircled{F}$

Ex



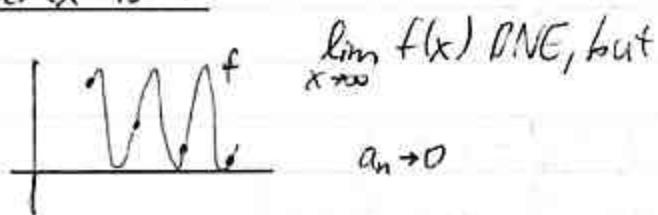
$$\lim_{x \rightarrow \infty} f(x) = \infty \Rightarrow \lim_{n \rightarrow \infty} a_n = \infty$$

WARNING  $\lim_{x \rightarrow \infty} f(x) \text{ DNE} \not\Rightarrow \lim_{n \rightarrow \infty} a_n \text{ DNE}$

(does  
(not imply))

(i.e., if  $\lim_{x \rightarrow \infty} f(x) \text{ ONE}$ ,  
then  $\lim_{n \rightarrow \infty} a_n$   
may or may not  
be ONE)

Counterex to  $\Rightarrow$



Thm 2.8 (p. 60) extends to seqs.

limit of sum = sum of limits (if exist), etc.

## ⑥ Geometric Seqs.

Ex  $\{(0.9)^n\}$

$$0.9, 0.81, 0.729, \dots \rightarrow 0$$

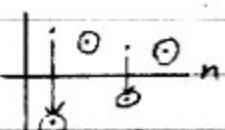
(10% discounts)

Like K-mart  
west to do  
A dress costs \$1  
Marge

What effect do  
you think  $(-1)^n$  has?  
Factor out  $-1$   
is here; take  
 $(-1)^n$  out of the  
whole thing.  
Start w/  $(0.9)^n$

If bare it in there  
narrow band  
Edwards 535 pf.  
 $|r^n| = |(r^m)^{n/m}|$   
 $0 < r < 1$   
 $\sim \frac{1}{r^m} = (1/a)^m$   
 $> 1/m a$   
 $0 < r^n < \frac{1}{l/m a}$   
 $\therefore$

How can you say?  
using  $\frac{1}{l/m a}$   
Get used to this!  
More compact;  
books do it.



$$\lim_{n \rightarrow \infty} r^n = 0 \text{ if } -1 < r < 1$$

$(|r| < 1)$

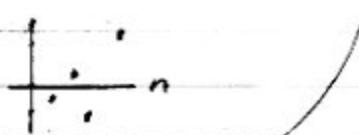
Ex  $\{2^n\}$

$$2, 4, 8, \dots \rightarrow \infty$$



Ex  $\{(-2)^n\}$

$$-2, 4, -8, 16, \dots \rightarrow (\text{DNE})$$



Ex  $\{\underbrace{(-2)^n}_{\infty}\}$  same as  $\{2^n\}$

$$\lim_{n \rightarrow \infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ \text{DNE} & \text{if } r < -1 \end{cases}$$

$$\lim_{n \rightarrow \infty} |r^n| = \infty \text{ if } |r| > 1$$

Makes all terms "+"

Geom. bcc. really  
 $\{2^n\}$

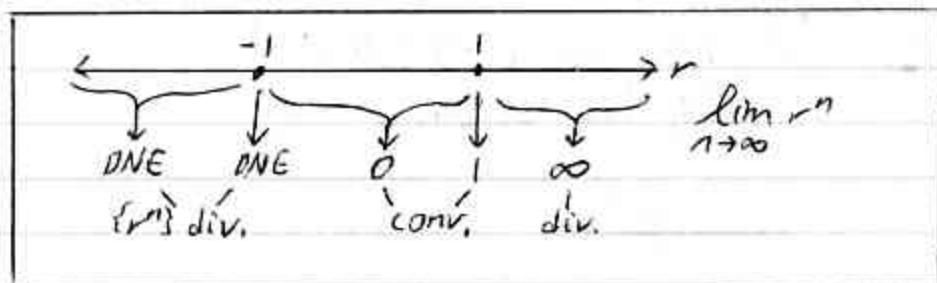
$2^n, 3^n, \dots$   
 $(-2)^n, (-3)^n, \dots$

$$\underline{r=1} \quad \{1^n\}$$

$1, 1, 1, \dots \rightarrow 1$

$$\underline{r=-1} \quad \{(-1)^n\}$$

$-1, 1, -1, 1, \dots \rightarrow (\text{DNE})$



Ex (Geom. seq.)

2, 6, 18, what next?  
19 test

If we want to write  
gen. nth term...  
 $\{2(3)^n\}$   
we have to  
modify if  
starting with 1.  
 $-2(\infty) = -\infty$

$$2, 6, 18, 54$$

$\overset{+3}{\nearrow} \quad \overset{+3}{\nearrow} \quad \overset{+3}{\nearrow}$   
 $2 \cdot 3^0 \quad 2 \cdot 3^1 \quad 2 \cdot 3^2$

$$\{2(3)^{n-1}\}$$

$\underbrace{\qquad \qquad}_{\infty} \quad \underbrace{n \rightarrow \infty}_{\infty}$

or a  
 $a_1 = 2$   
 $r = 3$  (common ratio)

Turns out,  $r=0$   
 $a \neq 0 \Rightarrow \text{then, } \infty$   
 conv for  $|r| > 1$

General form:  $\{a, r^{n-1}\}$  ( $a \neq 0$ )

Ex Principal = \$1000

4% ann. comp. interest  
(annually compounded)

$$\underbrace{\{1000(1.04)^n\}}_{\$ \text{ after } n \text{ yrs.}} \text{ or } \{1040(1.04)^{n-1}\}$$

(following the form  
strictly)

$$1040, 1081.6, \dots \rightarrow \infty$$

Manhattan Islands  
Indian -  
got restitution  
but not paid  
yet (by 2003?)  
Contract was  
illegal

Books never discuss

① Sign Alternators

$n =$	1	2	3	4
$(-1)^n$	-	+	-	+
$(-1)^{n+1}$	+	-	+	-

What if I want +  
How do I adjust?  
change parity  
 $n \pm 3, 5, \dots$

Ex 3, -6, 12, -24, ...

What's the first term?  
common ratio?

$$\{3(-2)^{n-1}\} \text{ or } \{(-1)^{n-1}3(2)^{n-1}\}$$

Terms  $\rightarrow 0$   
their  $|l|s \rightarrow 0$

$$\textcircled{E} \lim_{n \rightarrow \infty} a_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} |a_n| = 0$$

$\Rightarrow$  (clear:  $|l|$ , anyway)  
 $\Leftarrow$  Proven using Squeeze Thm.  
 $-|a_n| \leq a_n \leq |a_n|$

Ex  $\left\{\frac{(-1)^n}{n}\right\}$

$$-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots$$

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$\text{OR } \frac{-1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n} \quad (n \geq 1) \quad \begin{matrix} \downarrow & & \downarrow \\ 0 & & 0 \end{matrix} \quad \begin{matrix} \downarrow & & \downarrow \\ \text{as } n \rightarrow \infty & & \end{matrix}$$

$|l|s \rightarrow 0$ , so  
terms themselves  
 $\rightarrow 0$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$

$$\begin{array}{c} \cdot \frac{1}{n} \\ \cdot \cdot \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \cdot \\ \cdot \frac{1}{n} \end{array}$$

Circled:  $\frac{(-1)^n}{n}$ 

$\pm 1 \pm \frac{1}{2} \pm \frac{1}{3} \dots$   
Regardless of  
sign you pick.

In fact,  $1 \frac{1}{2} \frac{1}{3} \frac{1}{4} \dots \rightarrow 0$ 

if flip any signs

Can do HW  
(all)

$$\begin{array}{c} \cdot \cdot \cdot \cdot \cdot \\ \hline \cdot \cdot \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \cdot \end{array}$$

1, 1, 2, 3, 5, 8, 13...  
what is this

## (F) Recursively Defined Seqs.

### Ex (#50)

Dictionary of  
Math:  $a_1 = 2$ ,  
 $a_2 = 1$   
same recursion  
→ LUCAS #5  
 $f_n \rightarrow \sqrt{5}$   $r_n \rightarrow \tau$  (Conway)  
also 113  
People have trouble →

### Fibonacci seq.

$$\begin{cases} a_1 = 1 \\ a_2 = 1 \\ a_{k+1} = a_k + a_{k-1} \quad (k \geq 2) \end{cases}$$



Note

Fib. used to model rabbit progeny  
c. 1175-1250  
(Leonardo of Pisa)  
introduced Arabic # system  
to Europe!

In: almost all leaf arr., pineapples, cacti,  
sunflowers, pine cones, seashells

$$\text{Closed form for } a_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right]$$

New seq.  $\{r_n\} = \left\{ \frac{a_{k+1}}{a_k} \right\}$

(\* ratio of successive terms)

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \dots \rightarrow \tau \text{ ("tau")}$$

(assume exists)

GREAT  
BOOK:  
The  
Book of  
Numbers  
by Conway  
and Guy

Note (May see in Math 245: Discrete Math)

Find  $\tau$

really interested  
by this...

$\div a_n$

$$\frac{a_{n+1}}{a_n} = \frac{a_n}{a_{n-1}}$$

Use  
notation

$$r_n = 1 + \frac{1}{\frac{a_n}{a_{n-1}}} = r_{n-1}$$

Take  
limits

$$\lim_{n \rightarrow \infty} r_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{r_{n-1}}\right)$$

$r_n \rightarrow \tau$

$$\tau = 1 + \frac{1}{\tau}$$

$$\tau^2 = \tau + 1$$

$$\tau^2 - \tau - 1 = 0$$

Use QF: Quadratic formula

$$\tau = \frac{1 \pm \sqrt{5}}{2} \quad (\approx 1.618)$$

Note

Golden ratio

Show  $\tau - 1$  solves:

$$\begin{aligned} \frac{1+x}{1} &= \frac{1}{x} \\ x^2 + x - 1 &= 0 \\ x &= \frac{-1 \pm \sqrt{5}}{2} \\ &= \tau - 1 \end{aligned}$$

Conway 112

Knuth

Conway 184

Note / "  $r_n \rightarrow \tau$ " Disc. by Kepler

Ancient Gr. considered it most pleasant

ratio in art

diag side of reg. pentagon  $\square$

Cardr!



Discuss

(Important in proofs, higher math)

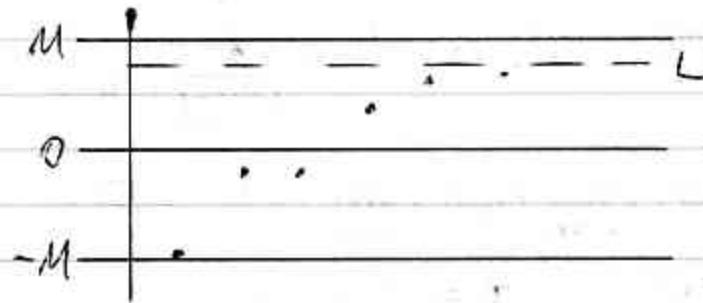
⑥ A bounded, monotonic seq. converges

$$|a_n| \leq M$$

real #  
for  $n=1, 2, 3, \dots$

$$\{a_n\} \text{ never } \nearrow (\nearrow) \quad \text{or never } \searrow (\searrow) \quad a_{n+1} \geq a_n$$

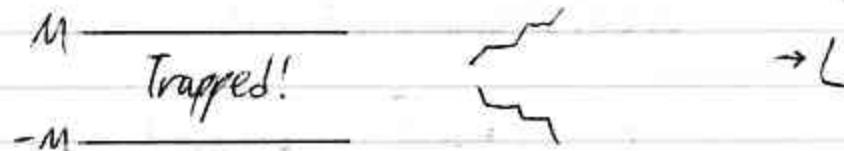
for  $n=1, 2, 3, \dots$



Bounded

Monotonic

Converges



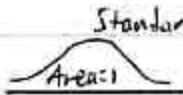
These  
don't  
have to  
be tight  
bounds

## 11.2: SERIES

What did you  
have a hard  
time believing?  
You had a  
hard time  
believing this!

