

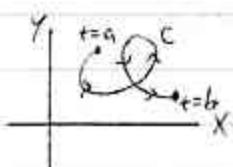
CH.13: PLANE CURVES, POLAR COORDS.

Know your precalc!

13.1: PLANE CURVES

Fails VCT, MLT

Ex



Parametric eqs. for C

Show this in related rates.  
need not be closed  
As  $t \rightarrow b$ ,  
 $\rightarrow$

Wizard of Oz -  
Man behind curtain  
(invisible)

$t$  is a parameter  
(underlying indep. var.).  
Think "time."

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad \begin{array}{l} \text{if } (a, b) \text{ cl.} \\ \text{q cont. for all } t \\ \text{in } [a, b] \end{array}$$



We can:

- ① describe complicated curves ("Maybe can't say  $y=f(x)$ ,  $x=f(y)$ .)
- ② trace motion along C:  
direction (orientation)  
speed

Ex  $\begin{cases} x = -t \\ y = 2t \\ -1 \leq t \leq 1 \end{cases}$

Draw C.

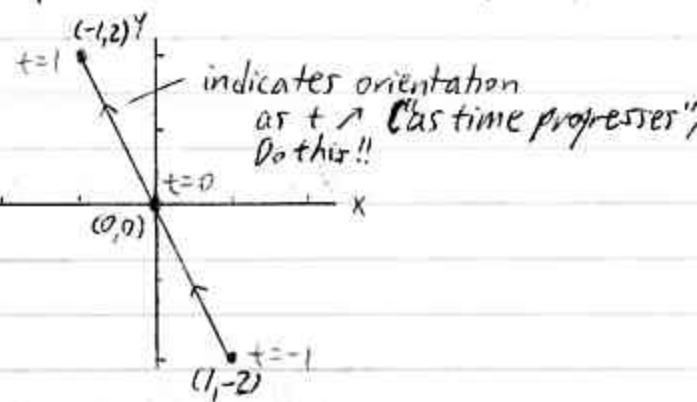
(should not be your only tool; assumes "simplicity")

Tool #1

Point - plotting

$t$	$x$	$y$
-1	1	-2
0	0	0
1	-1	2

Should find pts. corresp. to endpoints of  $t$ -interval and maybe  $t=0$



Tool #2

Analyze sign of  $\frac{dx}{dt}$  and/or  $\frac{dy}{dt}$  (helps w/ orientation)

$$\frac{dx}{dt} = -1 \Rightarrow x \downarrow \text{wrt } t \quad (-1 \leq t \leq 1)$$

$$\frac{dy}{dt} = 2 \Rightarrow y \uparrow \text{wrt } t \quad (-1 \leq t \leq 1)$$

and restrictions

with respect to

We go up and to the left (UFK).

Tool #3

Eliminate the parameter "t" (ETP)

$$\begin{aligned} x = -t &\xrightarrow{\text{① Solve for } t} t = -x \\ y = 2t &\xrightarrow{\text{② Sub}} y = -2x \end{aligned}$$

Cartoon 652

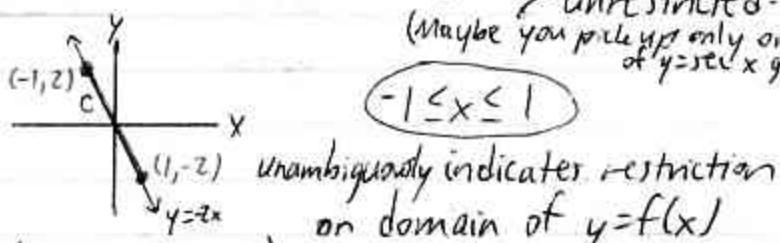
When we go from  
par. eqs.

A rectangular eq. (Helps us draw C)

Its graph includes C

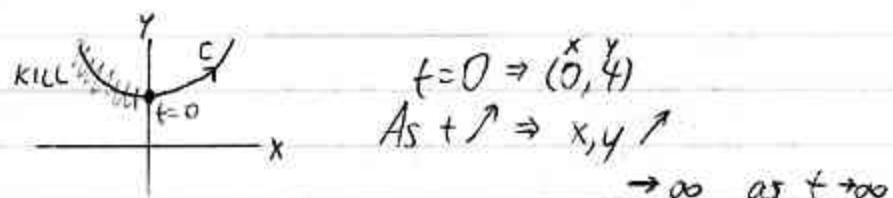
BUT Lose sense of motion, and

③ We may need to restrict the graph (even if t is unrestricted:  $t \in (-\infty, \infty)$ ).  
(Maybe you pick up only one "U" of  $y = \sec x$  graph.)



