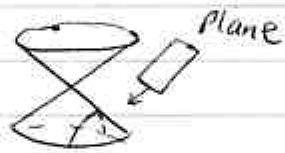
## 12.1: PARABOLAS



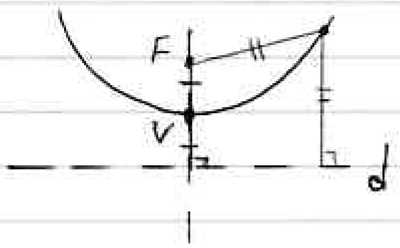

Rt. circ. cone

### A Locus Def'n

{ all pts. in fx equidistant from

1 a fixed line (directrix, $d$)
and 2 a fixed pt. (focus, $F$) }
not on $d$

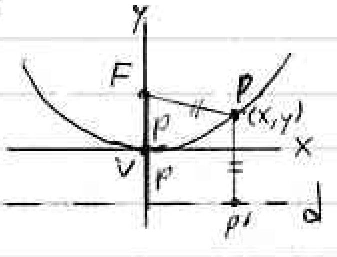
Ex




V: Vertex

### B If V is (0,0)

1 $y = ax^2 \ (a > 0)$




$\boxed{p = \frac{1}{4a}}$ for 1-4

Why?

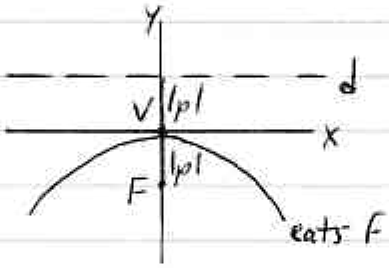
$d(P,F) = d(P,P')$

$\Rightarrow y = \left(\frac{1}{4p}\right)x^2$

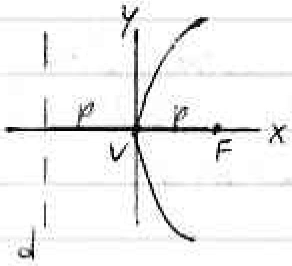
a

after some work (see book)

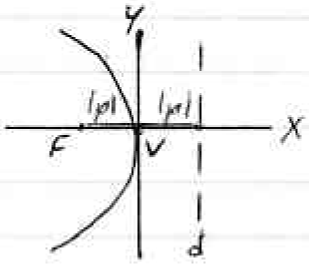
② $y = ax^2$ $(a < 0)$



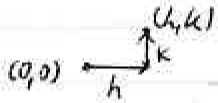

③ $x = ay^2$ $(a > 0)$




④ $x = ay^2$ $(a < 0)$




ⓒ If V is $(h, k)$




Replace $x$ w/ $(x-h)$
$\quad\quad\;\; y \quad\; (y-k)$ } Translations

①,② $y = ax^2 \rightarrow (y-k) = a(x-h)^2$

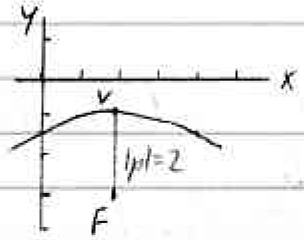
Completing the Square $\overset{CTS}{\Leftrightarrow}$ $y = ax^2 + bx + c$ $(a \neq 0)$

③,④ $x = ay^2 \Rightarrow (x-h) = a(y-k)^2$

$\Leftrightarrow x = ay^2 + by + c$ $(a \neq 0)$

① Ex

Find an eq. of the parabola w/
V: (2, −1), F: (2, −3).
h k




Case ② ∩

$(y-k) = a(x-h)^2$
$y-(-1) = a(x-2)^2$
$y+1 = a(x-2)^2$
?