### (A) Intro Ex

From 10.3



$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$$

### ① $\left\{ \left(\frac{1}{2}\right)^n \right\}$ , the original sequence $\{a_n\}$

$$\begin{aligned} a_1 &= \left(\frac{1}{2}\right)^1 = \frac{1}{2} & \boxed{\square} \\ a_2 &= \left(\frac{1}{2}\right)^2 = \frac{1}{4} & \boxed{\square} \\ a_3 &= \left(\frac{1}{2}\right)^3 = \frac{1}{8} & \boxed{\square} \\ &\vdots \end{aligned}$$

### ② $\{S_n\}$ , the sequence of partial sums (associated)

$$S_1 = a_1 = \frac{1}{2} \quad \boxed{\square}$$

$$S_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \quad \boxed{\square}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \quad \boxed{\square}$$

$S_k$  = the  $k^{\text{th}}$  partial  
(cumulative) sum

=  $a_1 + a_2 + \dots + a_k$   
 $= \sum_{n=1}^k a_n$   $\nearrow$  successively  
replace  
 $n$  with  $1, 2, \dots, k$   
and add the  
results

### ③ $S$ , the sum of the infinite series $\sum_{n=1}^{\infty} a_n$

too sloppy  
for Diff. Eq.!

(to 1st)

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots}_{S_1 = \frac{1}{2}}$$

$$\underbrace{S_2 = \frac{3}{4}}_{S_3 = \frac{7}{8}}$$

$$\Rightarrow S = 1 \quad \boxed{\square}$$

We approach  
filling in the  
entire box

$$\approx \frac{1}{10.3} = 1$$

Book says  $\sum a_n$  (a bit sloppy)

$$\sum_{n=1}^{\infty} a_n = \underbrace{a_1 + a_2 + a_3 + \dots}_{\text{has a num. (real #)}}$$

↳ series converges

↳  $\{S_n\}$  conv.

↳  $\lim_{n \rightarrow \infty} S_n = S$  (real #)

else series diverges (har no sum)

(Recap)

$$\begin{aligned} \{a_n\} &= \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \\ \{s_n\} &= \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots \rightarrow S = 1 \\ \sum_{n=1}^{\infty} a_n &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1 \end{aligned}$$

The series has sum 1.  
Is  $\sum a_n$  a sum?  
Well... does it have a sum?

## ⑤ Telescoping Series

Ex (#2)

$$\sum_{n=1}^{\infty} \frac{5}{(S_n+2)(S_n+7)}$$

a) Find  $S_k$  ( $k^{\text{th}}$  partial sum)

(PFD)

$$\frac{5}{(S_n+2)(S_n+7)} = \frac{A}{S_n+2} + \frac{B}{S_n+7}$$

What makes there = 2?  
We didn't see there in Ch. 9  
- could have.

$$5 = A(S_n+7) + B(S_n+2)$$

$$n = -\frac{2}{5}: \quad 5 = A(0) + B\left(-\frac{2}{5}+2\right)$$

$$5 = -5B$$

$$B = -1$$

$$n = -\frac{7}{5}: \quad 5 = A\left(-\frac{7}{5}+7\right) + B(0)$$

$$5 = 6A$$

$$A = 1$$

Stewart 7.14

for conv. series and  
for finite series,  
 $\sum (a_n - b_n)$

$$= \sum a_n - \sum b_n$$

Don't need abs. conv.  
If partial sums  
 $s_k = s_k + t_k$   
 $\downarrow$   
 $s$

Can't find nice  $s_k$  for  
many regrs.

$$\begin{aligned} S_k &= \sum_{n=1}^k \frac{5}{(S_n+2)(S_n+7)} = \sum_{n=1}^k \left( \frac{1}{S_n+2} - \frac{1}{S_n+7} \right) \\ &= \underbrace{\left( \frac{1}{2} - \frac{1}{7} \right)}_{n=1} + \underbrace{\left( \frac{1}{7} - \frac{1}{12} \right)}_{n=2} + \underbrace{\left( \frac{1}{12} - \frac{1}{17} \right)}_{n=3} + \dots + \underbrace{\left( \frac{1}{S_k+2} - \frac{1}{S_k+7} \right)}_{n=k} \end{aligned}$$

Collapsing telescope

$$= \left( \frac{1}{2} - \frac{1}{S_k+7} \right)$$

What do I do to  
find  $S$ , if  $\exists$ ?

### (b) Find $S$ (sum of the series)

1st see if  $\lim S_n$   
exists say " $S$ "

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left( \frac{1}{7} - \frac{1}{5n+7} \right) \xrightarrow{0} \frac{1}{7}$$

$S$  or  $S_\infty$

$$\Rightarrow S = \sum_{n=1}^{\infty} \frac{1}{(5n+2)(5n+7)}$$

$$= \boxed{\frac{1}{7}}$$

Up to  $S$

### (c) Geometric Series

Geom. seq.:  $\{ \underbrace{ar^{n-1}}_{a_n} \}, a \neq 0$

Don't say  
 $a_n \rightarrow 0$  it's like  
too generic  
When did this  
seq. conv. to 0?

$$(a, ar, ar^2, ar^3, \dots \rightarrow 0) \Leftrightarrow -1 < r < 1 \quad (|r| < 1)$$

$$\text{Ex } \frac{7}{1}, \frac{7}{2}, \frac{7}{4}, \dots \quad (a=7, r=\frac{1}{2})$$

Geom. Series Test for Conv. or Diver.

Geom. series:  $\sum_{n=1}^{\infty} a_n = a + ar + ar^2 + \dots + \underbrace{ar^{n-1}}_{n^{\text{th}} \text{ term}} + \dots$

conv.  $\Leftrightarrow -1 < r < 1 \quad (|r| < 1)$

Then, the sum,  $S = \frac{a}{1-r}$  (Pf What Words I-p.120)

(Note:  $S_k = \frac{a - ar^k}{1-r}$ , if  $r \neq 1$ . If  $|r| \geq 1 \Rightarrow (ark \rightarrow 0, S_k \rightarrow \infty)$ )

else diverges

$$\text{Ex } 7 + \frac{7}{2} + \frac{7}{4} + \dots + 7\left(\frac{1}{2}\right)^{n-1} + \dots$$

$$a = 7$$

$$r = \frac{1}{2} \Rightarrow \text{series conv. w/ sum}$$

$$S = \frac{a}{1-r} = \frac{7}{1-\frac{1}{2}} = \frac{7}{\frac{1}{2}} = \boxed{14}$$

Note: Except for  
0+0+... arithmetic  
series never have a  
sum. Ex 3+7+11+15+...  
diverges

Mead

What kind of a  
# is this?  
← Grade school

Ex  $0.\overline{1987} \Rightarrow$  nice fraction  
(rational)

$0.198719871987\dots$

Odd man out

$$\begin{aligned} &= 0.1 \quad \left\} \frac{1}{10} \\ &+ 0.0987 \quad \left\} \text{Geom series!} \\ &+ 0.000987 \quad \left\} a = 0.0987 \\ &\vdots \quad r = \frac{1}{1000}, \text{ or } 0.001 \\ &\Rightarrow S = \frac{a}{1-r} \quad \left\langle \begin{array}{l} = \frac{0.0987}{1-0.001} \leftarrow \text{by } 10,000 \\ = \frac{0.0987}{0.999} \leftarrow \div 10 \\ = \frac{987}{9990} \leftarrow \div 3 \\ = \frac{329}{3330} \end{array} \right. \\ &= \frac{329}{3330} \\ &\text{Don't forget!} \\ &= \frac{1}{10} + \frac{329}{3330} \\ &= \boxed{\frac{331}{1665}} \end{aligned}$$

Duh!  
Now, you can  
finish grade  
school....

Ex Find the sum of  $\sum_{n=1}^{\infty} 2^{n+3} 5^{-n}$

(Mostly  
"rewriting")

$$= \sum_{n=1}^{\infty} \frac{2^{n+3}}{5^n}$$

$$= \sum_{n=1}^{\infty} \frac{2^n \cdot 2^3}{5^n}$$

$$= \sum_{n=1}^{\infty} 8 \left(\frac{2}{5}\right)^n$$

$$\sum_{n=1}^{\infty} 8 \left(\frac{2}{5}\right)^n \quad \text{or} \quad \sum_{n=1}^{\infty} 8 \left(\frac{2}{5}\right) \left(\frac{2}{5}\right)^{n-1}$$

$$\begin{aligned} &= 8 \left(\frac{2}{5}\right) + 8 \left(\frac{2}{5}\right)^2 + 8 \left(\frac{2}{5}\right)^3 + \dots \\ &\quad \left| \begin{array}{l} a = \frac{16}{5} \\ r = \frac{2}{5} \end{array} \right. \\ &= \sum_{n=1}^{\infty} \underbrace{ar^{n-1}}_{\text{ar}^{n-1}} \quad (a = \frac{16}{5}, r = \frac{2}{5}) \quad \text{"fitting the form"} \end{aligned}$$

Married into  
form that looks  
like geom.

r pretty  $\frac{2}{5}$   
clearly  $\frac{2}{5}$   
r not the bare  
if  $r^{n-1}$ , though.

$|r| < 1 \Rightarrow$  sum exists

$$S = \frac{a}{1-r}$$

$$= \frac{16/5}{1 - 2/5}$$

$$= \frac{16/5}{3/5}$$

$$= \boxed{\frac{16}{3} \text{ or } 5.\bar{3}}$$

Up to 15

### ① $n^{th}$ -Term Test (for Div.)

For a geom. series,

$$\sum_{n=1}^{\infty} a_n \text{ conv} \Leftrightarrow \underbrace{(a_n \rightarrow 0 \text{ as } n \rightarrow \infty)}_{\substack{\text{"the terms"} \\ \text{"implied by the series"}}} \quad \text{i.e., } |r| < 1$$

BUT, in general,

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \not\Rightarrow (a_n \rightarrow 0) \quad \begin{matrix} \text{(does not} \\ \text{imply)} \end{matrix}$$

" $a_n \rightarrow 0$ " is a necessary but not sufficient condition for  $\sum_{n=1}^{\infty} a_n$  to converge.

(i.e., we need " $a_n \rightarrow 0$ ", but it's not enough!)

As a minimal requirement  
like eligibility  
to get 225  
but you  
also need  
800M

Carson 570

### Ex (Harmonic Series)

General Harmonic Series:  $\sum_{n=1}^{\infty} \frac{1}{n}$   
 In music, strings w/same diameter, tension, material whose lengths form a harmonic series produce harmonic tones.