(Also:  $x = -\frac{1}{2}y$ ,  $-2 \leq y \leq 2$ )

Ex (#8)  $\begin{cases} x = \sqrt{t} \\ y = 3t + 4 \\ t \geq 0 \end{cases} \xrightarrow{\text{ETP}} \begin{array}{l} t = x^2 \\ y = 3x^2 + 4 \end{array}$  restriction imposed by "x =  $\sqrt{t}$ "



Tool #4

Tool #5

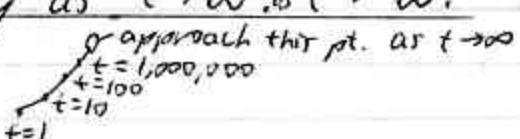
Tool #6

Keep restrictions on x, y in mind (Sign, extreme values)

(Related to Tool #2): As  $t \uparrow$ , does  $x \uparrow$  or  $\downarrow$ ?  $y \uparrow$ ?

What happens to x, y as  $t \rightarrow \infty$ ? as  $t \rightarrow -\infty$ ? when?

Maybe we have:

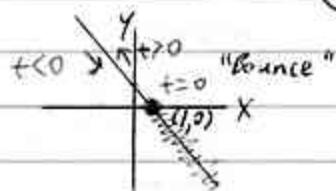


Ex (#26b)  $x = 1 - t^2 \Rightarrow x = 1 - y \Rightarrow (x + y = 1)$

ETP  
 $y = t^2$   
 $t \in \mathbb{R}$

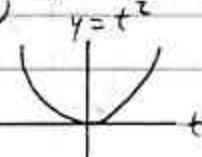
Sub in!  
why solve for  $t$ ?

Use fingers  
to trace  
graphs  
simultaneously.



Tool #7

Graph  $x$  and/or  $y$  against  $t$ .



Notice:  
 $y \geq 0$  (for all real  $t$ )

Tool #2:  $\frac{dy}{dt} = 2t$

$\begin{cases} < 0 & \text{for } t < 0 \\ = 0 & \text{at } t = 0 \\ > 0 & \text{for } t > 0 \end{cases}$	$\begin{cases} y \rightarrow -\infty & \text{as } t \rightarrow -\infty \\ y = 0 & \text{at } t = 0 \\ y \rightarrow \infty & \text{as } t \rightarrow \infty \end{cases}$
--	--

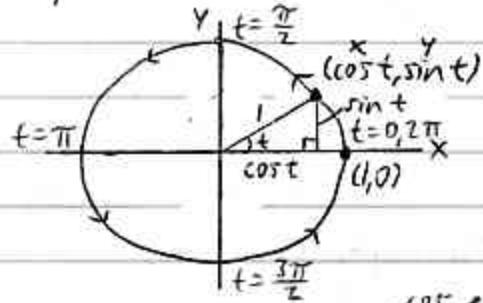
Graphing  $\frac{x}{t}$  can be awkward,  
because  $x$  corresponds to vertical words.)

Don't use  $\pm$   
implies  $\neq$

Exs ( $t \in \mathbb{R}$ )

①  $x = \cos t$  (Think:  $t$  in radians.)  
 $y = \sin t$

Connects  
unit circle,  
 $\text{rt.}-\text{A}/\text{SOH-CAH-TOA}$   
approaches to  
trig.

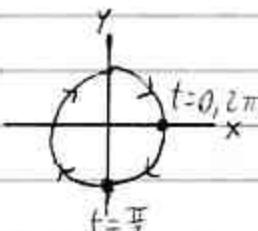


If you just want one revolution,  
you can restrict  $t$ :  $0 \leq t < 2\pi$ .

$t \rightarrow -t$   
reverse time  
always  
reverses orientation!

②  $x = \cos(-t)$   
 $y = \sin(-t)$

$$\begin{aligned} &\stackrel{\text{cos even}}{=} \cos t \\ &\stackrel{\text{sin odd}}{=} -\sin t \end{aligned}$$



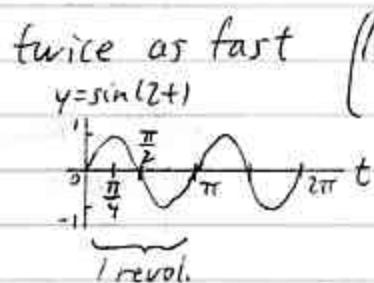
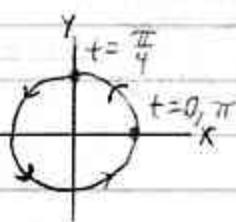
We reverse orientation,  
(like "reversing time")

$t \rightarrow 2t$   
always twice  
as fast?

