

11.6: POWER  $\Sigma$ A) ExEx Poly:  $1 + x + x^2$ Ex Power  $\Sigma$ :  $1 + x + x^2 + \dots + x^n + \dots$ 

This also happens to be a geom.  $\Sigma$  with  $a=1, r=x$ . (Not all power  $\Sigma$  are.)

Like a poly that goes on and on in powers.  
Not all geom  $\Sigma$  are power  $\Sigma$   
 $1 + \frac{1}{x} + \frac{1}{x^2} + \dots$  has no  $x$  (variable).  
If  $x$  is some # w/labs. value  $\infty$ , what do we know about this  $\Sigma$ ?

(Case 1) If  $|x| < 1 \Rightarrow \Sigma$  conv. w/sum  $S = \frac{a}{1-r} = \frac{1}{1-x}$

Ex If  $x = \frac{1}{10}$ ,

$$\begin{aligned} & 1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \dots + \left(\frac{1}{10}\right)^n + \dots \\ \text{i.e., } & 1 + .1 + .01 + \dots \end{aligned}$$

(converges) w/sum

$$S = \frac{1}{1 - \frac{1}{10}} \left( = \frac{1}{\frac{9}{10}} \right) = \frac{10}{9} (\approx 1.1)$$

(Case 2) If  $|x| \geq 1 \Rightarrow \Sigma$  div., no sum

Ex If  $x = 5$ ,

$$1 + 5 + (5)^2 + \dots \quad \text{div}$$

## B) Power $\Sigma$ in $x$

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

coeff charac'terize  
the power  $\Sigma$

If you have a  
4th-deg poly,  
do you have to  
have an  $x^2$   
term? No.  
Book: r

- Notes
- (a) Know  $\{a_n\}$   $\Rightarrow$  Know  $\Sigma$
  - (b)  $x^0 = 1$ , even if  $x=0$ . (Normally,  $0^0$  gives us pause.)
  - (c) Maybe not geom.  $\Sigma$
  - (d) Coeffs can be 0.

## C) $R$ , the Radius of Convergence, and $I$ , the Interval of Conv.

... for which  
 $\sum a_n x^n$  yields  
a real # output  
like the ...  
of  $f$

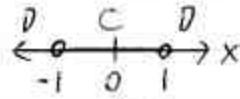
$$I = \left\{ \text{all real values of } x \text{ for which } \sum a_n x^n \text{ conv.} \right\}$$

= Domain of  $f$

$$\text{Ex } \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots \quad (\text{all } a_n = 1)$$

This conv.

$$\Leftrightarrow x \text{ is in } \underbrace{(-1, 1)}_I.$$



$\cancel{(R=1)}$  = half-width of  $I$

White shirt...

What can I look like?

3 possibl's:

①  $I = \{0\}$

is this an "interval"?  
- Don't worry.

$$R=0$$



②  $I = (-\infty, \infty)$

all real #'s

$$R=\infty$$

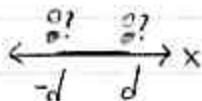


③  $I = [(-d, d)]$

or or

for some  $d > 0$

$$R=d$$



AC on  $(-d, d)$   
AC/CC/D at  $\sim d$ , d  
AC vs CC  
Not a big deal:  
we'll do  $\epsilon$ -  
= ln 2  $\underbrace{\text{CC}}$

- 44: Diverges  
 $\nexists \lim_{n \rightarrow \infty} x^n$   
 if AC then L, 0
- 45:  $I = (-r, r]$   
 c
- 46: AC at one  
 endpt.  $\Rightarrow$   
 AC at other

### ① Find I

Ex (#8)  $\sum_{n=1}^{\infty} \frac{1}{4^n \sqrt{n}} x^n$

$u_n$  depends on  $n, x$

Consider Ratio Test (or Root Test)  
 1st, for what  $x$  is  $L < 1$ ?

What  
 guarantees  
 that  $\sum$   
 conv.?  
 If  $L \dots$

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| \\ &\left( \text{Go ahead and write "L" even though "L}=\infty\text{" is inappropriate to write if } \lim_{n \rightarrow \infty} \dots = \infty \right) \\ &= \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{4^{n+1} \sqrt{n+1}} x^{n+1}}{\frac{1}{4^n \sqrt{n}} x^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{4^n \sqrt{n}}{4^{n+1} \sqrt{n+1}} \cdot \frac{x^{n+1}}{x^n} \right| \end{aligned}$$

$$= \lim_{n \rightarrow \infty} \left| \frac{1}{4} \sqrt{\frac{n}{n+1}} x \right|$$

$$= \frac{1}{4} |x|$$

Don't have to  
 write.

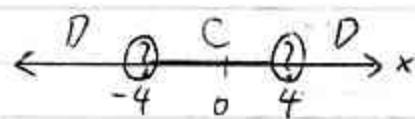
$$\begin{cases} \text{③ if } \frac{1}{4} |x| < 1 \\ |x| < 4 \end{cases} \quad \begin{cases} \text{④ if } \frac{1}{4} |x| > 1 \\ |x| > 4 \end{cases}$$

(R=4)

This L limit  
 value depends  
 on what  $x$  is.  
 (Sometimes,  
 it doesn't  
 matter;  
 $R=\infty$ )

What are the  
 only 2 values  
 of  $x$  we don't  
 know about  
 yet?

all 4 cases  
are possible



$$I = \left[ \begin{array}{c} (-4, 4) \\ \text{or} \\ (0, 4) \end{array} \right]$$

✓  $x = -4, 4$  (then,  $L = \frac{1}{4}|x| = 1 \Rightarrow$  Ratio Test inconclusive)

If  $x = -4$

$$\sum_{n=1}^{\infty} \frac{1}{4^n \sqrt{n}} (-4)^n = \sum_{n=1}^{\infty} \frac{1}{4^n \sqrt{n}} (-1)^n 4^n$$
$$= \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{\sqrt{n}}\right) \quad \text{① by AST}$$

alt. E.,  $\rightarrow_0$

If  $x = 4$

$$\sum_{n=1}^{\infty} \frac{1}{4^n \sqrt{n}} 4^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \quad \text{② by p-series } (p = \frac{1}{2} \leq 1)$$

Find I



$$I = (-4, 4)$$

What If...

$\lim_{n \rightarrow \infty} \frac{|x|}{n}$  is independent of  $n$ , can  $\cancel{p}$  out of  $\lim_{n \rightarrow \infty}$

$$\textcircled{1} \quad L = \dots = \underbrace{\lim_{n \rightarrow \infty} \frac{|x|}{n}}_{R=\infty} = 0 < 1 \text{ for all real } x$$

Up to 33/49 (w/e)  $\textcircled{2} \quad L = \dots = \lim_{n \rightarrow \infty} n|x| = \begin{cases} 0 & \text{if } x=0 \\ \infty & \text{if } x \neq 0 \end{cases}$   
 $R=0$ , S conv. only for  $x=0$  (Im OK with  $I = \{0\}$ .)

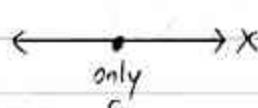
⑤ Power  $\Sigma$  in  $(x-c)$  (before:  $c=0$ )

$$\sum_{n=0}^{\infty} a_n (x-c)^n \quad \text{"Power } \Sigma \text{ centered at } c"$$

What value of  $x$   
guarantees conv.  
regardless of " $a_n$ 's".  
Not really an  
interval here

Possible "I"s

①

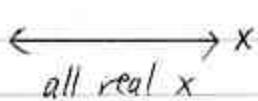


$$R=0$$

$$I = \{c\}$$

Is this an interval?  
Isn't it just a point!