Reciprocals of natural #s

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots \quad (\text{Sum of reciprocals of the natural #s})$$

$$a_n = \frac{1}{n} \rightarrow 0$$

Pf in Ex. 3  
 $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$   
 $1 + \left(\frac{1}{2}\right) + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{\frac{1}{2}} + \dots$   
 $\underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{\frac{1}{2}}$   
 $S_{2k} > S_{2k+1} \leftarrow \frac{1}{2}$   
 $\int_1^{\infty} \frac{1}{x} dx \text{ diverges (1st Test)}$   
 Pf. p. 538. converges  $\Rightarrow S$   
 $a_n = S_n - S_{n-1}$   
 $\lim_{n \rightarrow \infty} a_n = S - S = 0$

BUT more work needed to see if

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ conv. or } \text{div.}$$

turns out (Ex. 3 in your book)  
 (or 5 Test: II.3)

(Handout: Proof without Words - II 116)

$$\boxed{\sum_{n=1}^{\infty} a_n \text{ conv.} \Rightarrow (a_n \rightarrow 0)}$$

Contrapositive of " $A \Rightarrow B$ "

Daddy Man

equiv.

is "not  $B \Rightarrow \text{not } A$ "

not man not daddy

$$\boxed{(a_n \rightarrow 0) \Rightarrow \sum_{n=1}^{\infty} a_n \text{ div. (n-th-Term Test)}} \quad (\text{can't use } \varepsilon \text{ conv.})$$

Can't say  
 $\text{sum} = 0$

Ex  $\sum_{n=1}^{\infty} \underbrace{\sin(\pi n - \frac{\pi}{2})}_{a_n} = \sin \frac{\pi}{2} + \sin \frac{3\pi}{2} + \dots$

$$= 1 - 1 + 1 - 1 + \dots$$

$$\underbrace{s_1 = 1}_{s_2 = 0}$$

$$\underbrace{s_3 = 1}_{s_4 = 0}$$


$$\lim_{n \rightarrow \infty} s_n \text{ DNE} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ div.}$$

Shortcut  $\{a_n\}$ :  $1, -1, 1, -1, \dots \rightarrow 0$   
 $\Rightarrow \sum_{n=1}^{\infty} a_n \text{ div. (by n-th-Term Test)}$

## (E) Altering Series

Length of square  
 $\int_0^{\infty} \frac{1}{x} dx$  vs.  $\int_0^{\infty} \frac{1}{x^2} dx$

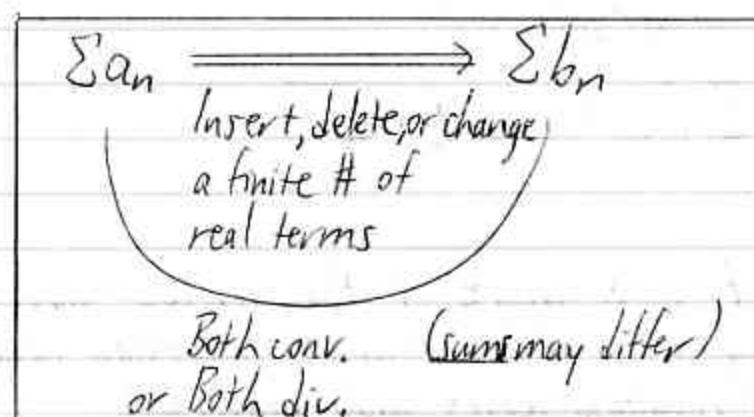
Thinking in a  
finite # of terms  
does not change  
the situation  
when we do.

No  $\frac{1}{n}$

Up to 39

Turns out:  $1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{n^2} + \dots$  conv.

$\Rightarrow 100 + 7 + 1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{n^2} + \dots$  conv.



## (F) Linearity Properties

Assume  $\sum_{n=1}^{\infty} a_n$ ,  $\sum_{n=1}^{\infty} b_n$ ,  $\sum_{n=1}^{\infty} c_n$  conv.  
 Sum = A      B      C

Then,  $\sum_{n=1}^{\infty} (7a_n + b_n - 9c_n)$  conv.  
 w/ sum  $7A + B - 9C$ .

$(2x+3)y$  is a linear  
combo of  $x, y$ .

If I mult each  
term in the  
 $a_n$  series by 7,  
the resulting  
 $E$  will conv.  
w/ sum 7A.

I wrote wouldn't  
pull out 7  
until he knew  
 $E(\frac{1}{n})^n$  conv.

Ex  $\sum_{n=1}^{\infty} 7\left(\frac{1}{n}\right)^n = 7 \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n}_{\text{conv.}} = 7(1) = \boxed{7}$

$$\begin{aligned} & 7\left(\frac{1}{2}\right) + 7\left(\frac{1}{4}\right) + \dots \\ & = 7 \underbrace{\left(\frac{1}{2} + \frac{1}{4} + \dots\right)}_{=1} \\ & = 7 \end{aligned}$$

If exactly one of the series is div, then

$$\sum_{n=1}^{\infty} (7a_n + 6b_n - 9c_n)$$

linear combo w/non-0 coeff. on "div guy"

If div guy has 0 coeff, we don't have to worry about him!

Ruins the party but 2 nuts can knock each other off.

Linear combo weights: 7, 1  
"ice" X-Files

Ex  $\sum_{n=1}^{\infty} \frac{1}{n}$  div.  $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$  conv.

$$\sum_{n=1}^{\infty} \left[ \frac{1}{n} + \left(\frac{1}{2}\right)^n \right] \quad \boxed{\text{div.}}$$

(On HW, if 2 or more of the series diverge, then they may interact in an interesting way, and the resulting series may converge.)

## 11.3/11.4: TESTS FOR CONV./DIV. OF POSITIVE-TERM SERIES

all terms  $a_n > 0$  "eventually" (i.e.,  $\forall n \geq \text{some } N$ )

(There's a point of no return after which the terms are always "+")

11.3

### ① S Test

why are we  
starting w/ 2  
instead of 1?  
↓

Ex (#8) Does  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$  conv. or div.?

$$\frac{1}{2(\ln 2)^2} + \frac{1}{3(\ln 3)^2} + \dots$$

Does this have a sum?

comes to smoothify  
interpolating func.  
A main comp. set  
of real #s.  
We'll formalize  
later (term  
deletion)

Since requires  
 $f(x) > 0$  for  
 $x \geq 2$  (proof  
assumes this),  
S test can be  
easily modified:  
 $f(x) > 0$  for  $x \geq M$   
Same for "+"

Don't have  
to explain why  
see L11-10F

We're going to  
find  $f'(x)$  and  
show it's what?  
now

Quot. or Recip.  
Rule  
 $\lim_{i \rightarrow 0} \frac{f_i}{g_i} = 0$

① Let  $f(x) = \frac{1}{x(\ln x)^2}$ .

② For  $x \geq 2$ , verify f is...

a) "+" valued [eventually]

$$\frac{1}{x(\ln x)^2} > 0 \quad \checkmark$$

b) cont.  $\checkmark$

c) decreasing ( $\searrow$ ) [eventually]

$$f'(x) = -\frac{D_x[x(\ln x)^2]}{[x(\ln x)^2]^2} \quad \left. \begin{array}{l} \text{Use Product Rule} \\ \text{Quotient Rule or} \\ \text{Reciprocal Rule (p.114)} \\ (\frac{1}{g})' = -\frac{g'}{g^2} \end{array} \right\}$$

$$\begin{aligned} &= -\frac{(1)(\ln x)^2 + (x)[2\ln x \cdot \frac{1}{x}]}{x^2(\ln x)^4} \quad \left. \begin{array}{l} \text{Maybe easier} \\ \text{to analyze} \end{array} \right\} \\ &= -\frac{\ln x + 2}{x^2(\ln x)^3} \quad (\ln x = 0 \Rightarrow \ln x > 0 \text{ for } x \geq 2) \\ &< 0 \quad \checkmark \end{aligned}$$

Rule:

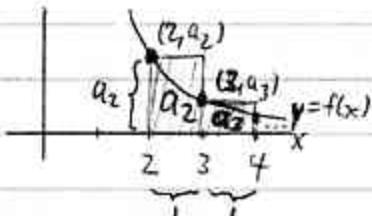
$$\textcircled{3} \quad \text{If } \int_2^{\infty} f(x) dx \text{ conv.} \Rightarrow \sum_{n=2}^{\infty} a_n \text{ conv.}$$

$\int \rightarrow \infty$ ?

div.  $\Rightarrow$  div.

Idea

Circumscribed  
rects,



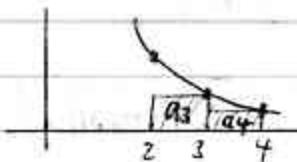
(like a Left-Hand Riemann Sum)

If little bro  
can't conv.  
then so many,  
neither can  
big bro.

$\int \text{div.} \Rightarrow \sum \text{div.}$  (Area of rect.  
have no sum.)

little bro div.  $\Rightarrow$  big bro div.

Inscribed  
rects.



(like a Right-Hand Riemann Sum)

$$\int \text{conv.} \Rightarrow \sum_{n=3}^{\infty} a_n \text{ conv.} \Rightarrow \sum_{n=2}^{\infty} a_n \text{ conv.}$$

big bro conv.  $\Rightarrow$  little bro conv.

throw in a\_2  
(doesn't change  
conv. vs. div.)

Does  $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx$  conv. or div.?

$$\int x \frac{1}{(\ln x)^2} dx = \dots = -\frac{1}{\ln x} + C$$

$$\left. \begin{array}{l} u = \ln x \\ du = \frac{1}{x} dx \end{array} \right\} \begin{array}{l} \text{If see } \ln n, \frac{1}{n} \Rightarrow \\ \int \text{test} \\ \text{best?} \end{array}$$

Are we going  
to have to  
split the  $\int$ ?  
Form still important!

$$\int_2^\infty \frac{1}{x(\ln x)^2} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x(\ln x)^2} dx$$

(cont. on  
(2,  $\infty$ )  
(we've done this,  
already.)

$$\stackrel{\text{FTC}}{=} \lim_{t \rightarrow \infty} \left[ -\frac{1}{\ln x} \right]_2^t$$

$$= \lim_{t \rightarrow \infty} \left( -\frac{1}{\ln t} - \left[ -\frac{1}{\ln 2} \right] \right)$$

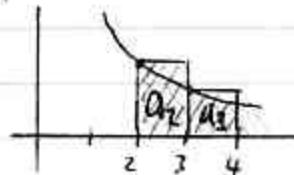
$\downarrow$   
 $\ln t \rightarrow \infty$

$$= \frac{1}{\ln 2} \quad \text{NOT } S!!$$

 $\approx 1.4427$  $\approx 2.1097$  $\Rightarrow \int$  conv.

$$\Rightarrow \sum_{n=2}^{\infty} a_n \text{ conv.}$$

(Actually an underestimate)



$$\sum_{n=2}^{\infty} a_n = \text{sum of areas of rectrs.}$$

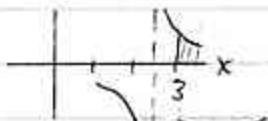
$$> \frac{1}{\ln 2}$$

$$\frac{1}{2} \Delta x$$

Ex 1 (p. 546) Harmonic series div.

$$\text{Ex } \sum_{n=1}^{\infty} \frac{1}{n-2.5} a_n$$

$$f(x) = \frac{1}{x-2.5} \quad (\text{There's a problem w/this interpolating func.})$$

 $f(+, \text{cont.}, \Delta x)$ on  $(3, \infty) \Rightarrow$  Can use  $\int$  Test thereCan show  $\sum_{n=3}^{\infty} a_n$  div.

(Better: BCT from 11.3 (c))

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ div.}$$

throw in  $a_1, a_2$

Up to 11  
Also  $\rightarrow$   
if  $f$  rtf  
Construct  $f$  that  
div.  
OK to have  
even "cont."  
eventually  
provided you  
restrict your  
analysis to  
 $\int_3^{\infty}$ . throw in  
other terms later.  
if  $f$  discontinuous at 2  
 $f(x) = \frac{1}{x-2}$ ,  $a_2$  undefined.

Up to 11

What's  $S_3$ ?