$p = -2$ (∩)
$a = \frac{1}{4p}$
$= \frac{1}{4(-2)}$
$= -\frac{1}{8}$

$$\boxed{\begin{aligned} y+1 &= -\tfrac{1}{8}(x-2)^2 \\ \text{or} \quad y &= -\tfrac{1}{8}x^2 + \tfrac{1}{2}x - \tfrac{3}{2} \end{aligned}}$$

## Ⓔ Find V

 $\frac{dy}{dx}=0$ 

$$y = \underbrace{ax^2+bx+c}_{f(x)} \quad (a \neq 0)$$

$\frac{d^2y}{dx^2} = 2a$
$a>0$ CU
$a<0$ CD

$$\frac{dy}{dx} = 2ax + b \overset{\text{set}}{=} 0 \qquad \text{(Critical \# Idea)}$$

$$2ax = -b$$

$$x = -\frac{b}{2a} \quad \text{(in QF)}$$

$$\boxed{V: \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)}$$

$\frac{dx}{dy}=0$ 
$\left(\frac{dy}{dx} \text{ und.}\right)$

$$x = \underbrace{ay^2+by+c}_{f(y)} \quad (a \neq 0)$$

$$\frac{dx}{dy} = 2ay + b \overset{\text{set}}{=} 0$$

$$\vdots$$

$$y = -\frac{b}{2a}$$

$$\boxed{V: \left(f\left(-\frac{b}{2a}\right), -\frac{b}{2a}\right)}$$

F Ex

$$2y^2 - 4y + 6x - 10 = 0$$

Find V, F, d. Sketch graph.

Solve for non-squared var. (here, x)

$$6x = -2y^2 + 4y + 10$$
$$x = \underbrace{-\tfrac{1}{3}y^2 + \tfrac{2}{3}y + \tfrac{5}{3}}_{f(y)}$$

Find V

$$y = -\frac{b}{2a} = -\frac{\frac{2}{3}}{2(-\frac{1}{3})} = 1$$

$$x = f(1)$$
$$= -\tfrac{1}{3}(1)^2 + \tfrac{2}{3}(1) + \tfrac{5}{3}$$
$$= 2$$

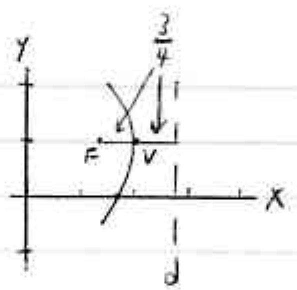
$$\boxed{V: (\overset{h}{\underset{x}{2}}, \overset{k}{\underset{y}{1}})}$$

Find F

$$x = -\tfrac{1}{3}y^2 \ldots \quad )$$

$$a = -\tfrac{1}{3}$$
$$p = \frac{1}{4a} = \frac{1}{4(-\frac{1}{3})} = -\frac{3}{4}$$

Which coord is diff?




$\boxed{V: (2, 1)}$

$F: (2 - \frac{3}{4}, 1)$

$(1\frac{1}{4}, 1)$

$\boxed{F: (\frac{5}{4}, 1)}$

$d: x = 2 + \frac{3}{4}$

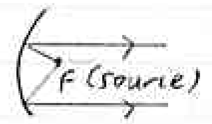
$\boxed{d: x = \frac{11}{4}}$

## ⑥ Applics

Rain

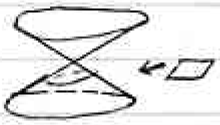
Road surfaces, Projectiles (Galileo)

Reflective prop.




Headlight ① Paraboloid

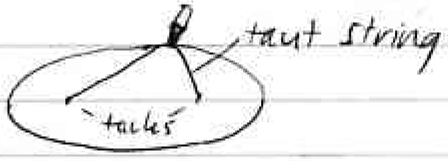
# 12.2: ELLIPSES



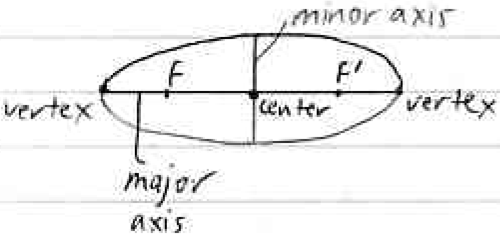
## (A) Locus Def'n

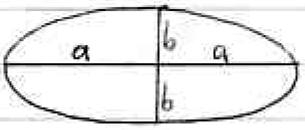
foh-sigh or leigh

2 foci


## (B) Terminology

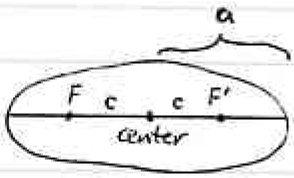



major axis = $2a$
minor axis = $2b$

$a > b > 0$

semimajor axis = $a$



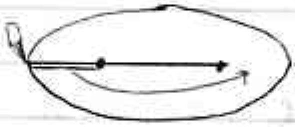

$$c^2 = a^2 - b^2$$

$c < a$

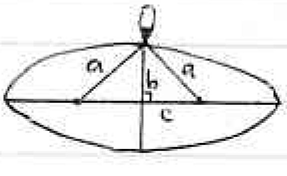
$c^2 > 0$

McKeague
not in book

Why?