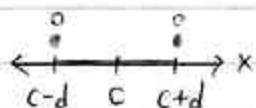
②



$$R=\infty$$

$$I = (-\infty, \infty)$$

③



$$R=d \quad (d>0)$$

$$I = [(c-d, c+d)]$$

Test+  
separately

Ex (#12) Find I for  $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} (x-2)^n$

For what  $x$  is "Ratio Test"  $L < 1$ ?  
(or Root)

$$L = \lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{(n+1)(n+2)} (x-2)^{n+1}}{\frac{1}{n(n+1)} (x-2)^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{n(n+1)}{(n+1)(n+2)} (x-2) \right|$$

$$= \lim_{n \rightarrow \infty} \left| \left( \frac{n}{n+2} \right) (x-2) \right|$$

$$= |x-2|$$

$\Sigma$  (C) if  $|x-2| < 1$

$$\begin{array}{l} |\alpha| < 1 \\ -1 < \alpha < 1 \end{array}$$

How can we  
rewrite about  
1?

$$\begin{array}{ccccccc} -1 & < & x-2 & < & 1 \\ +2 & & +2 & & +2 \end{array}$$

$$1 < x < 3$$

$$\xleftarrow{\quad D \quad ? \quad C \quad ? \quad D \quad} \qquad I = \underset{\text{or}}{[(1, 3)]} \underset{\text{or}}{}$$

If  $x=1$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} (1-2)^n = \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{n(n+1)} \right)$$

Alt. S  $\xrightarrow[n \rightarrow \infty]{\cdots}$   $\rightarrow 0$

(C) by AST

If  $x=3$ 

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} (3-2)^n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$(1)^n = 1$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2+n}$$

I thought  $1^\infty$  indet?  
 Not if the base  
 is just "1"!

BCT

$$\frac{1}{n^2+n} \leq \frac{1}{n^2} \quad (n \geq 1)$$

Big  $\mathcal{E}$  conv.  
 $\Rightarrow$  little  $\mathcal{E}$  C

I :



$$\boxed{[1, 3]}$$

Larson 6.22  
Many 17, 18c  
mathemrs:  
Gregory, Newton,  
John Wallis,  
Bernoulli, Leibniz,  
Euler, Lagrange,  
Fourier  
writers,

## 11.7: POWER $\Sigma$ REPRESENTATION OF $f(x)$

### (A) Exs (Geometric Template)

If you plug in  
 $x=0.1$

$$\text{Ex } \underbrace{1 + x + x^2 + \dots + x^n + \dots}_{\text{sum of } \Sigma} = \frac{1}{1-x} \text{ if } |x| < 1 \\ (\text{else } \emptyset)$$

a power  $\Sigma$  rep. for  
 $f(x) = \frac{1}{1-x}$  on  $\underbrace{(-1, 1)}_{(\leftarrow -1 \rightarrow 1)}$   
 $I = \text{Domain of } f$

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1}$$

for every  
 $x$  in  $\text{Dom}$ ,  
sum of series is a  
real #  
Outside  $I$ , the sum  
is und., too

If known c.v.  
unique power  $\Sigma$   
repnq

Larson  
6.7-1:  
Tricks for nat'l  
func: long ÷

$$3-2x \sqrt{1}$$

$$\text{PFD} \quad I = \bigcap I_i$$

(Find a power  $\Sigma$   
w/this sum.)

Ex (#4) Find a power  $\Sigma$  rep. for  $f(x) = \frac{1}{3-2x}$

Rewrite as  $\underbrace{(\text{or } x^k) \cdot \frac{1}{1-\frac{2}{3}x}}_{(\text{Easy to distribute } \Rightarrow \text{still have power } \Sigma)}$

$$f(x) = \frac{1}{3-2x} \quad \begin{matrix} \text{Want "1"} \\ \uparrow \end{matrix}$$

$$= \frac{1}{3(1 - \frac{2}{3}x)}$$

$$= \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}x}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}x\right)^n, \quad \left|\frac{2}{3}x\right| < 1$$

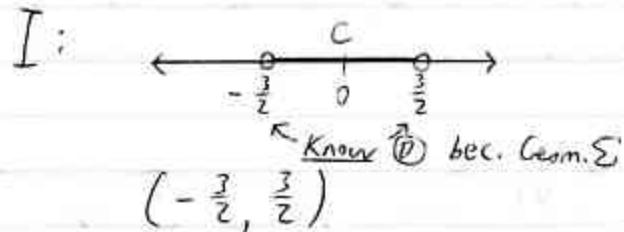
$$= \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n x^n, \quad \frac{2}{3}|x| < 1$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} x^n, \quad |x| < \frac{3}{2}}$$

$$R = \frac{3}{2}$$

Need another if  
develop another  
 $\Sigma$ . Larson 6.15

$$\frac{1}{1-x} \ln(x+1) - 1$$



Like #8, but  
replace  $-w/+$

Ex

$$\begin{aligned} f(x) &= \frac{x^3}{4+x^3} \\ &\text{Want "1"} \\ &= \frac{x^3}{4} \cdot \frac{1}{1+\frac{x^3}{4}} \\ &\quad \text{Want "+ -"} \\ &= \frac{x^3}{4} \cdot \frac{1}{1-\left(-\frac{x^3}{4}\right)} \end{aligned}$$

$$= \frac{x^3}{4} \cdot \sum_{n=0}^{\infty} \left(-\frac{x^3}{4}\right)^n, \quad \left|-\frac{x^3}{4}\right| < 1$$

$$= \frac{x^3}{4} \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{4^n}, \quad \frac{|x^3|}{4} < 1$$

$$|x^3| = |x \cdot x \cdot x| \\ = |x||x||x| \\ = |x|^3$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{4^{n+1}} \cdot x^{3n+3}, \quad \begin{aligned} |x^3| &< 4 \\ |x|^3 &< 4 \\ |x| &< \sqrt[3]{4} \end{aligned}$$

$\sqrt[n]{\text{monotonically increasing}}$   
 $\sqrt[n]{\text{every odd}}$   
 $\text{irrelevant for L1}$   
 $\text{Do } 1-7 \text{ odd, if any,}$   
 $\text{except 1b, 3b}$

$$R = \sqrt[3]{4} \\ I = (-\sqrt[3]{4}, \sqrt[3]{4}) \\ \text{bec. geom. } \Sigma$$

## (B) Finding $f'(x)$ Using Power $\Sigma$ (Important in Diff. Eqs.!!)

Review Ex

Always true  
if have finite  
# of terms,  
differently

$$\left( \begin{array}{l} D_x (x^2 + 7x^3) = D_x(x^2) + D_x(7x^3) \\ \text{linear operator} \\ = 2x + 7(3x^2) \\ = 2x + 21x^2 \end{array} \right)$$

If graph of  $f$  just  
consists of 1 pt,  
Literally no point  
in talking about  
derivs., etc.