Ex Approx.  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  by  $S_3$ , perform error analysis

CONV.  
by  $S$  test  
(later:  $p$ -series test)

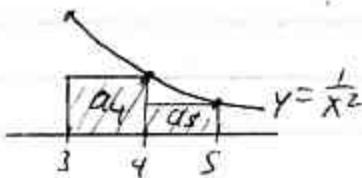
What are the first 3 terms?

$$S = 1 + \underbrace{\frac{1}{4} + \frac{1}{9}}_{S_3} + \underbrace{\frac{1}{16} + \frac{1}{25} + \dots}_{\text{error}}$$

$$\begin{array}{c} S_3 \\ \approx 1.3611 \end{array}$$

$$\begin{array}{c} \text{error} \\ = \sum_{n=4}^{\infty} \frac{1}{n^2} (> 0) \end{array}$$

Like Right-hand Riemann



Start  $S$  at ①

$\zeta$ , not  $S$ ;  
 $f$   $\vee$  func.  
Always gives

$$\begin{array}{c} \text{error} < \int_3^{\infty} \frac{1}{x^2} dx \\ \text{upper bound} \\ = \frac{1}{3} \end{array}$$

upper bound  
on error

Better (thinner)  
shorter) interval  
if  $S_{10}, S_{100}, \dots$

$S_k \uparrow$  but  $S_{k+1} \downarrow$   
eventually get  
4-dec place accuracy

Range game  
up to 11

Ahlfors 140,  
Ross 67  
Fourier E, Complex  
 $\sum_{n=1}^{\infty} \frac{1}{n^2}$  if even  
(not for odd)

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Euler

We really subst.  
from 1.6944

$$\Rightarrow 1.3611 < S < 1.3611 + \frac{1}{3}$$

$$\Rightarrow 1.3611 < S < 1.6944$$

$$\begin{array}{c} \text{lower bound} \\ \text{for } S \\ \text{upper bound} \\ \text{for } S \end{array} \quad \begin{array}{c} \text{Euler} \\ (\text{WOW!}) \\ \text{in} \end{array}$$

Note / What if we use  $S_4$  instead of  $S_3$ ?

$$S_3: 1.3611 < S < 1.6944$$

$$\begin{array}{c} \text{both} \\ + \frac{1}{a_4 + \frac{1}{16}} \end{array} \quad \begin{array}{c} \text{BUT} \\ - \frac{1}{a_4^{1/2}} \end{array} \quad \begin{array}{c} \text{since error bound} \\ \rightarrow S_3 \rightarrow S_4 \end{array}$$

$$S_4: 1.4236 < S < 1.6736$$

relative to 1.6944

$\left\{ \frac{1}{a_4^{1/2}} + \frac{1}{16} \right\}^{-1/2} = -\frac{1}{48}$

contained within  $S_3$ -based interval  $1.3611 < S < 1.6944$

## (8) p-Series Test

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad (p > 0)$$

conv.  $\Leftrightarrow p > 1$   
 div.  $\Leftrightarrow p \leq 1$

Proof by S Test

Not præsen  
by f-test  $\rightarrow$   
 $f(x) \nearrow$

Note  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(n^2)} \underset{\infty}{\searrow}$  div. by  $n^{\text{th}}$ -Term Test

Exs  $p = \frac{1}{2}$ :  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{3}} + \dots$  div

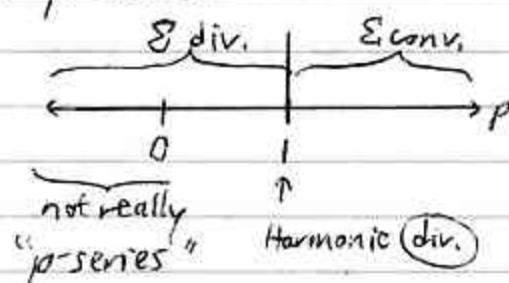
$\left( \begin{array}{l} \text{the harmonic} \\ \text{series is the} \\ \text{"smallest" } p\text{-S} \\ \text{that div.} \end{array} \right)$   $p = 1$ :  $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$  div

$p = 1 + \epsilon \Rightarrow \text{conv.}$   
( $\epsilon > 0$ )

$p = 1,1$ :  $\sum_{n=1}^{\infty} \frac{1}{n^{1,1}} = 1 + \frac{1}{2^{1,1}} + \frac{1}{3^{1,1}} + \dots$  conv.

$p = 2$ :  $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots$  conv.

Line of p-series:



## ⑥ Basic Comparison Tests (BCT)

Like when  
you were young  
makes noise.

$\sum b_n$  "big brother" dominates

$\sum l_n$  "little brother"

Assume for all  $n \geq$  some  $N$ ,  
(eventually, past some pt. of no return)

$$l_n \leq b_n$$

$$l_n, b_n > 0$$

Otherwise, in  
magnitude,  $b_n$   
dominant

a) If  $\sum b_n$  conv.  $\Rightarrow \sum l_n$  conv.  
(big bro fits  $\Rightarrow$  so does little bro)

b) If  $\sum l_n$  div.  $\Rightarrow \sum b_n$  div.  
(little bro too big  $\Rightarrow$  so is big bro)

Ex Does  $\sum_{n=3}^{\infty} \frac{4}{ln n + 7^n}$  conv. or div?

~~$\sum_{n=3}^{\infty} \frac{4}{ln n + 7^n}$  conv!~~

Try to compare this to a geom. series  
or a p-series (easy to analyze).

We know  $\sum_{n=3}^{\infty} \frac{4}{7^n} = \sum_{n=3}^{\infty} 4\left(\frac{1}{7}\right)^n$  conv.

Geom. series

$$\left|\frac{1}{7}\right| = \frac{1}{7} < 1$$

incl. error analysis  
Stewart 734

p-test

You could  
compare to  
a  $\sum$  where  
you need  
a  $\sqrt{n}$  test, but  
works!

What kind of  
 $\sum$  is  $\sum \frac{4}{7^n}$ ?

Not penitent

$\sum \frac{4}{ln n}$  div.

$$\frac{4}{ln n} > \frac{4}{n}$$

but  $\frac{4}{7^n} > \frac{4}{ln n + 7^n}$   
wrong way!

Everything's "+"

$$\frac{4}{(\ln n) + 7^n} \leq \frac{4}{7^n} \quad (\text{for } n \geq 3)$$

so  
so does little bro.  $\Sigma$

BCT  $\Rightarrow \sum_{n=3}^{\infty} \frac{4}{\ln n + 7^n}$  [conv.]

Big bro.  $\Sigma$  conv.  $\Rightarrow$

What if  
Up to 17

WARNING If big bro.  $\Sigma$  div.  
If little bro.  $\Sigma$  conv.  $\Rightarrow$  can't use BCT

Stewart  
732

Ex  $\sum_{n=1}^{\infty} \frac{\ln n}{n} a_n$

see  $\ln n, \frac{1}{n} \Rightarrow$   
 $\ln(\ln n) = 1$

Test OK:  $\int_3^{\infty} \frac{\ln x}{x} dx$

Hypotheses are satisfied for  $x \geq 3$ :  
 ①  $f(x) > 0$  for  $x > 1$  ( $\ln 1 = 0$ )  
 ②  $f$  cont. for  $x > 0$   
 ③  $f'(x) = \frac{1 - \ln x}{x^2}$  ( $\ln e = 1$ )  
 $\left. f'(x) \right|_{x=e} = 0$  for  $x > e \approx 2.7$   
 $\Rightarrow f \downarrow$  there

Turns out  $\int_3^{\infty} f(x) dx$  div.

(Test)  
 $\Rightarrow \sum_{n=3}^{\infty} a_n$  div.  
 $\Rightarrow \sum_{n=1}^{\infty} a_n$  [div.]

Up to 17

BCT quicker:

Everything's "+" for  $n > 1$ , or  $n \geq 2$  ( $\ln 1 = 0$ )

$$\frac{1}{n} \leq \frac{\ln n}{n} \quad \text{holds for } n \geq 3$$

$(\ln e = 1)$

$\Sigma$  can be sloppy  
ignore

little bro.  $\Sigma \frac{1}{n}$  div.  $\stackrel{\text{(BCT)}}{\Rightarrow}$  big bro.  $\Sigma \frac{\ln n}{n}$  div.

Review n! factorial (See Ch.10 Notes on Gamma Func.)

$$\begin{aligned}
 0! &= 1 && \leftarrow (\text{by definition; convenient for series, } \binom{n}{k} \text{ combinations,}\right. \\
 1! &= 1 && \left. \text{Binomial Thm, etc.}\right) \\
 2! &= (2)(1) = 2 && \text{Consistent w/ gamma func.} \\
 3! &= (3)(2)(1) = 6 && 2 \cdot 4 \\
 4! &= 4(3!) = 24 && 3 \cdot 5 \\
 5! &= 5 \cdot 24 = 120 && \\
 \vdots & && \\
 n! &= n(n-1)(n-2) \cdots (1), n \text{ is a natural #}
 \end{aligned}$$

Ex (#20)  $\sum_{n=1}^{\infty} \frac{1}{n!}$  (You'd use the Ratio Test : II. 4)

What's your intuition?  
Denom. blowing up

$\approx \frac{1}{5}$  know about induction.  
If you know mathematical induction, it's much more powerful!  
Otherwise, pick a better test!

We guess it conv.  
Compare to an easy conv.  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ,

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{1}{n!} &= 1! + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots
 \end{aligned}$$

$$= 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \dots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots$$

Can ignore 1st few terms

for  $n \geq 4$ ,

$$\frac{1}{n!} \leq \frac{1}{n^2}$$

Big bro,  $\sum \frac{1}{n^2}$  conv.

Little bro  $\sum \frac{1}{n!}$  conv.

Ask me if you can assume! For something like this, convince me this is true! Otherwise, your use of BC is suspect! Here, you can use mathematical induction. Possible to do HW test? what Basis step  $(n=4)$

$$\begin{aligned}
 \frac{1}{4!} &\leq \frac{1}{4^2} \\
 \frac{1}{24} &\leq \frac{1}{16} \checkmark
 \end{aligned}$$

Inductive step  
Let  $k \geq 4$  (k int.) Assume  $\frac{1}{k!} \leq \frac{1}{k^2}$

$$\begin{aligned}
 (i.e., k! &\geq k^2) \\
 \text{Show } \frac{1}{(k+1)!} &\leq \frac{1}{(k+1)^2} \\
 (i.e., (k+1)! &\geq (k+1)^2)
 \end{aligned}$$

$$\begin{aligned}
 (k+1)! &= (k+1) \cdot k! \\
 &\geq (k+1) \cdot k^2 \text{ by } \# \\
 &\geq (k+1)(k+1) \\
 &= (k+1)^2
 \end{aligned}$$

Rec.  $(k+1)^2 \geq (k+1)$  for  $k \geq 1$   
I believe this is true; can use.

Note  $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots = e$

We'll show in II.7

We'll see in II.7  
Up to 19  
True for  $k \geq 2$ ,  
but  $\frac{1}{2!} > \frac{1}{2^2}$

## ① Limit Comparison Test (LCT)

Why?

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \quad \text{Try BCT:}$$

$$\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} \quad (\text{Everything is } +, \text{ same num., smaller denom.})$$

↑ Big  $\sum \frac{1}{\sqrt{n}}$  div. ( $p$ -series,  $p = \frac{1}{2} \leq 1$ )

(BCT fails.)

↑ larger or smaller?

Frustrating!  
So close!

How can we easily compare  $\sum \frac{1}{\sqrt{n+1}}, \sum \frac{1}{\sqrt{n}}$ ?