How does graph  
differ? What changes? Period!  
lowest t value  
of t that takes  
us here

$$\textcircled{3} \quad x = \cos(2t)$$

$$y = \sin(2t)$$



twice as fast (Note:  $\frac{dx}{dt} = -2\sin(2t)$ )

$$y = \sin(2t)$$

$$\frac{dy}{dt} = 2\cos(2t)$$

↑ reflect doubling  
of speed  
relative to \textcircled{1}

\textcircled{1}-\textcircled{3} have same rect. eq.!

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$(x^2 + y^2) = 1$$

Cover \textcircled{1}-\textcircled{3}  
what's different  
about them  
compared to  
\textcircled{0}-\textcircled{3}?

$$\textcircled{4} \quad x = 2 \cos t + 1 \Rightarrow \frac{x-1}{2} = \cos t$$

$$y = 2 \sin t - 3 \Rightarrow \frac{y+3}{2} = \sin t$$

New radius (resizes) New center (translates)

$$\cos^2 t + \sin^2 t = 1$$

$$\left(\frac{x-1}{2}\right)^2 + \left(\frac{y+3}{2}\right)^2 = 1$$

$$\frac{(x-1)^2}{4} + \frac{(y+3)^2}{4} = 1$$

$$(x-1)^2 + (y+3)^2 = 4$$

standard form for Eq. of a circle in  $\mathbb{R}^2$ :  
 $(x-h)^2 + (y-k)^2 = r^2$

Circle w/ center  $(h, k)$ , radius  $r$  ( $r > 0$ )



Circle  
 $r=2$

Be prepared  
to reverse  
this  
process  
in  
Calc II,  
III..

Memorize:  
 $x = r \cos t + h$   
 $y = r \sin t + k$

Tool #8

Use template!

$(h, k)$

radius

Thought process:  
 $\frac{1}{x} = \sec^2 t$   
 $(\frac{1}{x})^2 = \sec^4 t$

Tool #9

See HW #29 for ellipses.  
(Section 12.2.)  $\left( \begin{matrix} x & \frac{b}{a} \\ y & 0 \end{matrix} \right)$  or  $\left( \begin{matrix} a & b \\ 0 & y \end{matrix} \right)$  ( $a > b$ ) L13-3  
You may want to use an ID or some other relationship connecting  $x$  and  $y$ . semimajor axis 13.1

(can do  
HW 13.1)

$$\text{Ex } x = \cos^3 t \quad y = \sec^6 t \Rightarrow y = \frac{1}{x^2}, \text{ since } \sec^6 t = \frac{1}{(\cos^3 t)^2} = \frac{1}{\cos^6 t}$$

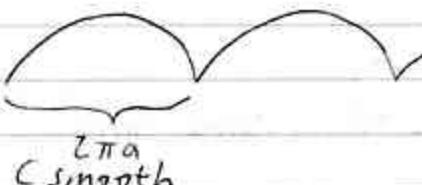
## Cycloids

Speed doesn't matter.  
Param. determines speed.

Move  $\rightarrow$



P traces out:



$$x = a(t - \sin t)$$
$$y = a(1 - \cos t)$$

Inverted arch of cycloid  $\curvearrowleft$  solves both the

### ① Tautochrone Problem

tautology  
redundancy

Gr.: tautos = identical  
chronos = time

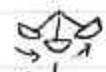
Larson 658  
Galileo noticed  $\rightarrow$   
Huygen designed  
constructed  
pen. clock



-faster  
-slower

$\approx$  time to complete a full swing

Pendulum clock



(boost to get same arc on return,)  
(so  $\approx$  not a problem)

What curve  $\Rightarrow$  time for each full swing?

M  
inv. cycloid

Swing:  
A at origin  
B at bottom

$\frac{1}{2}$  swing?

## brachistochrone

Solved by  
John Bernoulli  
in 1696 Newton  
Leibniz  
N, L, C, H,  
John James B  
all solved

## ② Brachistochrone Problem

History

Gr: brachys = short  
chronos = time

Solve:

A at origin

Caron, st.  
Breed not be  
at bottom

People on a slide.  
Use fingers to  
trace A  $\rightarrow$  B  
simult-w/C  $\rightarrow$  B.