length of string $= 2a$

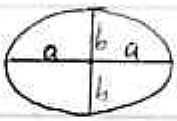



$$b^2 + c^2 = a^2$$
$$\Rightarrow \quad c^2 = a^2 - b^2$$

## Ⓒ If Center is (0,0)

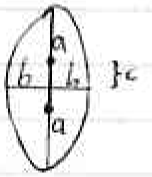
① $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ "x-long"

larger den.
under $x^2$




② $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ "y-long"

larger den.
under $y^2$



## (D) If Center is (h,k)

① ⇒ $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$


② ⇒ $\underbrace{\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2}}_{\text{Center makes this 0.}} = 1$


## (E) Eccentricity "e"

$\boxed{e = \dfrac{c}{a}}$


e=0 e=0.3 e=0.8

Stewart 692
$e = \dfrac{|PF|}{|Pl|}$
See 12.2.30

Ellipse: $0 < e < 1$
(Parabola: $e = 1$
Hyperbola: $e > 1$)

## (F) Ex

Graph $4x^2 + 27y^2 - 16x + 54y + 7 = 0$
Find the center, vertices, foci, and e.

Group terms

$$\underbrace{(4x^2-16x)}_{\substack{\text{Factor out}\\ \text{leading coeff.}=4}} + (27y^2+54y) = -7$$

$$4(x^2-4x\underbrace{+4}_{CTS}) + 27(y^2+2y\underbrace{+1}_{CTS}) = -7\underbrace{+4(4)+27(1)}_{\text{Be fair!}}$$

Factor!

$$4(x-2)^2 + 27(y+1)^2 = 36$$

Need 1
$\div$ thru by 36

$$\frac{\cancel{4}^{1}(x-2)^2}{\cancel{36}_{9}} + \frac{\cancel{27}^{3}(y+1)^2}{\cancel{36}_{4}} = 1$$

$$\frac{(x-2)^2}{9} + \frac{3(y+1)^2}{4} = 1$$

$$\underbrace{\frac{(x-2)^2}{9} + \frac{(y+1)^2}{\frac{4}{3}}}_{\boxed{\text{Center: }(2,-1)}} = 1$$

makes left side = 0

Helpful when find c

$a^2 = 9$ (larger)
$a = \pm 3$

$b^2 = \frac{4}{3}$
$b = \pm\sqrt{\frac{4}{3}}$
$b = \frac{2}{\sqrt{3}}$
$b = \frac{2\sqrt{3}}{3} \approx 1.15$

Find c

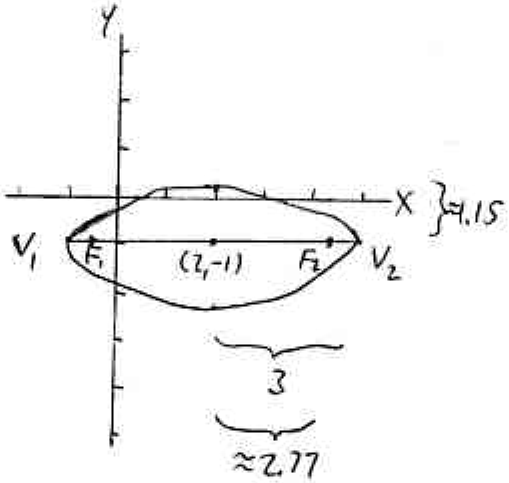
$$c^2 = a^2 - b^2$$
$$= 9 - \frac{4}{3}$$
$$= \frac{23}{3}$$
$$c = \sqrt{\frac{23}{3}}$$
$$c = \frac{\sqrt{69}}{3} \approx 2.77$$

Find e

$$e = \frac{c}{a}$$
$$= \frac{\frac{\sqrt{69}}{3}}{3}$$
$$\boxed{e = \frac{\sqrt{69}}{9} \approx 0.92}$$

Graph

"x stuff" has larger denom.




Fine-it looks like a lemon :)
What coord. changes?