### LCT Idea

Every term matters for sum!  
BCT wouldn't work, but...  
Num. of  $a_n$ :  
 $7 - (\frac{1}{10})^n$   
In the long run,  
what's happening?

$$\begin{aligned} & \text{compare } \sum a_n = \frac{6.9}{2} + \frac{6.99}{4} + \frac{6.999}{8} + \dots \text{ (conv.) w/sum?} \\ & \text{to } \sum b_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \text{ conv. w/sum } S^{(=1)} \\ & \text{Note: } \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \text{ conv. w/sum } 7S^{(=\infty)} \\ & a_n \text{ terms } \rightarrow 7 \cdot b_n \text{ terms} \\ & \frac{a_n}{b_n} \rightarrow 7 \end{aligned}$$

Me: Don't need  
 $a_n, b_n > 0$  ↪  
(another proof  
earlier)

or  $c > 0$ . Div. Rule Categories:  
 $Ean \approx \infty$  vs  $a_n$   
If  $\lim a_n = 0$ , then how  
more complicated  
Look at S1, S2  
 $\lim = 0$   
 $\Sigma D C \Rightarrow \Sigma C$

$\lim = \infty$   
 $\Sigma D D \Rightarrow \Sigma N D$   
(Pf Idea: S2)  
 $\exists M: 1 + k > M$ ,

$$\begin{aligned} \frac{a_n}{b_n} &> 1 \\ \Rightarrow a_n &> b_n \\ (\text{S1}): \frac{a_n}{b_n} &< 1 \end{aligned}$$

### LCT

If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$  non-0 real # (like 7, say) book assumes "+" term series, so it says "c > 0."  
implies non-0, real #.

$\Rightarrow \sum a_n, \sum b_n$  both conv. or both div.

Given  $\sum a_n$ , how pick  $\sum b_n$ ?

$$a_n = \text{_____}$$

Take dominant terms,  
Make constant factors = 1

$$\Rightarrow b_n$$

(old) Ex  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$

Why care if  
the limit is?  
or  $\frac{1}{7}$ ?  
well, tell me the  
lim, anyway,  
though.  
0, oo messes up,  
anyway.  
Order matters  
if you expand  
#5, #2

Let  $a_n = \frac{1}{\sqrt{n}+1}$   
 $b_n = \frac{1}{\sqrt{n}}$   $\leftarrow$  can switch

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n}+1}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n}+1} \leftarrow \frac{\sqrt{n}}{\sqrt{n}} \quad \begin{matrix} \leftarrow \sqrt{n} \\ \text{Type 5! } \sqrt{n+1} \end{matrix} \quad \begin{matrix} \text{How to Ace p. 40} \\ \text{Type 5! } \sqrt{n+1} \end{matrix}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{1 + \left(\frac{1}{\sqrt{n}}\right)}$$

$$= 1 \\ > 0, \text{ real} \\ (\text{an terms} \rightarrow b_n \text{ terms})$$

We know  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$  div. ( $p$ -series,  $p = \frac{1}{2} \leq 1$ )

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1} \boxed{\text{div.}}$$

Like 31 1137  
unassigned

Ex  $\sum_{n=1}^{\infty} \frac{3^n+1}{n+2^n}$

$$a_n \\ b_n = \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n$$

orange shirt  
green pants  
vt. green  
orange  
Change pants!  
(mult. comment.)

expl's kirk  
polyr' butt

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{3^n+1}{n+2^n}}{\frac{3^n}{2^n}} &= \lim_{n \rightarrow \infty} \frac{3^n+1}{n+2^n} \cdot \frac{2^n}{3^n} \\ &= \lim_{n \rightarrow \infty} \frac{3^n+1}{3^n} \cdot \frac{2^n}{n+2^n} \leftarrow \frac{3^n}{3^n} \leftarrow \frac{2^n}{2^n} \\ &= \lim_{n \rightarrow \infty} \left(1 + \left(\frac{1}{3^n}\right)\right) \left(\frac{1}{\frac{n}{3^n} + 1}\right) \\ &= 1 \\ &> 0, \text{ real} \end{aligned}$$

We know  $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$  div. (Geom. series,  $k = \frac{3}{2} > 1$ )  
 $\Rightarrow \sum_{n=1}^{\infty} \frac{3^n + 1}{n + 2^n}$  [div.]

Ex  $\sum_{n=1}^{\infty} \left( \frac{7}{\sqrt[3]{(4n^2+1)(n^3+5)}} \right)$

$a_n (> 0)$

$\sqrt[3]{4n^5} + \dots$   
 dominant,  
 ignore 4  
 $\sqrt[3]{n^5} = n^{5/3}$

$$b_n = \frac{1}{n^{5/3}} (> 0)$$

$$\lim_{n \rightarrow \infty} \frac{7}{\sqrt[3]{(4n^2+1)(n^3+5)}} = \lim_{n \rightarrow \infty} \frac{7n^{5/3}}{\sqrt[3]{(4n^2+1)(n^3+5)}}$$

$$= \lim_{n \rightarrow \infty} \frac{7(\sqrt[3]{n^5})}{\sqrt[3]{(4n^2+1)(n^3+5)}}$$

domin. term  
for BCT!

In 11.5, they  
call this →  
(lower order terms)  
Dominant

"Dominant term analysis":

$$= \lim_{n \rightarrow \infty} 7 \left( \sqrt[3]{\frac{n^5}{4n^5 + \dots}} \right)$$

lower-order terms  
rational,  
same degree

$$\frac{n^5}{4n^5 + \dots}$$

ratio of leading  
coeffs =  $\frac{1}{4}$

More precisely,

$$= \lim_{n \rightarrow \infty} 7 \left( \sqrt[3]{\frac{n^5}{(4n^2+1)(n^3+5)}} \right) \quad \begin{matrix} \leftarrow n^5 \\ \leftarrow n^5 \end{matrix}$$

$$= \lim_{n \rightarrow \infty} 7 \left( \sqrt[3]{\frac{1}{\left(\frac{4n^2+1}{n^2}\right)\left(\frac{n^3+5}{n^3}\right)}} \right)$$

$$= \lim_{n \rightarrow \infty} 7 \left( \sqrt[3]{\frac{1}{\left(4\left(\frac{1}{n^2}\right)\right)\left(1+\left(\frac{5}{n^3}\right)\right)}} \right)$$

Painfully  
rotating

$$= 7 \left( \sqrt[3]{\frac{1}{4}} \right)$$

> 0, real

We know  $\sum_{n=1}^{\infty} n^{\frac{1}{p-1}}$  conv. ( $p$ -series,  $p = \frac{5}{3} > 1$ )  
 $\Rightarrow \sum_{n=1}^{\infty} a_n$  [conv.]

11.4: RATIO, ROOT TESTS

Not comparison tests

(Deals w/ratios of successive terms)

LCT ( $a_1 + a_2 + a_3 + \dots$ )  
 (Deals w/ratios of corresponding terms)  
 $b_1 + b_2 + b_3 + \dots$

Geom. Series (Core of proofs here)

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \underbrace{\left(\frac{1}{2}\right)^n}_{a_n} + \dots$$

What's & are  
 $\frac{1}{2}, \dots$   
 actually, you'd  
 just use  
 geom. series  
 test

$$\textcircled{1} \quad \frac{a_{n+1}}{a_n} = r = \frac{1}{2}$$

$| \frac{1}{2} | < 1$ , so  $\sum$  conv.  
 by Ratio Test

$$\textcircled{2} \quad \sqrt[n]{a_n} = \sqrt[n]{\left(\frac{1}{2}\right)^n} = \frac{1}{2}$$

$| \frac{1}{2} | < 1$ , so  $\sum$  conv.  
 by Root Test

Generalize!Assume all  $a_n \neq 0$  (eventually)

① Proof  
 If  $L \leq 1$   
 Let  $r \in (0, L)$   
 $\forall n \geq N$

$\frac{a_{n+1}}{a_n} < r$   
 $a_{N+1} + a_{N+2} + \dots$   
 $< a_N r + a_N r^2 + \dots$   
 conv.

If  $L > 1$  then  $a_n$   
 $\frac{a_{n+1}}{a_n} > r > 1$   
 $a_{n+1} > a_n$   
 $\text{nth term test}$

②  $a_n \neq 0$  (cc1)  
 Already excluded  
 by first "if"  
 Like  $n^{th}$  term  
 Test when  
 $a_n \rightarrow 0$

① Ratio Test

If  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$  or  $\infty$   
 Don't need  
 if all  $a_n > 0$

② Root Test

If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$  or  $\infty$

If  $L < 1 \Rightarrow \sum a_n$  conv.  
 $L > 1$  or  $\infty \Rightarrow$  div.  
 $L = 1 \Rightarrow$  ② Need more  
 ("or DNE") info!  
 Inconclusive!

} same rules, though  
 $L$  for Root Test may  
 differ from  $L$  for  
 Ratio Test

} Note  
 ①

Maybe  $L = .8$  for ①  
 $L = .9$  for ②  
 Never have  $i$  test  $\rightarrow$  conv.  
 Other  $\rightarrow$  div.

Notes ①  $L$  for ①, ② may differ.

⑥ If  $L = 1$ , the tests "fail" (are inconclusive).

For ① and ②,  $L = 1$  for  $\sum \frac{1}{n}$ ,  $\sum \frac{1}{n^2}$

$$= \sqrt{n}$$

(div.) (conv.)

$L = 1$  if  $\sum$  algebraic (rational w/ maybe roots),  
 incl. p-series

\* ⑦ If ② fails  $\Rightarrow$  ① fails. If ① fails  $\Rightarrow$  ② often fails. (They tend to fail together)

⑧ If  $a_n$  has  $n!, 2^n$ , products, ..., try ① Ratio.  
 $(\underbrace{\quad}_{\text{including}})^n, 2^n, \dots$ , try ② Root.

⑨ Proofs use BCT to compare  $\sum a_n$  w/ a geom. series. (for conv.,) and  $n^{\text{th}}$ -term Test (for div.)

$$\begin{aligned} & 1+2+4+8+\dots \\ & \text{Ratio: } L=2 \end{aligned}$$

If you're curious!

\* Note on Note (Koss, Elementary Analysis, p. 14)

Root works, Ratio fails for:  $\sum 2^{(-1)^{n+1}-n}$

$$\text{Conv. by BCT: } 2^{(-1)^{n+1}-n} \leq 2^{1-n}$$

$$\begin{cases} 2^{1-n}, n \text{ even} \\ 2^{-1-n}, n \text{ odd} \end{cases}$$

$\sum$  conv.  
 (big base)

$$\sum_{n=1}^{\infty} 2^{1-n}$$

Root works: even:  $\sqrt[n]{2^{1-n}} = \sqrt[n]{\frac{2}{2^n}} = \frac{\sqrt[n]{2}}{2} \rightarrow \frac{1}{2}$  (sweep away)  
 n odd:  $\sqrt[n]{2^{-1-n}} = \sqrt[n]{\frac{1}{2 \cdot 2^n}} = \frac{\sqrt[n]{1}}{\sqrt[n]{2} \cdot 2} \rightarrow \frac{1}{2}$  ('odd n')

$$\Rightarrow L = \frac{1}{2}$$

Ratio fails: n even:  $\frac{a_{n+1}}{a_n} = \frac{2^{-1-(n+1)}}{2^{1-n}} = \frac{2^{-2-n}}{2^{1-n}} = \frac{2^{-2}}{2^1} \cdot \frac{2^{-n}}{2^n}$

$$\begin{aligned} n \text{ odd: } \frac{a_{n+1}}{a_n} &= \frac{2^{1-(n+1)}}{2^{1-n}} = \frac{2^{-n}}{2^{1-n}} = \frac{1}{2^1} \cdot \frac{2^{-n}}{2^n} \\ &= 2 \end{aligned}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \text{ DNE (inconclusive)}$$

$$\text{Ex (#10)} \sum_{n=1}^{\infty} \frac{n!}{(n+1)^5}$$

$n!$  → Use Ratio Test?