Can  $D_x$  a Power  $\Sigma$  term-by-term  
(Need  $R \neq 0$ )

$\left( \begin{array}{l} \text{If } R=0, \\ f(x) \\ \text{No } f' \end{array} \right)$

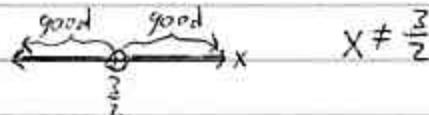
$$\text{Ex (#4)} \quad f(x) = \frac{1}{3-2x}$$

$$\text{Math 150} \quad f(x) = (3-2x)^{-1}$$

or Recip. or  
Power rules

$$f'(x) = - (3-2x)^{-2} (-2)$$

$$= \frac{2}{(3-2x)^2} \quad \text{A}$$



We found a power  $\Sigma$  rep. for  $f(x)$  in  $\mathbb{D}$ .

$$\frac{1}{3-2x} = \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} x^n, |x| < \frac{3}{2}$$

$(R = \frac{3}{2})$

Note  $\frac{1}{3-2x} = \left(\frac{1}{3}\right) + \frac{2}{9}x + \frac{4}{27}x^2 + \dots$  } Works if  
 n=0 term  
 $D_x \left(\frac{1}{3-2x}\right) = \underset{\substack{\downarrow \\ \text{start} \\ w/n=1}}{0} + \frac{2}{9} + \frac{8}{27}x + \dots$  } the "n=0" term is a constant.

$$D_x \left(\frac{1}{3-2x}\right) = D_x \left(\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} x^n\right)$$

↑ can switch for power  $\Sigma$  (can  $D_x$  term-by-term)

$$= \sum_{n=1}^{\infty} D_x \left( \frac{2^n}{3^{n+1}} x^n \right)$$

indep. of  $x$

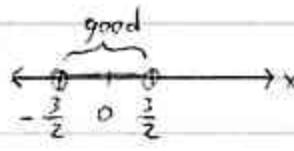
$$= \sum_{n=1}^{\infty} \frac{2^n}{3^{n+1}} \cdot nx^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{n2^n}{3^{n+1}} x^{n-1}$$

Again,  $R = \frac{3}{2}$ . } for new  
 Turns out  $I = \left(-\frac{3}{2}, \frac{3}{2}\right)$ . }  $\Sigma$

Must check endpoints.  
 New  $\Sigma$  not geom.

Sum of new  $\Sigma = \frac{2}{(3-2x)^2}$  if  $x$  is in  $I$ .



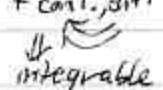
$f'(x)$  as the  $\Sigma$  has a smaller domain!

Stewart: 11.9.36  
 $\sum \frac{\sin nx}{n^2}$  conv. b/c  
 but  $\partial f_n'$  div.  
 when  $x = 2n\pi$ ,  
 Can't switch  $\Sigma$  b/c  $\Sigma$  is int.

$x = \frac{\pi}{2} \Rightarrow \sum \frac{1}{n^2}$   
 $x = -\frac{\pi}{2} \Rightarrow \sum (-1)^{n-1} \frac{1}{n^2}$   
 $D_x f$  (geom)  $\Rightarrow$  not geom.  
 not abs.  $\Rightarrow$  geom.

In practice, we do, but this is an easy fix of  $D_x$  a  $\Sigma$ .  
 See (6)

Can't use geom template easily  
 $\frac{1}{3-2x} \frac{1}{3+2x} = \Sigma \Sigma$   
 CTS?

Carson 6.11:  
 For  $x \in (-d, d)$   
 $(R=d)$ ,  
 $f$  cont., diff'ble  
  
 integrable

### ③ Finding $\int f$ Using Power E

Can  $\int$  a Power E term-by-term  
 (Need  $R \neq 0$ )

Non-shifted  
 parts E must  
 conv. at 0  
 gives AD w/  $C=0$   
 if  $\int$  Power E

In ② we found  
 power E rep  
 In ③ we  $\int f_x$   
 Now, in ④ we  $\int$

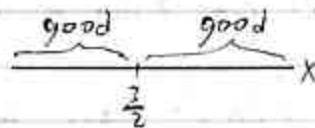
Book:  $\int_0^x f(t) dt$

gives one antideriv. of  $f(x)$ .

$$\text{Ex (#4)} \quad f(x) = \frac{1}{3-2x}$$

$$\text{Math 150} \quad \int \frac{1}{3-2x} dx \quad u = 3-2x$$

$$= -\frac{1}{2} \ln |3-2x| + C$$



$\int_a^b \frac{1}{3-2x} dx$   
 OK if  $\frac{3}{2}$  not in  $[a, b]$   
 Can use FTC directly.

Use power E rep. in ①

$$\int \frac{1}{3-2x} dx = \int \sum_{n=0}^{\infty} \underbrace{\frac{2^n}{3^{n+1}}}_{\text{const}} x^n dx, \quad |x| < \frac{3}{2} \quad (R = \frac{3}{2})$$

$$= \sum_{n=0}^{\infty} \underbrace{\frac{2^n}{3^{n+1}}} \int x^n dx \quad \text{indep. of } x$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} \cdot \left[ \frac{x^{n+1}}{n+1} \right] + C$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}(n+1)} x^{n+1} + C \quad \begin{cases} \text{represent same} \\ \text{family of funcs.} \\ (\text{converg. to } f \text{ on } I) \\ \text{on } I \end{cases}$$

$\left( = -\frac{1}{2} \ln |3-2x| + \hat{C} \text{ on its } I \right)$

Stewart 11.9.37  
 $\sum_{n=2}^{\infty}$  at  
endpts. but  
 $f'(0)$

Larson 612  
 $f'(0)$  at  $\pm d$   
 $\Rightarrow$  so does  $f$

$P_x, S$  (geom)  $\Rightarrow$   
not geom

Up to 7  
Up to 9 - it  
depends (no)

Again,  $R = \frac{3}{2}$   
Turns out  $I = \left[-\frac{3}{2}, \frac{3}{2}\right]$  } for new  
had to check endpts.  
separately

(When you  $P_x$  or  $S$  a Power  $\sum$ ,  $\Rightarrow$  same  $R$ , but  
the behavior at the endpts. of  $I$   
may change.)

In MISD we  
used Newton's  
Method to  
approx.  $\pi$ .  
4.8.25:  
 $f(x) = \tan x$   
 $x_{n+1} = x_n - \tan x_n$   
but how  
compute?

① Find a power  $\sum$  rep. for  $\tan^{-1}x$ , and use it to  
approx.  $\pi$  to 2 decimal places.

(See Ex 5 on p. 576 - we're simplifying this.)

$$f(x) = \tan^{-1}x$$

How can we use the Geometric Template?  
(Another method in 11.8)

$$\begin{aligned} f'(x) &= \frac{1}{1+x^2} \\ &= \frac{1}{1-(-x^2)} \end{aligned}$$

$$\text{Use: } \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} (-x^2)^n, |-x^2| < 1 \\ &\quad |x| < 1 \\ &\quad (R=1) \end{aligned}$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$f(x) = \int f'(x) dx$  (actually, one member  
of this family of antiderivs.)

$$\tan^{-1} x = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \int x^{2n} dx$$

Stewart does  
+C and C+

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C \quad (\text{Some write } C + \sum \dots)$$

(can pull out, if you want)

Solve for C

If had  $\sum_{n=0}^{\infty} (x-3)^{2n+1}$   
use  $x=3$ .

$$x=0 \Rightarrow \text{(we know sum of } \sum \text{)}$$

$$\tan^{-1}(0) = 0 + C$$

$$0 = C$$

$$C = 0$$

Stewart 7.7:  
Called  
Gregory's  
Series after  
James Gregory  
(Scottish;  
1638-1675)  
who discovered  
it in 1671.  
Madhava  
(Indian)  
knew of it  
in 1400!!

History

$$\Rightarrow \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$\{2n+1\} \Rightarrow \text{odd } \#$   
 $\{2n\} \Rightarrow \text{even } \#$

(R=1) for this  $\sum$ , also.