$a = 3$
$b = \frac{2\sqrt{3}}{3} \approx 1.15$
$c = \frac{\sqrt{69}}{3} \approx 2.77$

$V_1: (2-3, -1)$
$\boxed{(-1, -1)}$

$V_2: (2+3, -1)$
$\boxed{(5, -1)}$

$F_1: \boxed{\left(2 - \frac{\sqrt{69}}{3}, -1\right)}$

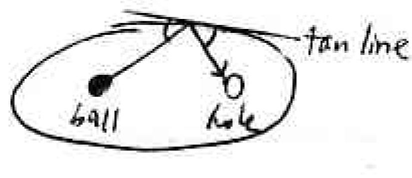
$F_2: \boxed{\left(2 + \frac{\sqrt{69}}{3}, -1\right)}$

can wait...

## (C) Applications

Reflecting Property

Pool Table




Sen. Sucher

Statuary Hall in U.S. Capitol

J.Q. Adams eavesdropped.

What happening surrounding tissue

Lithotripsy - destroying kidney stones


Kepler's Laws of Planetary Motion (1609)

small effect on temp. true!

Copernicus arched to revise calendar - not accurate w/ geocentric



(can have parabolic, hyperbolic orbit)

equal areas in equal times

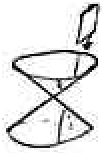
Larson 646

Comets disc. before 1970
482 parabolic
402 ell.
112 hyp.

Halley $e=0.97$
Kahoutek: $\approx .999925$

## 12.3: HYPERBOLAS

have 2 branches )(

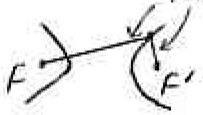


Manip.

### (A) Locus Def'n

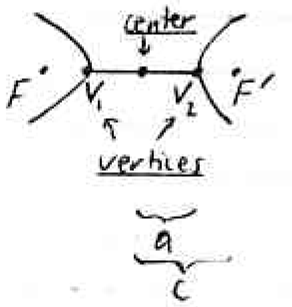
In an ellipse...

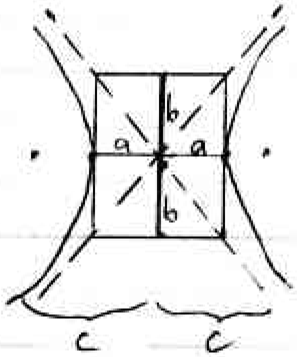
2 foci




### (B) Terminology

Some of you are




Transverse axis = 2a
Conjugate axis = 2b

$c^2 = a^2 + b^2$
Notice: $c > a$

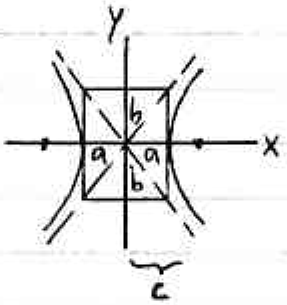
Scarface 1st you draw the box...

Draw: (1) Box
(2) Asymptotes
(3) Hyperbola

## Ⓒ If ... Center is (0,0)

① $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

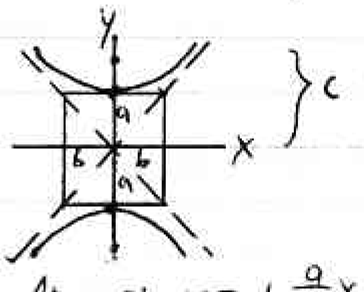
$a^2$ on left (not necessarily $> b^2$)




$\frac{\text{rise}}{\text{run}}$

Asyms: $y = \underbrace{\pm \frac{b}{a}}_{\text{slopes}} x$

② $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$




Asyms: $y = \pm \frac{a}{b} x$

## Ⓓ If ... Center is (h,k)

① $\Rightarrow \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$


Asyms: $y-k = \pm \frac{b}{a}(x-h)$

② $\Rightarrow \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

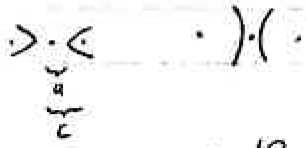

Asyms: $y-k = \pm \frac{a}{b}(x-h)$

## Ⓔ Eccentricity

$\boxed{e = \frac{c}{a}}$

Is $c>a$ or $c<a$?