A. What path minimizes the time  
to A  $\rightarrow$  B?

In fact,



A  $\rightarrow$  B

C  $\rightarrow$  B in same time!

(A, C as people on a  
slide; A can go fast.)

Stewart 50, 4

Bézier curves used to rep. letters in laser printers.

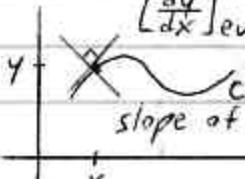
Section  
Title:  
Tangent Line and Arc length

## 13.2: (PLANE CURVES and CALCULUS)

Assume  $C$  is smooth.

$\frac{dx}{dt}, \frac{dy}{dt}$  cont.  
never both 0  
(maybe at endpts.)

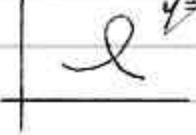
## (A) Tangent, Normal Lines

$y$  |   $\left[ \frac{dy}{dx} \right]_{\text{eval at } (x,y)} = \text{slope of tan line to } C \text{ at } (x,y)$   
 $x$  | slope of normal line =  $- \frac{1}{\frac{dy}{dx}} = - \frac{dx}{dy}$  (see 7.1, p. 380)  
 (neg recip.)

"Find  $\frac{dy}{dx}$ "  
What if

① hard to ETP  $\Rightarrow y = f(x)$

$$f(x, y) = c$$

②   $y = f(x)$  (Imp. Diff.?)

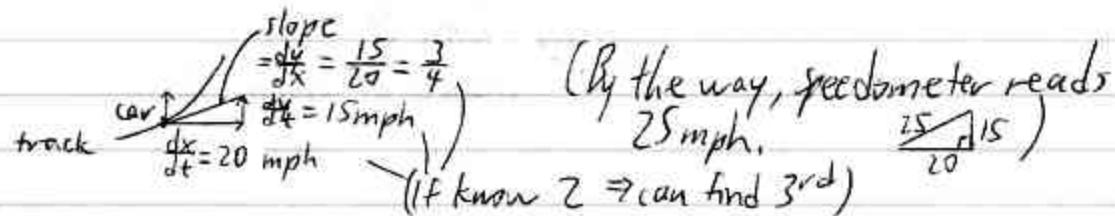
③ hard to find  $f'(x)$

What can we do?

$\frac{dy}{dx}$  what?

Chain Rule  $\Rightarrow \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$  (if exist)

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad (\leftarrow \text{if } \neq 0)$$



$$\text{Let } y' = \frac{dy}{dx}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

Where  $m=4$ ?  
is #10 in HW  
set

Ex  $x = t^2 + t$   
 $y = 5t^2 - 3$   
 $t \in \mathbb{R}$

① Find slope of tan line at pt. where  $t=1$ .

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{10t}{2t+1}$$

$$\left[ \frac{dy}{dx} \right]_{t=1} = \frac{10(1)}{2(1)+1}$$

$$= \boxed{\frac{10}{3}}$$

b) Find slope of normal line there.  $-\frac{3}{10}$  (neg. reciprocal)

c) Find pts. where tan line is horiz., vert.

We assumed  
there were cont.  
(always  
defined)

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \stackrel{\text{set } t=0}{=} \frac{10t}{2t+1}$$

(we assumed  
never simultaneously  
are cont.)

$$\text{Horiz: } 10t = 0 \\ t = 0$$

$$\Rightarrow x = (0)^2 + (0) = 0 \\ y = S(0)^2 - 3 = -3$$

$$\boxed{(0, -3)}$$

$$\text{Vert.: } 2t + 1 = 0 \\ t = -\frac{1}{2}$$

$$\Rightarrow \boxed{\left(-\frac{1}{4}, -\frac{7}{4}\right)}$$

d) Find  $\frac{d^2y}{dx^2}$

$$= \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

?