Find I for this  $\sum$

$$\leftarrow \underset{-1}{\overset{?}{0}} \underset{?}{c} \underset{1}{\overset{?}{0}} \rightarrow$$

$$\text{If } x = -1 \quad \sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^{3n+1} \frac{1}{2n+1}$$

Alt.  $\sum \quad \dots \rightarrow 0$

(C) by AST

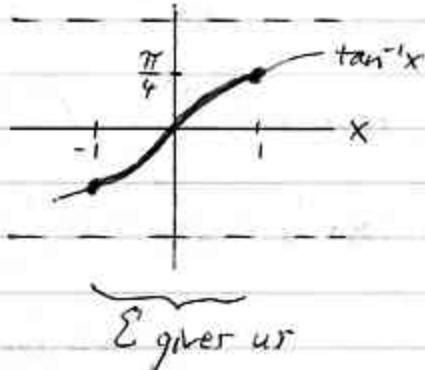
$$\text{If } x = 1 \quad \sum_{n=0}^{\infty} (-1)^n \frac{1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

Alt.  $\sum \quad \dots \rightarrow 0$

(C) by AST

New  $I = [-1, 1]$

For any  $x$  in  $I$ , sum of  $\sum = \tan^{-1} x$ .



Note odd func., so  
odd-only exponents in  $\sum$   
not a coincidence

Remember,  $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$

Plug in  $x=1$

$\pi \approx 3.14159265358979\dots$   
Now,  $\pi$  nice!

Stewart 758:  
Leibniz formula for  $\pi$ .  
Replugged in  
 $x=1$ ; independently  
discovered  
 $\tan^{-1}$  series in  
1674, three  
years after  
Gregory.  
The issue of  
plugging in  
 $x=1$  is dealt  
with in a  
Note.

History

$$\begin{aligned}\tan^{-1}(1) &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \\ \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \quad \left( \pi = \sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1} \right)\end{aligned}$$

Irrational!      All terms rational!  
Nice!

(The sum of a finite # of  
rational #'s must be rational.)

Approx.  $\pi$  to 2 dec. places

Need  $|\text{error}| < 0.005 \quad (= \frac{1}{200})$

Turns out you  
need 400 terms...

5<sub>400</sub>? We  
start w/ 10.

$$\pi = \underbrace{4 - \frac{4}{3} + \dots - \frac{4}{799} + \frac{4}{801} - \dots}_{400 \text{ terms}}$$

$$(|\text{error}| \leq |\frac{4}{801}| \approx 0.00499 < 0.005)$$

3.13909

$$\boxed{\pi \approx 3.14}$$

Annot not  
shrink fast  
enough  $\frac{1}{n^2}$

Beckmann  
140  
Like Spear -  
good looking but  
mostly useless

Larson 8.9, 48, 44  
Edwards 572-3  
 $\tan^{-1}(\frac{1}{5}) \approx \text{geom.}$

Note Like the alt. harmonic  $\sum$ ,  
this  $\sum$  converges slowly!  
Need  $\approx 10^{50}$  terms for 100-digit  
accuracy!

In 1706, John Machin used  
the following to approx.  $\pi$   
to 100 places:

$$\frac{\pi}{4} = 4 \tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right)$$

CAN SKIP!!

Highly  
Technical

Note Stewart (5<sup>th</sup> ed, 758) observes:

"We have shown that Gregory's ( $\tan^{-1}x$ ) series is valid when  $-1 < x < 1$ , but it turns out (although it isn't easy to prove) that it is also valid when  $x = \pm 1$ ."

Alternate PT.  
for  $\ln 2$ :  
178  
 $\ln 2 = \ln(1+x)$   
 $R_n(1) \rightarrow 0$

Post 147  
Hard to  
prove

The key is Abel's Theorem:

Let's say  $R = d$ .

If  $\sum S$  conv. at  $d$ , then  $\sum S$  & continuous at  $d$ .

Here Since Gregory's  $\tan^{-1}x$   $\sum S$  conv. at  $x=1$ ,  
then, by Abel's Thm., the  $S$  is cont. at  $1$ .  
 $\tan^{-1}x$  is also cont. at  $1$ , so the  
 $S$  evaluated at  $x=1$  must agree  
w/ $\tan^{-1}(1)$ .

It's OK that  $\underbrace{\frac{1}{1+x^2}}$  only had  $I = (-1, 1)$ .  
our  $f'(x)$   $S$ .

Stewart 7.68:  
Newton often did this way!

## (E) Exs (Newton often did this way!)

Ex:  $\int x \tan^{-1}(x^3) dx$ ,  $|x| < 1$   
 (Can't do  $\int x^3 \cdot \tan^{-1}(x^3) dx$ )  
 Limited!

Part 1: No...  
 If you expand this in terms of  $x^3$ , you don't know what you're doing.  
 (Don't sweep it over to the int.)

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \text{ from (D)}$$

$$\begin{aligned} \tan^{-1}(x^3) &= \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{2n+1}, \quad |x^3| \leq 1 \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1} \end{aligned}$$

$(R=1)$

What's the one change we make?

Now we can do it!  
 Don't forget +C!!

$$x \tan^{-1}(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{2n+1}$$

$$\int x \tan^{-1}(x^3) dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+5}}{(2n+1)(6n+5)} + C \quad (R=1)$$

We can do it using  $\Sigma$ .

$$\text{Ex } \int_0^{0.9} x \tan^{-1}(x^3) dx \stackrel{\text{FTC}}{=} [\Sigma]_0^{0.9}$$

$$\approx 0.1101$$

$$\approx 0.1086$$

$$= \left[ \frac{x^5}{5} - \frac{x^{11}}{33} + \dots \right]_0^{0.9}$$

$$= \frac{(0.9)^5}{5} - \frac{(0.9)^{11}}{33} + \dots \quad \times$$

satisfies hypothesis of AIT

$$|\text{error}| \leq |\text{1st neglected term}|$$

How can we write this sum so we can expand as a  $\Sigma$ ?  
 Part 2: No if A!  
 Book doesn't mention  $\rightarrow$

$$\text{Ex } \int \frac{\tan^{-1} x}{1+x^5} dx = \int (\tan^{-1} x) \left( \frac{1}{1+x^5} \right) dx$$

(Book doesn't do!)

$$= \int \left( x - \frac{x^3}{3} \dots \right) \left( 1 - x^5 + x^{10} \dots \right) dx$$

Distribute (find all terms of degree  $\leq 20$ , say?)

$$\text{or Long } \div: 1+x^5 \overline{)x - \frac{x^3}{3} \dots} \quad (\text{Risky})$$

(Theory not as clean)

$\int$  term-by-term, + C

(R=1)  $\leftarrow$  Don't have to find here.

Still up to 7/9

If  $\int \Sigma \sim \Sigma - \int x$   
 If same R  
 $\Rightarrow$  same R for resulting  $\Sigma$

Grid of 8 funcs:  
So far,  $\frac{1}{1-x}$ ,  $\tan^{-1}x$

Larson 6.8:  
Modern calc devices  
use 8. (See L11-38)

Weird pf in 11.7  
formula in 11.8  
Newton's  
 $1+x = e^{\ln(1+x)}$   
 $= \sum_{n=0}^{\infty} \frac{(1+x)^n}{n!}$   
match coeffs.  
use if  $x < 0$

### (F) $e^x$

Turns out (1) will convince you in 11.8)

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots \\ &\text{for all real } x \quad (R=\infty) \\ &I = (-\infty, \infty) \quad (n! \text{ in den. "helps"}) \end{aligned}$$

calculator  
just needs  
 $+, -, \cdot, \div, \text{const.}$   
to approx.  
 $e^{2.91}$ , say

Note:  $e^0 = 1$

Shifting over  
1 position

$$\begin{aligned} D_x(e^x) &= 1 + x + \frac{x^2}{2!} + \dots \\ &= e^x \end{aligned}$$

where's  
the 1?

$$\begin{aligned} \int e^x dx &= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + C \\ &= e^x + C \quad \text{SPLIT out 1} \end{aligned}$$

$C=1+C$  confused them!