$c > a$, so $e > 1$




$e = 2$ $\quad$ $e = 10$

Ⓕ Ex (#14)

Graph $y^2 - 4x^2 - 12y - 16x + 16 = 0$
Find center, vertices, foci, and asymptotes.

Group terms

$$(y^2 - 12y) + \underbrace{(-4x^2 - 16x)}_{\text{Factor out } (-4)} = -16$$

$$(y^2 - 12y \underbrace{+36}_{\text{CTS}}) - 4(x^2 + 4x \underbrace{+4}_{\text{CTS}}) = -16 \underbrace{+36 - 4(4)}_{\text{Be fair!}}$$

Factor!

$$(y-6)^2 - 4(x+2)^2 = 4$$

Need a "1"
÷ thru by 4

$$\frac{(y-6)^2}{4} - \frac{(x+2)^2}{1} = 1$$

$\boxed{\text{Center}: (\overset{x}{\underset{h}{-2}}, \overset{y}{\underset{k}{6}})}$

"y stuff" on left, so $a^2 = 4$

$a^2 = 4$ $\quad$ $b^2 = 1$
$a = \pm 2$ $\quad$ $b = \pm 1$

## Find c

$$c^2 = a^2 + b^2$$
$$= 4 + 1$$
$$= 5$$

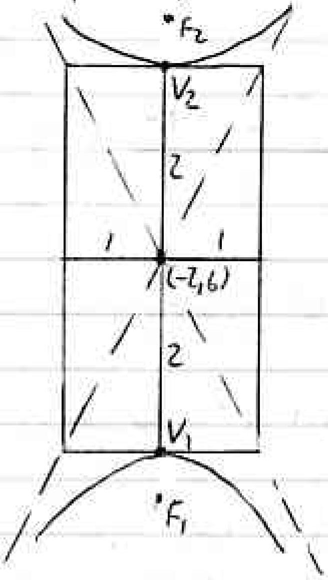
$$c = \sqrt{5}$$

## Asyms

$$y - k = \pm \frac{a}{b}(x - h)$$
$$\left(y - 6 = \pm \frac{2}{1}(x - (-2))\right)$$
$$\boxed{y - 6 = \pm 2(x + 2)}$$

## Graph Setup

$a = 2,\ b = 1,\ c = \sqrt{5} \approx 2.2$




$V_1$: $(-2, 6-2)$
$\boxed{(-2, 4)}$

$V_2$: $(-2, 6+2)$
$\boxed{(-2, 8)}$

$F_1$: $\boxed{(-2, 6-\sqrt{5})}$
$F_2$: $\boxed{(-2, 6+\sqrt{5})}$

## (6) Applics:

Intersect hyps
$\Rightarrow$ pinpoint

Radar

|diff| = const.
receives signal earlier

Telescopes CHW
Look at

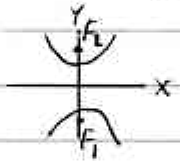
Reflecting property

(If there's time)

<u>12.3 #26</u>

<u>Ex</u> Find an eq. for the hyp. w/center at (0,0), foci at (0, ±10), and asyms. $y = \pm \frac{1}{3}x$

$c = 10$




$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Asyms.: $y = \pm \frac{a}{b} x$

$$\frac{a}{b} = \frac{1}{3}$$
$$b = 3a$$

How do I relate c to b and a?

$$c^2 = a^2 + b^2$$
$$(10)^2 = a^2 + (3a)^2$$
$$100 = a^2 + 9a^2$$
$$100 = 10a^2$$
$$a^2 = 10$$

$$c^2 = a^2 + b^2$$
$$100 = 10 + b^2$$
$$b^2 = 90$$

$$\boxed{\frac{y^2}{10} - \frac{x^2}{90} = 1}$$

## 12.4: ROTATION

Identification

### (A) ID Thm.

A, B, C matter
Can't switch A, C
$\cot(2\phi) = \frac{A-C}{B}$

The graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ may be a $\underbrace{\text{[rotated?]}}_{B \neq 0}$ conic. If it is, it's

Book: (1) ↔ (2)

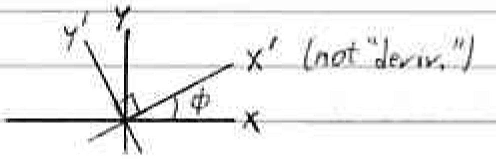
(1) an ellipse if $\Delta < 0$
(2) a parabola if $\Delta = 0$
(3) a hyperbola if $\Delta > 0$

Invariant under rotation

where the discriminant $\Delta = B^2 - 4AC$ (in QF)

### (B) If $B \neq 0$

Find an eq. in an $x'y'$-plane w/no cross-term ($x'y'$).