upto17

$$= \frac{\frac{10}{2t+1}}{2t+1}$$

$$= \boxed{\frac{10}{(2t+1)^3}}$$

$$y' = \frac{dy}{dx} = \frac{10t}{2t+1}$$

$$\frac{dy'}{dt} = D_t(y')$$

$$= \frac{(2t+1)(10) - (10t)(2)}{(2t+1)^2}$$

$$= \boxed{\frac{10}{(2t+1)^2}}$$

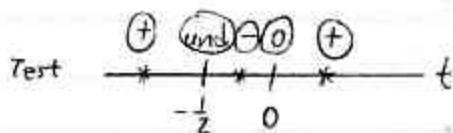
(optional)

e) Graph

$$H.Tan \text{ at } (0, -3) \quad (t=0)$$

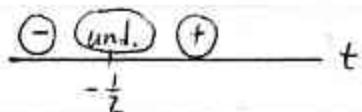
$$V.Tan \text{ at } (-\frac{1}{4}, -\frac{7}{4}) \quad (t=-\frac{1}{2})$$

$$\frac{dy}{dx} = \frac{10t}{2t+1}$$



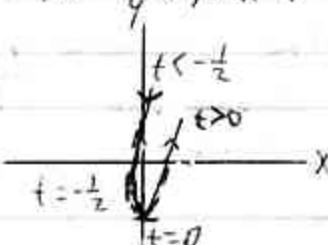
Curve  $\nearrow \searrow \nearrow$  with respect to  $x$  (not  $t$ )

$$\frac{d^2y}{dx^2} = \frac{10}{(2t+1)^3}$$



Curve  $\overset{CD}{\curvearrowleft} \overset{CU}{\curvearrowright}$ .

Look  $\rightarrow$  for  $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ .



Observe:

As  $t \rightarrow -\infty, t \rightarrow \infty$

$$x = t^2 + t \quad x \rightarrow \infty \quad x \rightarrow \infty$$

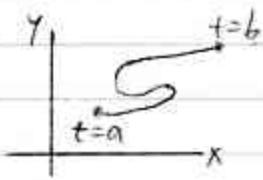
$$y = 5t^2 - 3 \quad y \rightarrow \infty \quad y \rightarrow \infty$$

$\left( \begin{array}{l} \text{Couldn't do in 150.} \\ \text{We don't have } y=f(x) \text{ or } x=f(y) \\ \text{unless we decompose the graph.} \end{array} \right)$

$x\text{-int: } -0.2, 1.4$   
 $y\text{-int: } 5, -3$   
 why not  $t = -\frac{1}{2}$ ?  
 ① At  $t \rightarrow \pm\infty$ ,  
 ② How get vert. tangent?  
 $\lim_{t \rightarrow 1/2} t$

## (B) Arc length "L"

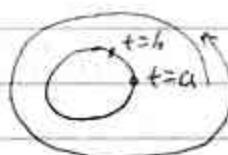
Smooth  $C$ :  $x = f(t)$   
 $y = g(t)$   
 $t \in [a, b]$



Books say no self-intersections, but isn't  $\ell$  OK?

No self-overlaps

NO



$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

In time  $dt = dt$

$$\frac{ds}{dt} = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}$$

$$(ds)^2 = (dx)^2 + (dy)^2 \\ \Rightarrow \sqrt{(dx)^2 + (dy)^2} dt = ds$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$= \frac{ds}{dt}$   
from Pyth. Thm.

### Special Case

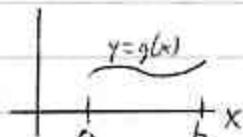
If  $x = t$   
 $y = g(t) = g(x)$

$$L = \int_{x=a}^{x=b} \sqrt{1 + [g'(x)]^2} dx \quad (\text{M150})$$

What's the deriv.  
of  $t$  wrt  $t$ ?

$$\frac{dt}{dt} = g'(t) = g'(x)$$

$\tilde{\text{No}}$  one remembers...  
In that corner...



Ex Find the length of

$$\begin{aligned} \text{Like H24, but in } [0, \pi] \\ t \in [0, \pi] \\ x = \cos^2 t - \sin^2 t \\ y = 2\sin t \\ \Rightarrow x + 2y = 1 \end{aligned}$$

$\rightarrow$  self-overlap!

$\checkmark$  Sol's manual is wrong.

On HW, assume no self-overlap

$$C: \begin{cases} x = \cos(2t) \\ y = \sin^2 t \\ 0 \leq t \leq \frac{\pi}{2} \end{cases}$$

Draw C (not on HW for arc length)

$$x = \cos^2 t - \sin^2 t \Rightarrow x + 2y = \cos^2 t + 2\sin^2 t = 1$$

(ex use PRTI:  $\sin^2 t = \frac{1-\cos(2t)}{2}$ )

$y = \frac{1-x}{2}$  (careful!  $x = \cos(2t)$ )

(If draw C  
→ could use  
distance formula  
to get L.)