Sum of recip  
of facs

upto 17

$$\begin{aligned} \text{If } x = 1 \\ e &= \sum_{n=0}^{\infty} \frac{1}{n!} \\ &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \dots \end{aligned}$$

Note: Although it's easy to compute  $2!, 3!, \dots$ , we like to indicate the pattern.

before sinh  
(+1) more  
convenient

## ⑥ cosh x

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad (\text{by def'n: p. 440})$$

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ e^{-x} &= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots \\ &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \end{aligned}$$

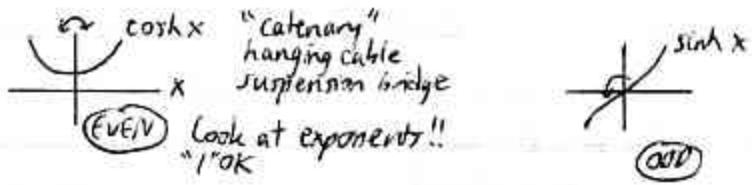
$$e^x + e^{-x} = 1 + 2 \cdot \frac{x^2}{2!} + 2 \cdot \frac{x^4}{4!} \dots$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} \dots$$

$\{2n\} \Rightarrow \text{even } \#s$

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \quad \text{for all real } x$$

Dictionary of Math:  
catenary:  
 $y = a \cosh(\frac{x}{a})$   
a: y-axis



## ⑦ sinh x

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Can prove  
using def'n

$$\text{Also, } D_x(\cosh x) = \sinh x$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$\begin{aligned} D_x \left( \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \right) \\ = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!} \end{aligned}$$

$$D_x(\cosh x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$\{2n+1\} \Rightarrow \text{odd } \#s$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \quad \text{for all real } x$$

Why can't we have a  $\Sigma$  for  $\ln x$  centered at 0?  
Instead,  
 $\ln(1+x)$ .

$$\text{Ex 3 p. 575} \quad \ln(1+x) = \int_0^x \frac{1}{1+t} dt$$

$$\textcircled{1} \quad \ln(1+x)$$

I want a  $\Sigma$   
How can I start?  
How can I use  
the geom template?

6 of 8 so far.  
last:  $\sin x$ , w/  $x$   
Geom. template  
not helpful.  
Not related directly  
to  $e^x$ .

$$f(x) = \ln(1+x)$$

$$f'(x) = \frac{1}{1+x}$$

$$= \frac{1}{1-(-x)}$$

$$= \sum_{n=0}^{\infty} (-x)^n, \quad |x| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$$

$$(R=1)$$

We're sacrificing  
precision  
for  
convenience.

$$\Rightarrow f(x) = \int \sum_{n=0}^{\infty} (-1)^n x^n dx \quad (\text{one member})$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + C$$

$$x=0 \Rightarrow$$

$$\ln 1 = 0 + C$$

$$0 = C$$

$$C = 0$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\text{or } \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$$

$$\begin{aligned} \text{Sub: } k &= n+1 \\ &\rightarrow n = k-1 \end{aligned}$$

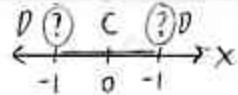
for what  $x$ ?

"Common trick in 2SS"  
Diff Eqs.

(Note:  $\Sigma$  is altern. for  $x > 0$ , not for  $x < 0$ .  
Plug in a value for  $x$  before you think  
about altern.  $\Sigma$ .)

## Find I

$R=1$ , again



If  $x = -1$

Is this an alt. E?

$$\sum_{n=0}^{\infty} (-1)^n \frac{(-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \underbrace{(-1)^{2n+1}}_{=-1} \frac{1}{n+1}$$

always odd

$$= - \sum_{n=0}^{\infty} \frac{1}{n+1} \text{ or } - \sum_{k=1}^{\infty} \frac{1}{k} \quad (\text{Harmonic E!!})$$

$(k=n+1)$

D



Note  $\ln(1+(-1)) = \ln 0$   
undefined!

L11-29f: Abel's Thm  
 $\sum_{n=0}^{\infty} a_n x^n$  converges  $\Rightarrow \sum_{n=0}^{\infty} a_n$

(is THIS an alt. E?)

$$\sum_{n=0}^{\infty} (-1)^n \frac{(1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$$

alt.  $\underset{n \rightarrow \infty}{\curvearrowleft}$

C by AST

Note: Plug in  $x=1 \Rightarrow \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$  becomes  
 $\ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

$\ln 2 = \text{sum of alt. harm. E}$	converges slowly, though!!
$\approx 0.693$	

$I = (-1, 1]$
---------------

$\sqrt{2}$  Newton's  
Method

Binomial E for  
 $(1+x)^k$ ,  $|x| < 1$

## 11.8: MACLAURIN (M) and TAYLOR (T) SERIES

Stewart 762  
 Colin MacLaurin  
 generalized  
 M. series in  
 1742 calc book  
 766  
 M. series for  
 $e^x$ ,  $\sin x$ ,  $\cos x$   
 disc by Newton  
 using diff. methods  
 Did Indian  
 astronomer  
 know  $\sin x$ ,  $\cos x$   
 &  $e^x$  century  
 before Newton?

History

### (A) M Series' for $f(x)$

Power E rep. of  $f(x)$  centered at 0

(11.7) (We found M. Series for 6 funcs.)

If  $f$  has a power E rep.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad (R \neq 0)$$

Then

seed at  $x=0$   
 sprinkle  
 deriv info  
 $\Rightarrow$  grow into  
 better,  
 higher -  
 degree poly  
 approx.  
 of  $f$  func.

These all exist:

$$\begin{aligned} f^{(0)}(0) &= f(0) \\ f^{(1)}(0) &= f'(0) \\ f^{(2)}(0) &= f''(0) \\ &\vdots \end{aligned}$$

which means

We hope  
 there's a  
 nice pattern  
 for our  $f$ .

IF  $R=0$ , just say  
 something like  
 $f(0)=10$ , and  
 whatever  
 do it  
 why bother  
 Even a small  $I \neq \{0\}$   
 is more interesting.

and

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad (\textcircled{A}) \quad (\text{Compact!!})$$

How can we  
 use this tech.  
 info at 0 to  
 construct our  
 M/power E?

Maybe  $0!$ ,  $1!$

$$\begin{aligned} &= \underbrace{f(0)}_{\text{const.}} + \underbrace{f'(0)x}_{\text{const.}} + \underbrace{\frac{f''(0)}{2!}x^2}_{\text{const.}} + \underbrace{\frac{f'''(0)}{3!}x^3}_{\text{const.}} \\ &\quad + \dots + \underbrace{\frac{f^{(n)}(0)}{n!}x^n}_{\text{const.}} + \dots \end{aligned}$$

Ex If  $e^x$  can be rep. by  $\sum_{n=0}^{\infty} a_n x^n$  ( $R \neq 0$ ),  
what must it be?  
i.e., find the M series for  $e^x$ ,  
assuming it exists.

$$f(x) = e^x \\ f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = e^0 = 1 \quad (n=0, 1, 2, \dots)$$

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \text{Always!}$$

$$\Rightarrow e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

for all real  $x$

Why not "I"?  
We're now  
talking about  
repr. of funcns,  
not just  
powers.