Ex (#4) $x^2 - xy + y^2 = 3$ (*) (conic)

Not necess, but good form
$x^2 - xy + y^2 - 3 = 0$

$A = 1, B = -1, C = 1$

ID Thm.

$\Delta = B^2 - 4AC$
$= (-1)^2 - 4(1)(1)$
$= -3$
$< 0 \Rightarrow$ ellipse

## Rotate axes

① Solve $\cot(2\phi) = \frac{A-C}{B}$ for $\phi$,

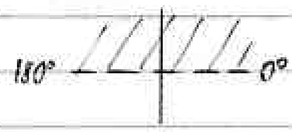
where $0° < 2\phi < 180°$

$\phi$ acute $\quad 0° < \phi < 90°$

Ex $\cot(2\phi) = \frac{1-1}{-1}$

$= 0$

$\tan(2\phi)$ is und.




$2\phi = 90°$

$\phi = 45°$

② Find $\sin\phi$, $\cos\phi$.

Ex $\sin\phi = \sin 45°$

$= \frac{\sqrt{2}}{2}$

$\cos\phi = \cos 45°$

$= \frac{\sqrt{2}}{2}$

(Do trig table)

## ③ Rotation of axes formulas

254:
Inv. matrix
$[- \ +]$

opposite from what you think

$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{Old coords.}} = \underbrace{\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}}_{\text{Rotation matrix}} \underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\text{New coords.}}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \begin{matrix} 2a+3b \\ 2c+3d \end{matrix}$$

$$x = x'\cos\phi - y'\sin\phi$$
$$y = x'\sin\phi + y'\cos\phi$$

Ex
$$x = \tfrac{\sqrt{2}}{2}x' - \tfrac{\sqrt{2}}{2}y'$$
$$y = \tfrac{\sqrt{2}}{2}x' + \tfrac{\sqrt{2}}{2}y'$$

or
$$x = \tfrac{\sqrt{2}}{2}(x' - y')$$
$$y = \tfrac{\sqrt{2}}{2}(x' + y')$$

## ④ Sub into Ⓐ, → Standard form

Ⓐ $x^2 - xy + y^2 = 3$

Book skips alg

$$\left[\tfrac{\sqrt{2}}{2}(x'-y')\right]^2 - \left[\tfrac{\sqrt{2}}{2}(x'-y')\right]\left[\tfrac{\sqrt{2}}{2}(x'+y')\right] + \left[\tfrac{\sqrt{2}}{2}(x'+y')\right]^2 = 3$$

Ignore ′ for now.

or factor $\frac{1}{2}$

$$\tfrac{1}{2}(x-y)^2 - \tfrac{1}{2}(x-y)(x+y) + \tfrac{1}{2}(x+y)^2 = 3$$
$$\tfrac{1}{2}(x^2-2xy+y^2) - \tfrac{1}{2}(x^2-y^2) + \tfrac{1}{2}(x^2+2xy+y^2) = 3$$
$$\cancel{\tfrac{1}{2}x^2} - \cancel{xy} + \tfrac{1}{2}y^2 - \cancel{\tfrac{1}{2}x^2} + \tfrac{1}{2}y^2 + \tfrac{1}{2}x^2 + \cancel{xy} + \tfrac{1}{2}y^2 = 3$$
$$\tfrac{1}{2}x^2 + \tfrac{3}{2}y^2 = 3$$
$$\frac{x^2}{6} + \frac{y^2}{2} = 1$$

Put in $'$

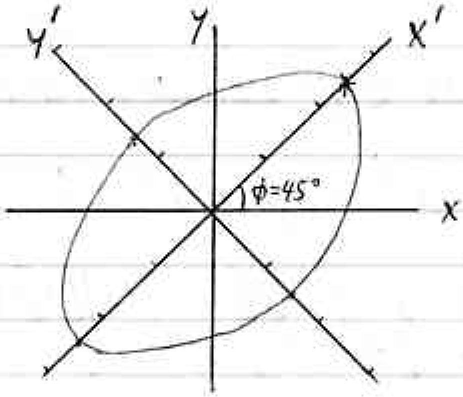
$$\frac{(x')^2}{6} + \frac{(y')^2}{2} = 1$$