$$L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

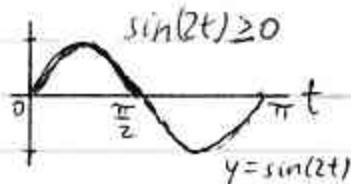
$$\begin{aligned} x &= \cos(2t) \\ \frac{dx}{dt} &= -2\sin(2t) \end{aligned}$$

$$\begin{aligned} y &= \sin^2 t \\ &= (\sin t)^2 \\ \frac{dy}{dt} &= 2\sin t \cos t \\ &= \sin(2t) \end{aligned}$$

$$= \int_0^{\pi/2} \sqrt{[-2\sin(2t)]^2 + [\sin(2t)]^2} dt$$

$$= \int_0^{\pi/2} \sqrt{5\sin^2(2t)} dt$$

$$= \sqrt{5} \int_0^{\pi/2} |\sin(2t)| dt$$



$$= \sqrt{5} \int_0^{\pi/2} \sin(2t) dt$$

$$= \sqrt{5} \left[ -\frac{1}{2} \cos(2t) \right]_0^{\pi/2}$$

$$= \sqrt{5} \left[ -\frac{1}{2} \underbrace{\cos \pi}_{(-1)} + \frac{1}{2} \underbrace{\cos 0}_{(1)} \right] \quad \text{if } \boxed{1}$$

$$= \boxed{\sqrt{5}} \quad (\text{Easier: } \quad )$$

Carron 615  
Cycloid arch = 8a  
Calc by Christ  
Wren in 1658  
(Architect, mathematician)  
rebuilt many  
bridges, churches in  
London, incl. St.  
Paul's Cathedral

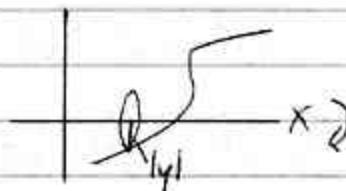
(SKIP)

## ① Surface Area "S"

Smooth C:  $x = f(t)$   
 $y = g(t)$   
 $t \text{ in } [a, b]$

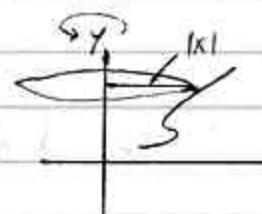
No self-overlaps

We S circumferences w/r respect to arc length.



$$S = \int_{t=a}^{t=b} 2\pi |y| ds$$

$$= \int_a^b 2\pi |g(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

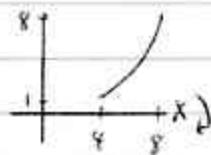


$$S = \int_{t=a}^{t=b} 2\pi |x| ds$$

$$= \int_a^b 2\pi |f(t)| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Ex (#30) Find  $S$  if

$$\begin{aligned} C: \quad & x = 4t \quad \Rightarrow \frac{dx}{dt} = 4 \\ & y = t^3 \quad \Rightarrow \frac{dy}{dt} = 3t^2 \\ & 1 \leq t \leq 2 \end{aligned}$$



$$S = \int_1^2 2\pi |y| \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= \int_1^2 2\pi |t^3| \sqrt{(4)^2 + (3t^2)^2} dt$$

$$= \int_1^2 2\pi t^3 \sqrt{16 + 9t^4} dt$$

$$u = 16 + 9t^4$$

$$\therefore \\ \approx 220.94$$

(See Ex. 5 on pp. 656-7 for  $S$  for a sphere.)

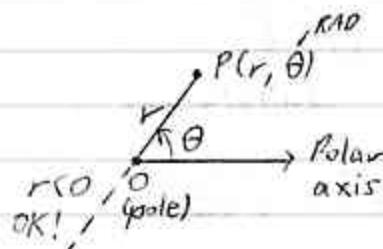
Larson-671  
James Bern.  
intro PG 5 in 1691  
but Newton  
may have used it  
Lil 361: 1st suggested  
by Viète (157)

L13-8  
13.3

### 13.3: POLAR COORDS (PCs)

#### (A) PCs

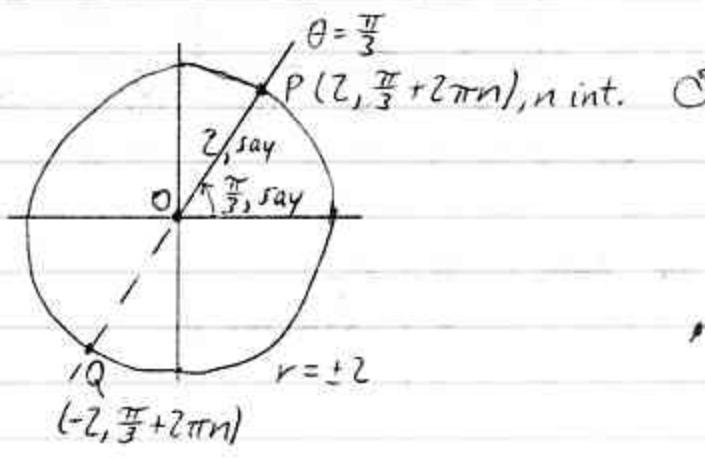
- I've buried a treasure chest. You can assume 2 rs.
- Rect/Lart. ...?
- Michael Galton fix.