Why? (Proof of  $\textcircled{A}$ )

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \quad (\text{We assume a M. series rep. exists})$$

Show:  $a_n = \frac{f^{(n)}(0)}{n!} \quad (n=0, 1, 2, \dots)$   
(formula for coefft of M Series)

$$f^{(n)}(x) = D_x^{(n)} ((a_0 + a_1 x + \dots + a_{n-1} x^{n-1} + a_n x^n))$$

$n^{\text{th}} \text{ deriv} = 0$

$$n^{\text{th}} \text{ deriv.} = n(n-1)(n-2)\dots(1)a_n x^0 \\ = n! a_n$$

$$( + a_{n+1} x^{n+1} + \dots )$$

Ex The 3rd deriv. of a 2nd-degree poly is 0.

Apply Power Rule  $n$  times  
 $\Rightarrow -n$  from exponent

$$= n! a_n + c_1 x + c_2 x^2 + \dots \quad ("c_i" \text{ s real #s})$$

$\swarrow \searrow$   
(turns out we don't care what these are.)

some # times  $x$

$$f^{(n)}(0) = n! a_n \quad \text{Solve for}$$

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Ex  $f(x) = \sin x$  has a M series. Find it.

	$f^{(n)}(0)$	for $n =$
$\sin x$	0	0 4
$\cos x$	1	1, 5 ...
$-\sin x$	0	2 6
$-\cos x$	-1	3 7

$$\sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$n=0 \Rightarrow 0 \text{ term}$   
 $n=2 \Rightarrow 0 \text{ term}$

$$= \frac{1}{1!} x^1 + \frac{(-1)}{3!} x^3 + \frac{1}{5!} x^5 + \frac{(-1)}{7!} x^7 + \dots$$

→  $\sin x = \sum_{\substack{n=0 \\ \text{ODD}}}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

*Careful!*  
 ← The  $n=7$  term here  
 ← is the  $n=3$  term here!!

for all real  $x$   
 (RAD)

The graph is sym about origin  
 If you replace  $x \leftrightarrow -x$ ,  
 same E, diff. form  
 we automatically  
 skip over 0 terms  
 we're numbering  
 the terms differently  
 use 'n' is  
 more efficient/compact  
 encoding of E,  
 calc. must be (n).

12.  $\cos(x^2)$

$$\begin{aligned} \text{Ex } \sin(x^2) &= \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!} \end{aligned}$$

Can  $\int \sin(x^2) dx$ . Can approx.  $\int_a^b \sin(x^2) dx$  for any real  $a, b$ .

Why not use M. reiter template  
 Repeatedly differentiating for  
 $\sin(x^2)$ , itself, would have  
 been messy. Works in  
 principle → same series.

$$\text{Ex } f(x) = \cos x$$

$$\cos x = D_x(\sin x)$$

$$= D_x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)$$

$$\Rightarrow \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

(EVEN)

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

for all real  $x$

$$\text{Also, } D_x\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$\sinh x$  vs  $\cosh x$   
 $\neq$  even  
 $\sinh x, \cosh x$  (11.7)

Do not start w/  $n=1$ .  
 Then  $n=0$  term is not a constant.

Another way:

$$\sinh x$$
  
 $\sinh 0 = 0$

$$\cosh x$$
  
 $\cosh 0 = 1$

← Think:  $\sin 0 = 0$   
 $\cos 0 = 1$

If  $f(x) = \sinh x \Rightarrow f^{(\text{even})}(0) = 0$  and  $f^{(\text{odd})}(0) = 1$ .

$$\sinh x = \frac{1}{1!}x^1 + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

(ODD)

EVEN

same as (M series) for  $\sin x, \cos x$ ,  
 except all "+" , no sign flipper.

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

for all real  $x$

what if you  
 forgot?  
 def'n?  
 $\sinh x = \frac{e^x - e^{-x}}{2}$

Don't say  
 pos-term  
 or 11; what  
 if  $x < 0$ ?

No sign  
 flipper /  
 alternator.  
 Never alt.  
 for any  $x$ .

The sign flipper  
 is periodicity  
 weird!  
 Who'd suspect  
 $\sin(0) =$   
 $\sin(0+2\pi) ??$

Note  $\sinh x$  is ↗ ; not periodic like  $\sin x$ .

✗  
 $x$

(Who'd suspect periodicity  
 from  $\Sigma$ ??)

11.7 method easier!

Ex  $f(x) = \ln(1+x)$

$$f'(x) = \frac{1}{1+x}$$

$$= (1+x)^{-1}$$

$$f''(x) = -(1+x)^{-2}$$

$$f'''(x) = 2(1+x)^{-3}$$

$$f^{(4)}(x) = -3 \cdot 2 (1+x)^{-4}$$

:

$$f^{(n)}(x) = (-1)^{n-1} (n-1)! (1+x)^{-n}$$

$$f(0) = \ln 1 = 0$$

$$f'(0) = 1$$

$$f''(0) = -1$$

$$f'''(0) = 2$$

$$f^{(4)}(0) = -3!$$

:

$$f^{(n)}(0) = (-1)^{n-1} (n-1)!$$

( $n = 1, 2, 3, \dots$ )

what I want!!

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f^{(0)}(0) = f(0) = 0$ . Just drop the  $n=0$  term.

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (n-1)!}{n!} x^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad \text{or} \quad \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}$$

$\underbrace{\phantom{\sum_{n=1}^{\infty}}}_{k=n-1}$

Ex  $f(x) = \tan^{-1} x$   
 $f^{(n)}(x)$  hard!!  
 Use 11.7

$$\begin{cases} f'(x) = \frac{1}{1+x^2} \\ f''(x) = -\frac{2x}{(1+x^2)^2} \\ f'''(x) = \frac{6x^2-2}{(1+x^2)^3} \\ f^{(4)}(x) = -\frac{24x(x^2-1)}{(1+x^2)^4} \end{cases}$$

YUK!!

Ex  $\tan x = \frac{\sin x}{\cos x} > \text{Expand, Long } \div \text{ (theory messier)}$   
 There is a pattern, but it's complicated!!

Stewart  
long =  
154 iter

calculator:  
 $\tan 1 = \frac{\sin 1}{\cos 1} ??$

Larson 598;  
Brook Taylor  
published early  
comprehensive  
work on poly  
approx of  
transc. func.  
1715

### (B) T Series for $f(x)$ at $c$ ( $x=c$ )

If  $f$  has a power  $\Sigma$  rep. centered at  $c$

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \quad (R \neq 0)$$

Then

$f(c), f'(c), f''(c), \dots$  all exist

and

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

Maybe  $0!, 1!$

$$= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots$$

M series;  $c=0$

Ex (#20) Find the T series for  $f(x) = e^x$  at  $c = -3$ .

$$f(x) = e^x \\ f^{(n)}(x) = e^x \quad f^{(n)}(-3) = e^{-3} = \frac{1}{e^3} \quad (n=0,1,2,\dots)$$

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!} (x - (-3))^n$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{1}{e^3 n!} (x+3)^n}$$

$$\left( \frac{1}{e^3} + \frac{1}{e^3} (x+3) + \frac{1}{e^3 \cdot 2!} (x+3)^2 + \dots \right)$$

Why?  
Later

expanding in powers of  $(x+3)$

Ex (#24) Find the first 3 terms of the T series for  $f(x) = \tan x$  at  $c = \frac{\pi}{4}$ .

$$f(x) = \tan x$$

$$f\left(\frac{\pi}{4}\right) = \tan \frac{\pi}{4} = 1$$

$$f'(x) = \sec^2 x \\ = (\sec x)^2$$

$$f'\left(\frac{\pi}{4}\right) = (\sec \frac{\pi}{4})^2 \\ = (\sqrt{2})^2 \\ = 2$$

$$f''(x) = 2 \sec x \cdot \sec x \tan x \\ = 2 \sec^2 x \tan x$$

$$f''\left(\frac{\pi}{4}\right) = 2(\sec \frac{\pi}{4})^2 (\tan \frac{\pi}{4}) \\ = 2(2)(1) \\ = 4$$

Don't need a general pattern!

$$\tan x = \frac{f\left(\frac{\pi}{4}\right)}{(0!)^1} + \frac{f'\left(\frac{\pi}{4}\right)}{(1!)^1} (x - \frac{\pi}{4}) + \frac{f''\left(\frac{\pi}{4}\right)}{(2!)^2} (x - \frac{\pi}{4})^2 + \dots$$

(may help)  $\rightarrow$

$$= 1 + 2(x - \frac{\pi}{4}) + 2(x - \frac{\pi}{4})^2 + \dots$$

Can do HW

Why Bother? (Why not just use M series?)