## ⑤ Graph in $x'y'$-plane

Center: $(0,0)$

$x'$-long $\quad a^2 = 6$
$\quad\quad\quad\quad a = \sqrt{6} \approx 2.4$

$b^2 = 2$
$b = \sqrt{2} \approx 1.4$




(Optional) What's * in $(x,y)$?

Rotation matrix perfectly suited

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{array}{c} \begin{bmatrix} \sqrt{6} \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow x' \\ \leftarrow y' \end{array}$$

$$= \begin{bmatrix} \sqrt{3} \\ \sqrt{3} \end{bmatrix}$$

REVIEW CH. 12

(12.4) $\underbrace{Ax^2 + Bxy + Cy^2} + Dx + Ey + F = 0$

$\Delta = B^2 - 4AC$

$<0$ ell.
$=0$ par.
$>0$ hyp.

If $B \neq 0$

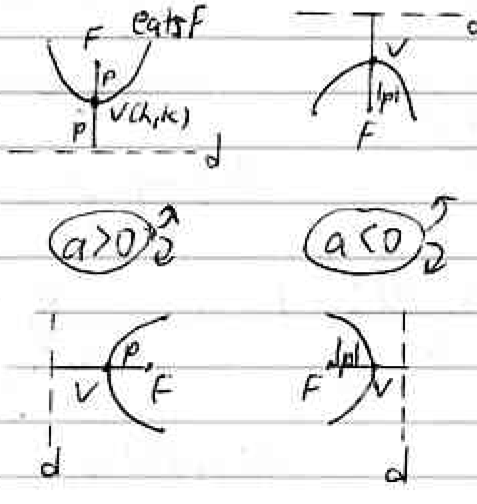
$\cot(\underbrace{2\phi}_{\text{in } (0°,180°)}) = \frac{A-C}{B}$

$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{Old}} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\text{New}}$$

$\Rightarrow \begin{cases} x = x'\cos\phi - y'\sin\phi \\ y = x'\sin\phi + y'\cos\phi \end{cases}$ Sub

$\Rightarrow$ Standard form (12.1–12.3)

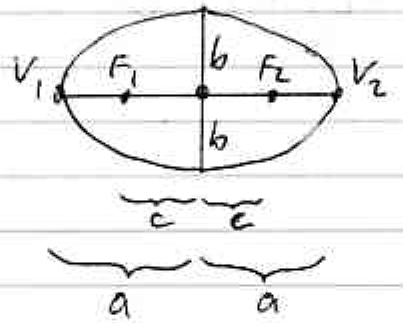
(12.1) Par.




$(y-k) = a(x-h)^2$
$a = \frac{1}{4p}$
$y = ax^2 + bx + c$ $\quad y = f(x)$
$V: \left(-\frac{b}{2a}, f(\ )\right)$

$(x-h) = a(y-k)^2$
$x = ay^2 + by + c$ $\quad x = f(y)$
$V: \left(f(\ ), -\frac{b}{2a}\right)$

## 12.2 Ell.




$c^2 = a^2 - b^2$

$$CTS \Rightarrow \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

larger denom.


$$\text{or} \quad \frac{(\quad)}{b^2} \quad \frac{(\quad)}{a^2} = 1$$

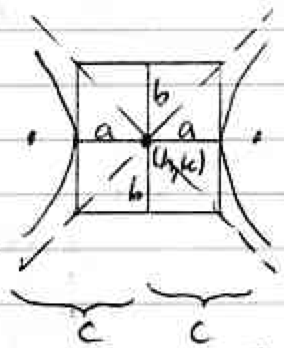
larger


$e = \frac{c}{a}$

## 12.3 Hyp.

$$CTS \Rightarrow \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

on left




$c^2 = a^2 + b^2$

Asyms: $y-k = \pm \frac{b}{a}(x-h)$

$$\text{or} \quad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$


Asyms: $y-k = \pm \frac{a}{b}(x-h)$