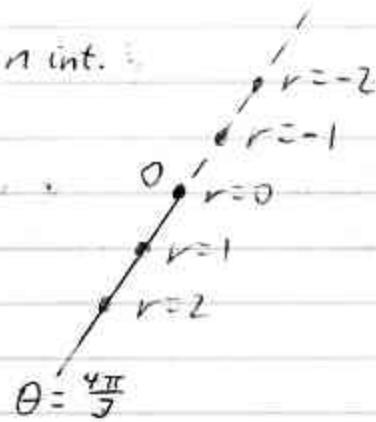
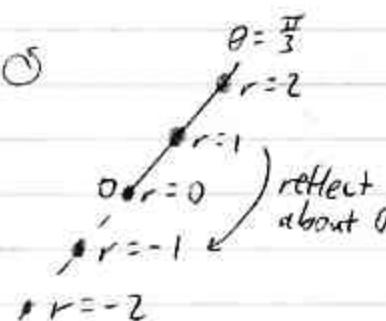
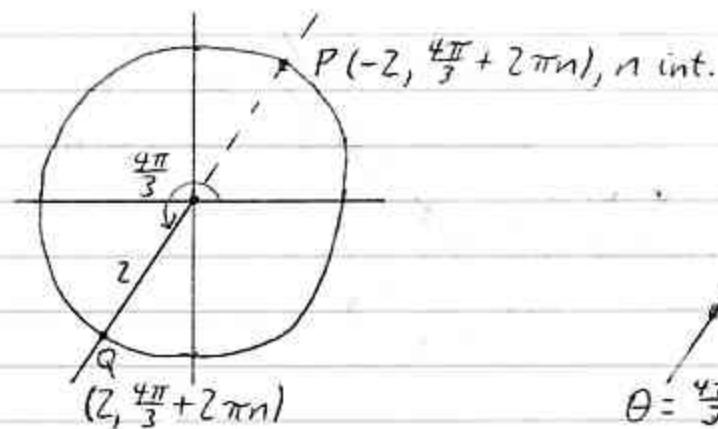
Pole O:  $(0, \theta)$   
*'any angle'*

P has  $\infty$  many PC reps.

Hand 1  
graph  $r=2$ ?  
what pts have  
 $r=2$ ?  
What figure does  
they form?  
This idea  
confused them!



or



## B) Graph of a Polar Eq.

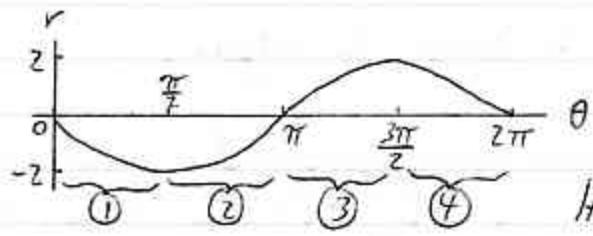
consists of all pts.  $(r, \theta)$   
Satisfy eq.

Usual form:  $r$  or  $r^2 = f(\theta)$

Ex (#6)  $r = -2 \sin \theta$

Stewart 673  
"in Cartesian words"

Graph  $r$  vs.  $\theta$  as Cartesian/rectangular coords.