Ex Approx.  $e^{-2.9}$ .

M series ( $c=0$ )

$$e^x = 1 + x + \frac{x^2}{2!} + \dots$$

$$e^{-2.9} = 1 + \underbrace{(-2.9) + \frac{(-2.9)^2}{2!} + \dots}_{\text{STRUGGLE!}}$$

$$e^{-2.9} \approx 0.30500$$

( $n!$  eventually "really wins out")

T series ( $c=-3$ )

We can approx.  $e$ , and we could, in principle, cube the approx. by hand, or with cheap calculator, and take reciprocal. Why not center  $e^x$  at  $-2.9$ ? Then, you'd have to know  $e^{-2.9}$ . The whole problem we're attacking is the interpretation of  $e^x$  between integer values of  $x$ .

$$e^x = \frac{1}{e^{-x}} + \frac{1}{e^{-x}}(x+3) + \frac{1}{e^{-x}2!}(x+3)^2 + \dots \quad (\text{A cheap calculator can approx. } e^3.)$$

$$= \frac{1}{e^{-3}} \left[ 1 + (x+3) + \frac{(x+3)^2}{2!} + \dots \right]$$

$$e^{-2.9} = \frac{1}{e^{-3}} \left[ 1 + (-2.9+3) + \frac{(-2.9+3)^2}{2!} + \dots \right]$$

$$= \frac{1}{e^{-3}} \left[ 1 + (0.1) + \frac{(0.1)^2}{2!} + \dots \right] \quad \text{STRUGGLE!}$$

$$\approx 0.05501$$

Terms to faster convergence

Turns out  $e^{-2.9} \approx 0.05502$

If you care about  $x \approx c$ , expand T series at  $c$ .

Sometimes, we can't use M series to find a func. value. Re-center!

Ex Approx.  $\ln 3$ , where  $x=2$ . Want  $\ln(4x)$ . M. series for  $\ln(1+x)$ ;  $I=(-1,1)$ . No 2!!

Try T. series for  $\ln x$  at  $e$ .  $\ln x = \ln e + \frac{1}{e}(x-e) - \frac{(x-e)^2}{2!} + \dots$

I need a center  $c$  so that  $f(c)$  is known.

1st term in expansion  $\ln x$ . Is there a # where for we know center  $c$  is neither  $e$ , nor  $1$ ?

## ③ M, T Polys.

If it exists, M. series for  $f(x)$

$$= \underbrace{f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n}_{P_n(x), \text{ the } n^{\text{th}}\text{-degree M. poly. for } f(x)}$$

best cubic approx.  
given by  $P_3(x)$ ,  
at least near  $x=0$ .

best  $n^{\text{th}}$ -deg. poly. approx.  
for  $f(x)$  near  $x=0$ .

I'm Mr. Laiin Mac  
Givemint at 0.

Ex (p. 591)  $f(x) = e^x = 1 + x + \frac{1}{2}x^2 + \dots$

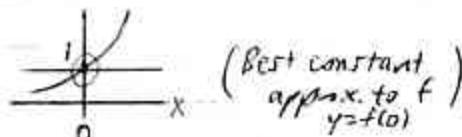
Gardening  
If you plant  
bulbs on seed,  
how does the poly  
approx. grow?  
A.I.

Use

Poly. Approx.

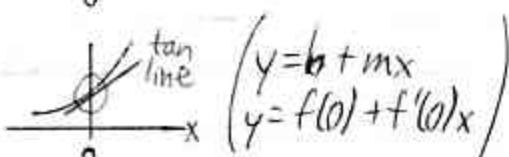
$$f(0) = 1$$

$$P_0(x) = 1$$



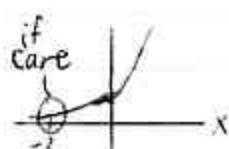
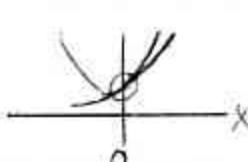
$$\text{and } f'(0) = 1$$

$$P_1(x) = 1 + x$$



$$\text{and } f''(0) = 1$$

$$P_2(x) = 1 + x + \frac{1}{2}x^2$$



Use T poly. at -3.

(Calc. use T poly. modified  
so that error is spread more  
evenly through an interval.)

Stewart 7.80

Handout

Where is  
valid?

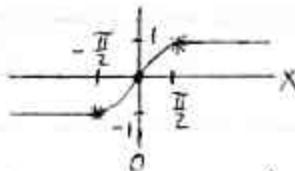
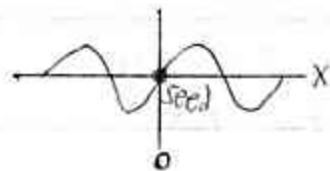
① For what  $x$  is  $f(x)$  rep. by its M series?

g from Larson 6.24

Ex

$$f(x) = \sin x$$

$$g(x) = \begin{cases} -1 & , x < -\frac{\pi}{2} \\ \sin x & , |x| \leq \frac{\pi}{2} \\ 1 & , x > \frac{\pi}{2} \end{cases}$$



Note:

All M series are power series.  
They have derivs.  
of all orders  
on  $(-R, R)$ ,  $R \neq 0$   
or  $(-\infty, \infty)$ ,  $R = \infty$ .  
If  $f(x)$  falls apart,  
the M series can't repeat  
there.

$$f^{(n)}(0) = g^{(n)}(0) \quad (n=0, 1, 2, \dots) \quad (\text{All orders of derivs. at } 0 \text{ match up.})$$

Based on these values, we construct  
the M series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

This represents  $f(x)$  for all real  $x$   
but  $g(x)$  for  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ , only.  
( $g'' \text{ DNE}$ )

Proof? Later!

Where?  
OK:  $\sin x$  somewhere  
I just here  
-1  
 $\partial_x^2(\sin x) = -\sin x$   
 $\overbrace{\partial_x^2(\sin x)}^{\text{OK}} = -\sin x$   
 $\rightarrow 0$   
Stopped dead in  
your tracks?

Let  $J$  be some interval containing  $0$ .

If  $f, f', \dots, f^{(n+1)}$  exist on  $J$ , (actually, if  $f^{(n+1)}$  exists  $\Rightarrow$  so do the previous ones)

Redundant:  
 $f^{(n+1)} \Rightarrow$  previous

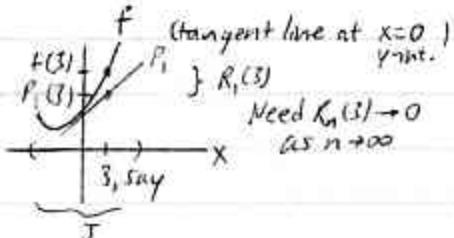
Lesson 602:  
MVT special case  
 $f(x) = f(a) + f'(c)(x-a)$

Reinforcement of AST  
(error analysis).