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)!}{\frac{n!}{(n+1)^5}} \\ \text{all } a_n > 0 \\ = \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+2)^5} \cdot \frac{(n+1)^5}{n!}$$

Orange shirt  
Green pants

$$= \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} \cdot \frac{(n+1)^5}{(n+2)^5} \quad (\text{Exchanging pants})$$

$$\text{Ex } \frac{2'}{2!} = \frac{1 \cdot 2 \cdot 1}{2 \cdot 1} = 3$$

$$\text{Ex } \frac{(n+1)!}{n!} = \frac{(n+1) \cdot n!}{n!} \\ = (n+1)$$

why?

$$= \lim_{n \rightarrow \infty} (n+1) \cdot \frac{(n+1)^5}{(n+2)^5}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^6}{(n+2)^5} \quad \leftarrow \text{polys in } n$$

$$= \lim_{n \rightarrow \infty} \frac{n^6 + \dots}{n^5 + \dots} \quad \leftarrow \begin{array}{l} \text{Only consider} \\ \text{dominant terms} \end{array} \\ (\text{could use L'H})$$

$$= \infty$$

$$\Rightarrow \sum \boxed{\text{div.}}$$

LCT: 00  
unless  
(except for  
(1, 3, 5, 2))  
Up to 9

$$\text{Ex } \sum_{n=1}^{\infty} \frac{n^4}{2^{1+3n}}$$

Ratio Test works.  
Root Test:

$$\lim_{n \rightarrow \infty} \sqrt[n]{\text{Term}} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^4}{2^{1+3n}}} \\ = \lim_{n \rightarrow \infty} \left( \frac{n^4}{2^{1+3n}} \right)^{1/n}$$

We know  
laws of exponents  
better than  
laws of radicals.

Mult. exponents by  $\frac{1}{n}$

$$= \lim_{n \rightarrow \infty} \frac{n^{4/n}}{2^{\frac{1+3n}{n}}}$$

$$= \lim_{n \rightarrow \infty} \frac{(n^{1/n})^4}{2^{\frac{1+3n}{n}}}$$

Can use  $n^{\frac{1}{n}} \rightarrow 1$  on HW tests.  
 $(\sqrt[n]{n})^{\frac{ln n}{n}}$  trick

$$(\sqrt[2]{2}, \sqrt[3]{3}, \sqrt[4]{4}, \sqrt[5]{5}, \dots \rightarrow 1)$$

$$= \frac{1}{2^3}$$

$$= \frac{1}{8}$$

Ratio Test  
may have diff. L.

$$L = \frac{1}{8} < 1 \Rightarrow \sum \boxed{\text{conv.}}$$

## 11.5: WHEN $\sum a_n$ CAN BE < 0

11.2:  $n^{\text{th}}$ -Term Test  
Now, for conv.  
11.5: Tests for conv.

### (A) Alternating Series Test (AST) for Conv.

"eventually" too  
riskier...  
need  $a_n > 0$   
 $|a_n| \rightarrow 0$  for my  
Exs.

Assume all  $a_n > 0$  (delete finite # of terms as necessary)

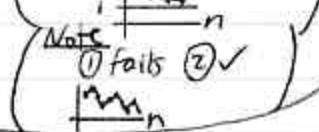
$$\text{Alt. } \sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 \dots \quad a_n = |a_n|$$

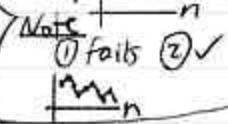
$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 \dots \quad (\leftarrow \text{I tend to use this in my Exs.})$$

(①) Have alt.  $\Sigma$  like above.)

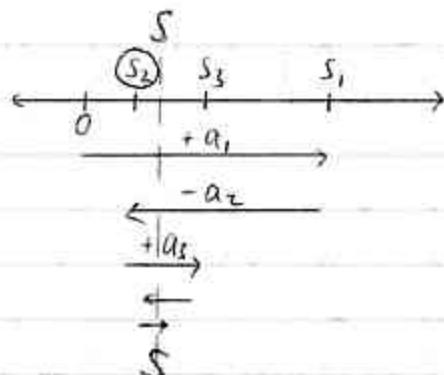
If  $\begin{cases} \text{① } a_n \searrow \text{(nonincreasing)} \\ \text{② } a_n \rightarrow 0 \end{cases}$

$\Rightarrow$  each  $\Sigma$  conv.

Note: ①  $\checkmark$ , ② fails:  


Note: ① fails ②  $\checkmark$   


Idea



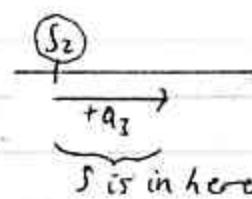
What do we do  
for  $-a_2$ ?

Finger wagging

Alt.  $\Sigma$ : Arrows keep reversing direction.  
 $a_n \searrow$ : Arrows never grow.  
 $a_n \rightarrow 0$ : Arrow lengths  $\rightarrow 0$ .

Ex Approx.  $S$  by  $S_2$   $\stackrel{\text{sum of whole } \Sigma}{\approx} a_1 - a_2$

$S$  is within  $a_3$  of  $S_2$ .  
i.e.,  $|S - S_2| \leq a_3$   
distance bet.  $S, S_2$



Note: Need  $\leq a_3$   
 $a_4, a_5, \dots$  assumed  
non-0, BUT  
could have:

In general Approx.  $S$  by  $S_N$

$$|\text{error}| \leq |\text{1st neglected term}|$$

$$|S - S_N| \leq a_{N+1}$$

(100%)  
Add 1st  $N$  terms.  
Peek at next  
term to see  
how far off you  
can be from  $S$ .

Harmonic div.  
This pappy conv.

### Ex (Alternating Harmonic $\Sigma$ )

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n}\right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots$$

If there's  
any work,  
it's here

Verify conditions of AST

① alt.  $\Sigma$  ✓

②  $\frac{1}{n}$  ... ↘

③  $a_{n+1} \leq a_n$  ?  
 $\frac{1}{n+1} \leq \frac{1}{n}$  ( $n \geq 1$ ) ✓

Hint at Ratio Test

✓ what?

or ④  $\frac{a_{n+1}}{a_n} \leq 1$  ?

$$\frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \leq 1 \quad (n \geq 1) \quad \checkmark$$

or ⑤  $f(x) = \frac{1}{x}$   
 $f'(x) = -\frac{1}{x^2} < 0 \quad (x \geq 1)$  ✓

For  $\frac{1}{n}$ , you  
don't have  
to show,  
but I'm  
showing  
you techniques  
for verifying  
harder cases.

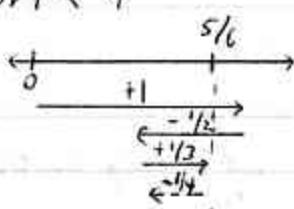
⑥  $\frac{1}{n} \rightarrow 0$  ✓ (I believe you! Show as appropriate.)  
 $\lim_{n \rightarrow \infty} a_n = 0$

∴  $\Sigma$  [conv.]  
(Therefore)

Approx.  $S$  by  $S_3$  (say)

$$S_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

|error| <  $\frac{1}{4}$



We last added,  
so upper bound  
on  $S$ .

$$\underbrace{\frac{5}{6}}_{1/12} - \frac{1}{4} \leq S \leq \underbrace{\frac{5}{6}}_{1/12}$$

$$0.5833 \leq S \leq 0.8333$$

In HW  
sheet

Since here

Turns out:  $S = \ln 2 \approx 0.6931$  (Wow!! We'll see)  
(To approx. to "4 dec. places," ensure |error| < 0.00005.)

(B) CC vs. AC  $\Sigma$  $\Sigma b_n$  is conditionally convergent (CC)  $\Leftrightarrow$  $|b_n| = a_n$  in  
aH.  $\Sigma (-1)^{n-1} a_n$  $\Sigma b_n$  conv., BUT  $\sum |b_n|$  div.corresp. absolute value series, all "+" terms  
(maybe 0)Ex (Alt. Harmonic  $\Sigma$ )

$$\sum_{n=1}^{\infty} \underbrace{(-1)^{n-1} \frac{1}{n}}_{b_n} = 1 - \frac{1}{2} + \frac{1}{3} - \dots \text{ is CC, because}$$

$\Sigma b_n$  conv. by AST (we saw this in (A))  
 BUT  $\sum |b_n| = \sum \frac{1}{n}$  div.

(Note In 11.7, we'll see that  
 $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \ln 2 \leftarrow \text{sum}$ )

$\Sigma b_n$  is absolutely convergent (AC)  $\Leftrightarrow$   
 $\Sigma b_n, \Sigma |b_n|$  both conv.

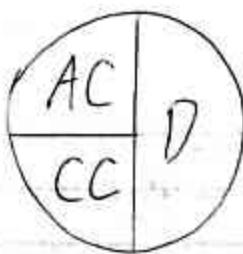
 $\Sigma \frac{1}{n}$  div, but what conv.?

$$\text{Ex } \sum_{n=1}^{\infty} \underbrace{(-1)^{n-1} \frac{1}{n^2}}_{b_n} = 1 - \frac{1}{4} + \frac{1}{9} - \dots \text{ is AC, because}$$

$\Sigma b_n$  conv. by AST  
 (alt.  $\Sigma a_n = \frac{1}{n^2} \rightarrow$  and  $\rightarrow 0$ )  
 AND  $\sum |b_n| = \sum \frac{1}{n^2}$  conv. ( $p$ -series,  $p=2 > 1$ )

In fact,  
 $\Sigma b_n$  conv.  
 Implies by  
 $\Sigma |b_n|$  conv.  $\circledcirc$

(Note Mathematica says  
 $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2} = \frac{\pi^2}{12}$   
 $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^4} = \frac{7\pi^4}{720}$ )

$\Sigma$ 

"+" from  $\Sigma$   
can't be what?

"+" term  $\Sigma$  can't be CC  
 $|b_n| = b_n$

Larson 8.5.53

Note

Alt. p-series

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$  conv. if  $p > 0$  (by AST)

In particular,  $\begin{cases} AC, & \text{if } p > 1 \\ CC, & \text{if } 0 < p \leq 1 \end{cases}$

Alt. Geom. Series

are geom.  $\Sigma$ , anyway! We saw in 11.2.

$$\underline{\text{Ex}} \quad \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} (-\frac{1}{2})^n$$

Geom.  $\Sigma$  is  $\begin{cases} AC, & \text{if } |r| < 1 \\ D, & \text{otherwise} \\ (\text{never CC}) \end{cases}$

### ③ Absolute Convergence Test (ACT) (for conv. only)

If  $\sum |b_n|$  conv.  $\Rightarrow \sum b_n$  conv. (in fact, AC)

Ex "+" term  $\sum |b_n| = \underbrace{1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots}_{\text{flip any signs (any sign configuration)}} \text{ conv. } (\sum_{n=1}^{\infty} \frac{1}{n^2}, n=2 \geq 1)$   
 $\Rightarrow$  resulting  $\sum b_n$  conv. by ACT

for instance,  $1 + \frac{1}{4} - \frac{1}{9} - \frac{1}{16} + \dots$   
 also conv. In fact, AC.

Note An "alt."  $\sum$  can't go  $++--++--\dots$   
 Can't use AST.

Note This is  $\sum_{n=1}^{\infty} (-1)^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{n^2}$ , where  
 $\lfloor \cdot \rfloor$  is the floor or greatest integer operator/function.  
 $\lfloor 4.2 \rfloor = 4$ .

ACT allows us to use the 11.2-11.4 tests for  
 "+" term  $\sum$  to possibly verify that a  
 mixed-sign  $\sum$  conv. and, in particular, AC.

I don't know  
 sum, Mathematica  
 failed me! Even  
 for approx.