Edwards 514  
"Lagrange form"  
for rem.  
1<sup>st</sup> appeared  
1797 book by L.

$$f(x) = P_n(x) + R_n(x)$$

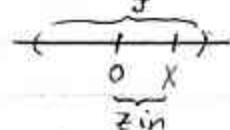
remainder  
or error



$$R_n(x) = \frac{f^{(n+1)}(\zeta)}{(n+1)!} x^{n+1}$$

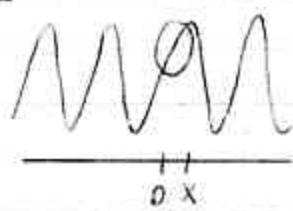
} looks like 1<sup>st</sup>  
neglected term in  
M'series, except  $0 \rightarrow \zeta$ .

where  $\zeta$  is some # betw.  $0, x$ .



## 11.9: Bound for $|R_n(x)|$

Idea



If  $|f^{(n+1)}|$  can be high,  
 $\Rightarrow |R_n(x)|$   
(Your poly. approx may  
"fly off" pretty quickly.)

If  $f^{(n+1)}$   
doesn't exist,  
see my notes on  
① L11-386.

If  $f, f', f'', \dots$  exist on  $J$ ,  
and  $R_n(x) \rightarrow 0$  for every  $x$  in  $J$ ,  
then  $f(x)$  is rep. by its M. series on  $J$ .

$$\left( \text{must be } \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \right)$$

Can  $J$  be  $I$ ? "Frequently"  
T. series at  $c$ : similar.

ROSS 11.850  
 $f$ , not always  
true for many functions.

Ex<sup>2</sup> (p.584) Prove:  $f(x) = \sin x$  is rep. by its M. series for all real  $x$

$$|R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right|$$

$$f^{(n+1)}(z) = \pm \cos z \\ \text{or } \pm \sin z$$

$$\Rightarrow |f^{(n+1)}(z)| \leq 1 \quad (\text{for any real } z)$$

Books: Stewart?  
 $e^x = \sum \frac{x^n}{n!} \quad \forall x$   
 $\Rightarrow \left( \frac{x^n}{n!} \rightarrow 0 \right)$

$$\Rightarrow |R_n(x)| \leq \underbrace{\left| \frac{x^{n+1}}{(n+1)!} \right|}_{\substack{\downarrow \\ 0}} \quad \downarrow \quad 0 \quad (\text{for any real } x)$$

$$\Rightarrow R_n(x) \rightarrow 0 \quad (\text{for any real } x)$$

$\sin \text{ odd} \Rightarrow 2n+1$   
 conv. breal  $x \neq 0$   
 periodic  $\Rightarrow (-1)^n$

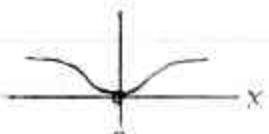
$$\therefore \text{M series} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Can plug in  $x=1/M$

represents  $\sin x$  for all real  $x$ .

Ross 181: Such  
 function vital to  
 theory of dist<sup>n</sup>s  
 related to  
 abv. diff eqs.  
 Fourier analysis

Ex 6 (pp. 586-7)  $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$



$$f^{(n)}(0) = 0 \quad (n=0, 1, 2, \dots)$$

Since there's no  
 interval containing 0, where valid  
 we say M. series not good.

M series = 0, valid only for  $x=0$ .  $R=0 \Rightarrow$  "No M Series"  
 $R_n(x) \rightarrow 0 \quad (x \neq 0)$

11.10: BINOMIAL SERIES (not on tests)

For  $(1+x)^k$   $\leftarrow$  real # (if whole #  $\Rightarrow$  get a poly from Binomial Thm. from Precalc),  $|x| < 1$ .

Why would we care  
about a  $\Sigma$  expansion  
for ...  
11.10.13

$$x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^n (2n+1)n!} x^{2n+1}$$

$$\text{Ex } [1 + (-x^2)]^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x^2}} = \sum \quad (\text{See #13.})$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$\uparrow$   
Get  $\Sigma$

( $\rightarrow$  fun w/ e<sup>x</sup> on Handout!)

REVIEW 11.6-8Basic M Series ( $\frac{1}{1-x}$ , and p. 587)

Know: 1st 4 non-0 terms

M series in  $\sum_{n=0}^{\infty}$  formI; it's  $(-\infty, \infty)$  unless otherwise specified (①, ⑦, ⑧)No " $1^n$ "  
in denom.

$$\textcircled{1} \quad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n, \quad (\text{Basic Geom. S})$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

= sum of ratios  
of  $x^n$

Alt. S  $\forall x \neq 0$  if  $x < 0 \Rightarrow$   
 $-+ -+$   
 after plug-in

$$\textcircled{2} \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (\text{D}_x(e^x) = e^x)$$

$$\textcircled{3} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\textcircled{4} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Alt.  $\sum$ ,  $\sin x$  is odd,  $\cos x$  is even  
 (= makes periodicity possible)

$$\textcircled{5} \quad \sinh x = (\text{same as } \sin x, \text{ except all } "+") = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\textcircled{6} \quad \cosh x = (\text{same as } \cos x, \text{ except all } "-") = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

In 0 is runct,  
 so  $\ln x$  has no  
 M series

$$\textcircled{7} \quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

$$\text{or } \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

From  $\int \frac{1}{1+x} dx, C=0$ Remember: When  $x=1 \Rightarrow \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ 

$$\textcircled{I} = (-1, 1]$$

$\ln 2 \approx$

sum of alt. harm. S

$$\textcircled{8} \quad \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

(for all  $x \neq 0$ )Alt.  $\sum$ ,  $\tan^{-1} x$  is oddNo " $1^n$ ":  $\tan^{-1}(1)$  converges slowly to  $\frac{\pi}{4}$ .

$$\textcircled{I} = [-1, 1]$$

From  $\int \frac{1}{1+x^2} dx, C=0$

Work out as  
last resort  
①-⑦

Not good  
for ⑧ →

$$\text{M Series: } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \left. \begin{array}{l} \text{Pattern?} \\ \text{Where valid?} \\ \exists f^{(n)} (n=0,1,2,\dots) \\ R_n \rightarrow 0 \end{array} \right\}$$

$$\text{T Series: } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

centered at  $c$

### Manipulations

Sub

$$\text{Ex } \sin(x^2)$$

$$\text{Ex } \frac{x^3}{4+x^3} = \frac{x^3}{4} \cdot \frac{1}{1 - (-\frac{x^3}{4})}$$

$\downarrow$        $\underbrace{\qquad}_{\substack{\text{can later} \\ \text{in later}}} \Rightarrow \text{Geom. } \sum, \left| -\frac{x^3}{4} \right| < 1 \Leftrightarrow |x| < \sqrt[3]{4}$

$f'(x), \int f(x) dx$  have same  $R$  as for  $f$   
 $\sum a_n x^n$        $+ C$   
 $\underset{n=0 \rightarrow n=1}{\text{if this term is constant}}$  I up to endpoints (if any)

### 11.6 Power $\sum$ in general

Find  $I$ : For what  $x$  is "Ratio Test  $L < 1$ " or Root  
✓ endpts. separately.

$$\frac{\sum a_n x^n}{\sum a_n (x-c)^n}$$

$$R=0 \quad \textcircled{O} \text{ for } \{0\}$$

$$R=\infty \quad (-\infty, \infty)$$

$$R=d>0$$

$$\{-d, d\}$$

$$\textcircled{C} \text{ for } \{c\}$$

$$(-\infty, \infty)$$

$$\{c-d, c+d\}$$

$$\text{Ex Solve } 3|x-2| < 1$$