Some tests were  
 OK for mixed-sign  
 $\sum$ , already:  
 (Telescoping/S)  
 Geom.  
 $n^{th}$ -Term  
 modified LCT  
 11.5 versions of  
 Ratio, Root

ACT

In high school

High school

has him

and all other

If you're  
E where  
you just  
curious!  
flip signs.

Also most likely  
to diverge.

From queen: all "+"

Dennis Miller E.

Hypnotic sarcastic,  
you may say -  
duh of course  
such &, but  
how do we know  
we have a sum?  
Why not  
oscillatory  
behavior for  $\Sigma b_n$   
like  $\sin n x$ ?

Idea

$\sum |b_n|$  is "most likely to div." (all "+")  
If conv.  $\Rightarrow$  any  $\sum$  where you just flip signs also conv.  
(AC)

ACT

Proof Idea

Assume  $\sum |b_n|$  conv.

$$b_n + |b_n| = 0 \text{ or } 2|b_n|$$

1	1	2
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
$-\frac{1}{9}$	$\frac{1}{9}$	0
$-\frac{1}{16}$	$\frac{1}{16}$	0
:	:	:

$$\begin{matrix} \sum_{\text{conv.}} \\ \text{Sum} = T \end{matrix} \Rightarrow \begin{matrix} \sum_{\text{conv.}} \\ \text{Sum} = U \end{matrix}$$

So,  $\sum_{\text{conv.}} \text{Sum} = U - T$

Idea  $\sum$  conv.  $\Leftrightarrow$  has a sum

Sum of  $\sum$  (call "+ terms")  
 $\textcircled{1}$  = Sum of  $\sum |b_n|$

Flip "+ terms" to  
"- terms".  
It's a nontrivial  
observation  
that  $S_k \rightarrow$  same #  
for each of these  
new  $\sum$ s. (Partial  
sums don't oscillate  
wildly forever.)  
 $\textcircled{2}$   
-  
 $\textcircled{-1}$  (all "- terms")  
"Sum of  $\sum (-b_n)$ "  
 $= -\sum |b_n|$

## D Ratio, Root Tests

$\left| \frac{a_{n+1}}{a_n} \right|$  for  
Ratio w/  
alt.  $\sum$

If  $\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = L$  or  $\infty$  ... see II.4

{ if  $L < 1 \Rightarrow \sum b_n$  conv. (AC)  
if  $L > 1$  or  $\infty \Rightarrow \sum b_n$  div.  
if  $L = 1 \Rightarrow$  (?)  
(or DNE)

Like in II.4, but use "||".  
For either, if  $L < 1 \Rightarrow \sum$  conv. (in fact, AC)  
(Ratio/Root)

SURPRISE! We can use Ratio, Root tests to verify that  
a mixed-sign  $\sum$  diverges !! AST, ACT couldn't do that.

(Note A CC  $\sum$  will yield  $L=1$  for both tests. (or DNE))  
BUT  $L=1$  for a test  $\Rightarrow$  (?) Could be AC, CC, or D!!  
inconclusive

REASON

## ⑥ Exs

Ex (#2) Does  $\sum_{n=1}^{\infty} (-1)^{n-1} n S^{-n}$  conv. or div? (CD)

$$\begin{aligned}
 &= \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{n}{S^n} \right) \\
 &\quad \text{an>0 (in our Alt.-S form)} \\
 &= \frac{1}{S} - \frac{2}{2S} + \frac{3}{12S} \dots \quad (\leftarrow \text{Not required})
 \end{aligned}$$

## Method 1 (AST) @ alt. S ✓

On HW, they're  
told to verify.  
1, 2

Verify ①  $a_n \rightarrow 0$ 

$$\frac{a_{n+1}}{a_n} \leq 1 \text{ (for } n \geq 1\text{?)} \quad \text{or if } f(x) = S^x$$

$$\Rightarrow f'(x) < 0 \text{ (for } x \geq 1\text{?)}$$

$$\frac{\frac{n+1}{S^{n+1}}}{\frac{n}{S^n}} = \frac{n+1}{S^{n+1}} \cdot \frac{S^n}{n}$$

$$f'(x) = \frac{S^x(1) - x(S^x \ln S)}{(S^x)^2}$$

$$= \frac{n+1}{n} \cdot \frac{S^n}{S^{n+1}}$$

$$= \frac{S^x - S^x x \ln S}{S^{2x}}$$

$$= \underbrace{\left(1 + \frac{1}{n}\right)}_{\leq 2} \underbrace{\left(\frac{1}{S}\right)}_{\leq 1 \text{ for } n \geq 1}$$

$$= \frac{\cancel{S^x}(1 - x \ln S)}{\cancel{S^{2x}}} \quad \cancel{x > 0}$$

$$\leq 1 \quad (n \geq 1) \quad \checkmark$$

$$\leq 0 \quad (x \geq 1)$$

②  $a_n \rightarrow 0$ 

exp/dr  
setup  
poly's

$$\lim_{n \rightarrow \infty} \frac{n}{S^n} \left( = \lim_{x \rightarrow \infty} \frac{x}{S^x} \right) \quad \begin{array}{l} \text{cont. interpolating func. } f(x) \\ \text{(unless DNE; } \infty, -\infty \text{ OK) } (\infty) \end{array}$$

$\lim_{x \rightarrow \infty} \frac{x}{S^x} = \lim_{x \rightarrow \infty} \frac{1}{S^{x-1}} = 0$

Can do 1, 3

$\therefore S$  by conv. by AST

## Method 2 (Ratio Test)

$$\lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^n \frac{n+1}{5^{n+1}}}{(-1)^{n-1} \frac{n}{5^n}} \right| \quad \begin{cases} (1 \text{ kills sign alternators}) \\ \text{if everything else "+."} \end{cases}$$

$$= \lim_{n \rightarrow \infty} \frac{\frac{n+1}{5^{n+1}}}{\frac{n}{5^n}} = \lim_{n \rightarrow \infty} \frac{n+1}{5} \cdot \frac{5^n}{n} = \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \frac{5^n}{5} \quad \begin{matrix} (\text{exchange} \\ \text{parts}) \end{matrix}$$

$$= \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right) \cdot \frac{1}{5} \\ = \frac{1}{5} \\ < 1$$

$\xrightarrow{\text{Ratio}}$   $\sum$  [conv.] (in fact, AC)

## Method 3 (Root Test)

$$\lim_{n \rightarrow \infty} \sqrt[n]{|b_n|} = \lim_{n \rightarrow \infty} \sqrt[n]{|(-1)^{n-1} \frac{n}{5^n}|} \\ = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n}}{5} \\ = \frac{1}{5} \\ < 1$$

$L$  for Root  
doesn't  
have to  
be  $L$  for Ratio

$\xrightarrow{\text{Root}}$   $\sum$  [conv.] (in fact, AC)

$$\text{Ex } \sum_{n=1}^{\infty} (4)(\frac{n+1}{n})^n = \sum_{n=1}^{\infty} \underbrace{(1 + \frac{1}{n})^n}_{e \neq 0} \quad \boxed{\text{div}} \text{ by } n^{\text{th}}\text{-Term Test}$$

$\xrightarrow{\text{DNE}}$   
No way  $\rightarrow 0$

Should be aware  
in any event

Ex (#8) Is  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+4}$  AC, CC, or div? AC | D  
CC

(We think it  
conv.  
alt,  $a_n \downarrow, \rightarrow 0$ )  
If div, AST  
can't verify  
div.  
(If AC, just test  
div. if you  
can.)

AST to verify conv.  
① Alt.  $\sum$  ✓  
Verify ①  $a_n \dots$

(Don't need if can show  $\sum$  AC  
using Ratio, Root, or otherwise showing  
Euler conv.)

$$\begin{aligned} f(x) &= \frac{x}{x^2+4} \\ f'(x) &= \frac{(x^2+4)(1) - (x)(2x)}{(x^2+4)^2} \\ &= \frac{-x^2+4}{(x^2+4)^2} \quad \leftarrow \text{if } x \geq 3 \\ &= \frac{4-x^2}{(x^2+4)^2} \quad \leftarrow + \end{aligned}$$

$$\langle 0 \quad (x \geq 3) \checkmark$$

$$\textcircled{2} \quad a_n = \frac{n}{n^2+4} \rightarrow 0 \checkmark$$

$\therefore \sum_{n=1}^{\infty} b_n$  conv. by AST

$\Rightarrow \sum_{n=1}^{\infty} b_n$  conv.

AC | X  
CC

Is it AC or CC?

$$\sum_{n=1}^{\infty} |b_n| (\equiv \sum_{n=1}^{\infty} a_n)$$

$$= \sum_{n=1}^{\infty} \left( \frac{n}{n^2+4} \right)$$

$$\text{Compare w/ } \frac{n}{n^2} = \frac{1}{n}$$

BCT fails:  $\frac{n}{n^2+4} \leq \frac{n}{n^2}$  or  $\frac{1}{n}$   
div. ? ?

$c_n$ ?

somewhat

$$\begin{aligned} \text{LCT} \quad \lim_{n \rightarrow \infty} \frac{\frac{n}{n^2+4}}{\frac{1}{n}} &= \lim_{n \rightarrow \infty} \frac{n^2}{n^2+4} \\ &= 1 \quad \begin{matrix} \uparrow \text{polys w/same degree} \\ \Rightarrow \text{take ratio of leading coeffs.} \end{matrix} \end{aligned}$$

carson 584  
 $\sum \frac{1}{n^2}$

$\therefore \sum_{n=1}^{\infty} b_n$  CC

We know  $\sum_{n=1}^{\infty} \frac{1}{n}$  div.  $\Rightarrow \sum_{n=1}^{\infty} \frac{n}{n^2+4}$  div.

Mead

## F) Rearrangements (AC vs. CC)

Larson  
Stewart 7.22  
Guido Ubaldus

Do we get a medal?

$$\text{Ex } 0 = 0 + 0 + 0 + \dots$$

$$= (-1) + (-1) + (-1) + \dots$$

$$\neq 1 + (-1+1) + (-1+1) + \dots$$

$$= 1$$

NO!

Regrouping can change sum.  
Assoc. Comm. laws  
of " $a+b$ " break down.

Guido Ubaldus thought this proved the existence of God: "something has been created out of nothing."

① If  $\sum b_n$  is AC with sum = S,

then any regrouping or reordering of terms

$\Rightarrow$  conv.  $\sum$  with sum = S. (When you add the order of the terms doesn't matter.)

If you consider all poss. reorderings, you can get all real sums.

Ref. Larson 5.86  
Gelbaum 5.4

Ex: may have to swap terms

② If  $\sum b_n$  is CC, and if r is any real #,

then there is a reordering  $\sum_{\text{new}}$  has sum = r. ("matter!)"

Stewart 11 (#39)

To if have time

Idea Consider  $\sum (+^n b_n)$ ,  $\sum (-^n b_n)$

① If AC  $\Rightarrow$  both conv.

② If CC  $\Rightarrow$  div.

Let r be a real #. Say  $r > 0$  (for our argument.)

Add "+" terms until  $s_n > r$  } constructing a new  $\sum$

Then, " $-$ " < r }

Repeat!

$s_n \rightarrow r$ , so sum of  $= r$

$\therefore - - - -$  Remember,  $\sum b_n$  conv., so  $(b_n \rightarrow 0)$ .

at least do graph

Riemann proved

Note Why  $s_n \rightarrow r$ ? Consider the " $b_n$ "'s that make  $s_n$  cross the r-line.

$$|s_n - r| \leq |b_n| \Rightarrow |s_n - r| \rightarrow 0$$

Distance betw.  $s_n, r$   $\downarrow$   $\Rightarrow s_n \rightarrow r$

Can get div.  $\sum$ ?

(My idea: Take a bunch of "+" terms until their sum  $>$  next neg. term  
(Can make  $\lim_{n \rightarrow \infty} s_n = \infty, -\infty$ , or P.N.E.)

Larson 5.86 has reference  
Gelbaum: "derangement"

Big Knopp 1.8, see 3.8  
Rearrange Alt. Harmonic  
 $\Rightarrow$  div.  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

Small Knopp 8.1 ds  
big Knopp 4.0 same

Gelbaum 5.4

⑥ Some Strategies for Alternating  $\sum b_n = \sum (-1)^{n+1} a_n$

$\begin{array}{|c|c|} \hline AC & D \\ \hline CC & \\ \hline \end{array}$  ?

$\underbrace{b_n}_{\geq 0}$

Is it really a geom.  $\sum$ ?  $\Rightarrow$  Geom.  $\sum$  Test

Do I think  $\sum b_n$  ...

D?

AC?

CC?

Try:  $n^{\text{th}}\text{-Term}$   
Ratio  
Root

Try: Ratio } fail if  
Root }  $a_n$  algebraic  
or Show  $\sum |b_n| (= \sum a_n)$   $\textcircled{C}$

$\Rightarrow \sum b_n \textcircled{C}, \textcircled{AC}$  by ACT

Try: AST to show  
 $\sum b_n \textcircled{C}$   
Then, use some  
test to show  $\sum a_n \textcircled{D}$ .

WARNING:  
Ratio, Root Tests  
fail if  $\sum b_n \textcircled{CC}$ .

ACT often linked  
w/ another  
test.

Note on AST hypotheses:

① Alt.  $\sum$  } Failure does not  
②  $a_n \rightarrow 0$  } imply  $\textcircled{D}$ .

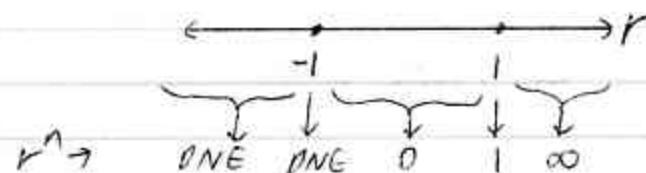
Failure implies that  
 $n^{\text{th}}\text{-Term Test}$  can be  
used to show  $\textcircled{D}$ .

Confirmed  
by a book →

The failure of the  
hypothesis to hold  
does not negate  
the conclusion.

Not true: "If hypothesis, then conclusion"  
"If I get an A, then I pass the class."  
(What about B, C?)  
"If I don't get an A, then I don't pass the class."

## 11.1-11.5: REVIEW

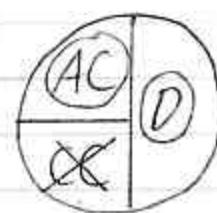
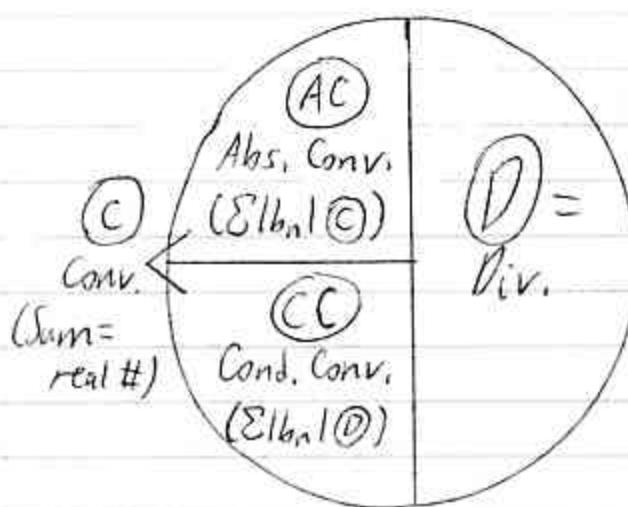
(11.1): SEQUENCES  $\{a_n\}$ Geom.  $\{ar^{n-1}\}$   $a \neq 0$ Can say Conv/Div.  
D D C C D $(-1)^{n[\pm 1]}$ : sign alternates
 $\lim_{n \rightarrow \infty} a_n$  Recall: Squeeze, L'H; Consider  $\lim_{x \rightarrow \infty} f(x)$   
 $(a_n \rightarrow 0) \Leftrightarrow (|a_n| \rightarrow 0)$  interpolating func.

$$\left(1 + \frac{1}{n}\right)^n \rightarrow e$$

(11.2-11.5): SERIES ( $\sum$ )

Chart p. S65

$$\sum b_n$$

 $\text{"+ Term } \sum$   
 $(\sum |b_n| \text{ same as } \sum b_n)$ 


$\textcircled{C}$ ,  $\textcircled{D}$  not affected by:

- (a) Inserting / dropping / changing a finite # of terms.  
("Eventually": for all  $n \geq$  some  $N$ )
- (b)  $\cdot, \div$  by a non-0 #

### Linear Combos

e.g.,  $\sum (7a_n + b_n - 9c_n)$

All  $\sum$   $\textcircled{C} \Rightarrow \textcircled{C}$

1  $\sum$   $\textcircled{D} \Rightarrow \textcircled{D}$

$\geq 2 \sum$   $\textcircled{D} \Rightarrow ?$

Tests for  $\textcircled{C}$  (State test when you use it.)

① Famous Families of  $\sum$

①a) Geom.  $\sum ar^{n-1}$  ( $a \neq 0$ )

$$\textcircled{C} \Leftrightarrow |r| < 1$$

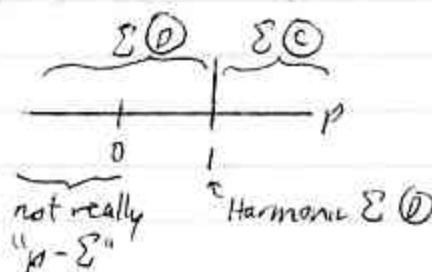
else  $\textcircled{D}$

Technically,  
 $p > 0$  for  
a "p-series"

①b) p-Series  $\sum \frac{1}{n^p}$

$$\textcircled{C} \Leftrightarrow p > 1$$

else  $\textcircled{D}$



## (2) $n^{\text{th}}$ -Term Test for D

If  $\sum a_n \Rightarrow D$

## (3) Comparison Tests (for "+" Term $\Sigma$ )

Given:  $\sum a_n$

Find:  $\sum b_n$  (Comparison  $\Sigma$ )

Usu. geom.,  $p$ -series

Take dominant terms from  $a_n$

For LCT, make constant factors = 1

Good if  $a_n$  algebraic (rational, though roots OK); w/p- $\Sigma$

To show  $\sum a_n$   $\textcircled{C}$ , pick  $\sum b_n$  that  $\textcircled{C}$

$\textcircled{D}$

$\textcircled{D}$

### (3a) BCT

If  $\Sigma$   $\Rightarrow$  use LCT

$$\text{Ex } \frac{1}{n^2+1} \textcircled{<} \frac{1}{n^2}$$

Big  $\Sigma$   $\textcircled{C}$

$\Rightarrow$  Little  $\Sigma$   $\textcircled{C}$

$$\text{Ex } \frac{1}{n} \leq \frac{1}{n-1}$$

Little  $\Sigma \textcircled{D} \Rightarrow$  Big  $\Sigma \textcircled{D}$

### (3b) LCT

If  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \text{non-0 real #}$  ("not special")

$\Rightarrow \sum a_n, \sum b_n$  both  $\textcircled{C}$

or both  $\textcircled{D}$

### (4ab) Ratio, Root Tests

$$L: \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \quad \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

Good if  
 $n!$ , products,  $2^n$ ,  $(\cdot)^n$

$$n^{1/n} \rightarrow 1$$

Bad if  $a_n$  algebraic, since  
 $\Rightarrow L=1$ . Use BCT, CCT?

Both:  $L < 1 \Rightarrow (\text{AC})$

$L = 1$ , "DNE"  $\Rightarrow (?)$

$L > 1, \infty \Rightarrow (\text{D})$

### (5) S Test (for "+ Term $\Sigma$ )

Ex  $\sum_{n=3}^{\infty} \frac{1}{n} \Rightarrow$  Consider  $\int_3^{\infty} \frac{1}{x} dx$

Need: can S interpolating func  $f(x)$   
 +, cont.,  $\downarrow$   
 (Show  $f'(x)(0)$ )

$$\text{If } S(\textcircled{1}) \Rightarrow \Sigma(\textcircled{1})$$

$$S(\textcircled{D}) \Rightarrow \Sigma(\textcircled{D})$$

Good if  $\ln n$  and  $\frac{1}{n}$ ?  
 u-sub

⑥ Test for ① if Terms may be "-"

⑥a) Alt. Series Test (AST) (for conv. of an alt.  $\Sigma$ )

$$\sum (-1)^{n[n+1]} \textcircled{a}_n$$

$$\underbrace{\qquad\qquad\qquad}_{>0} \textcircled{b}_n$$

If ① alt.  $\Sigma$

①  $a_n \dots \curvearrowleft$

②  $a_n \rightarrow 0$

(Show?)

(Show?)

$$\Rightarrow \sum b_n \textcircled{c}$$

⑥b) Abs. Conv. Test (ACT)

If I can show  $\sum |b_n| \textcircled{c}$   
 " + " Term  $\Sigma$

$$\Rightarrow \sum b_n \textcircled{c}, \textcircled{AC} \quad (\text{Don't need AST.})$$

(Also)  $n^{\text{th}}$ -Term Test for ①  
 Ratio, Root Tests for ①, ②  
 Alt. Geom.  $\Sigma$  are Geom.  $\Sigma$ .

CC<sub>T</sub>:  
 2 steps?

If we rearrange...

AC  $\Sigma \Rightarrow$  same sum

CC  $\Sigma \Rightarrow$  can get any real sum (or no sum)

## FINDING SUM, S

① Geom.  $\sum_{n=1}^{\infty} ar^{n-1}$

$$S = \frac{a}{1-r} \quad \begin{array}{l} \text{1st term} \\ \text{if } |r| < 1 \\ (\text{SC}) \end{array}$$

common ratio

② Telescoping  $\Sigma$

$$\Sigma a_n$$

rational

$$\text{Ex } \sum_{n=1}^{\infty} \frac{5}{(5n+2)(5n+7)}$$

Find  $S_k$  using pfd, canceling.

$$S = \lim_{k \rightarrow \infty} S_k \quad \begin{array}{l} (\text{exists} \Leftrightarrow \text{SC}) \\ \text{else, (D)} \end{array} \quad \} \text{True for all } \Sigma$$

## APPROX S by $S_N$

$$\textcircled{1} \sum_{n=1}^{\infty} a_n : S_N < S < S_N + \int_N^{\infty} f(x) dx$$

upper bound on error  
+ cont., >  
interpolating func. models  $\{a_n\}$

$$\textcircled{2} \text{Alt. } \sum (-1)^{n-1} a_n : |\text{error}| \leq \underbrace{|S - S_N|}_{a_{N+1}} \leq \underbrace{|\text{neglected term}|}_{\text{"looked at"}}$$

If you last added  $(+a_N)$ , then  $S_N$  is an upper bound for  $S$ . If last subtr.  $(-a_N)$ , then  $S_N$  is a lower bound.

$$\begin{aligned} \text{Mathematica} \Rightarrow \\ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2} = \frac{\pi^2}{12} \\ \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^4} = \frac{7\pi^4}{720} \end{aligned}$$

**COOL! (Don't worry for now)**

$$\left( \sum_{n=0}^{\infty} \frac{1}{n!} = e \right), \left( \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \ln 2 \right)$$

From Fourier series  
Advanced Complex Analysis  
Euler: we can find sums of  $p$ -series

$$\left( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \right), \left( \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \right) \dots$$