$$r: 0 \rightarrow -2 \rightarrow 0 \rightarrow 2 \rightarrow 0$$

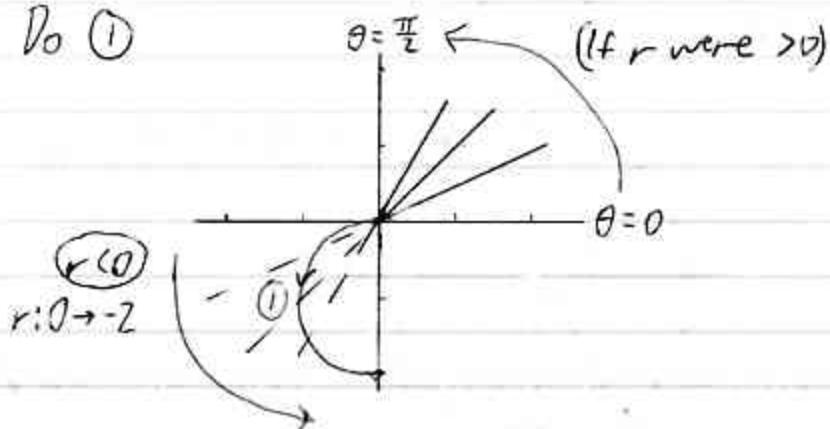
Here, "sectors" corresp.  
to quadrants, but  
be careful!

or Table:

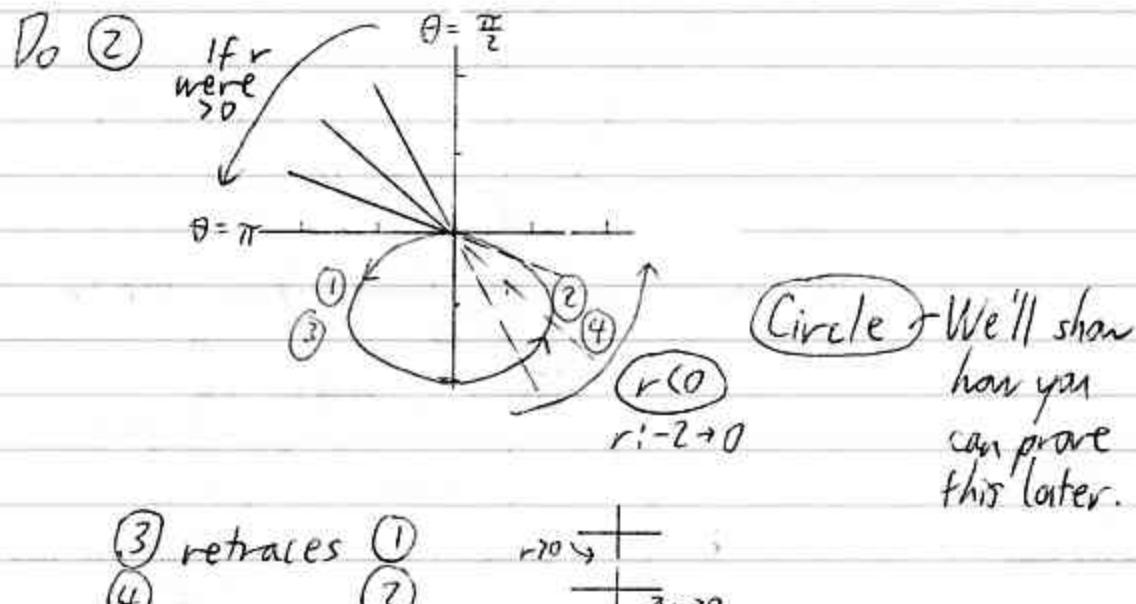
$\theta$	$r$
0	$-2 \sin(0) = 0$
:	

If  $r > 0$ , we'd  
be going thru  
QI

Do ①

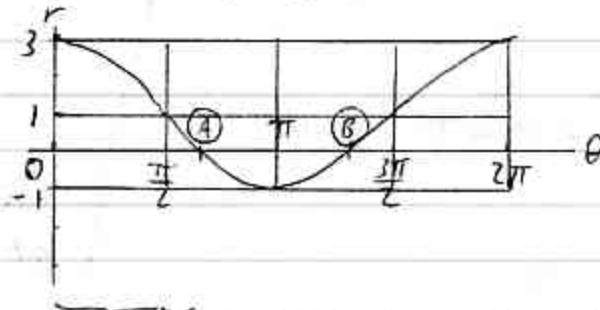


Trust me, it's  
a half-circle.  
Recognize  
basic forms



Ex (#10)  $r = 1 + 2 \cos \theta$

Note:  
 $-2 \leq 2 \cos \theta \leq 2$   
 $-1 \leq 1 + 2 \cos \theta \leq 3$



$r: 3 \rightarrow 1+0 \rightarrow -1+0 \rightarrow 1 \rightarrow 3$

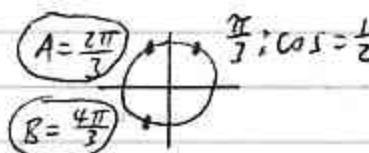
Find A, B

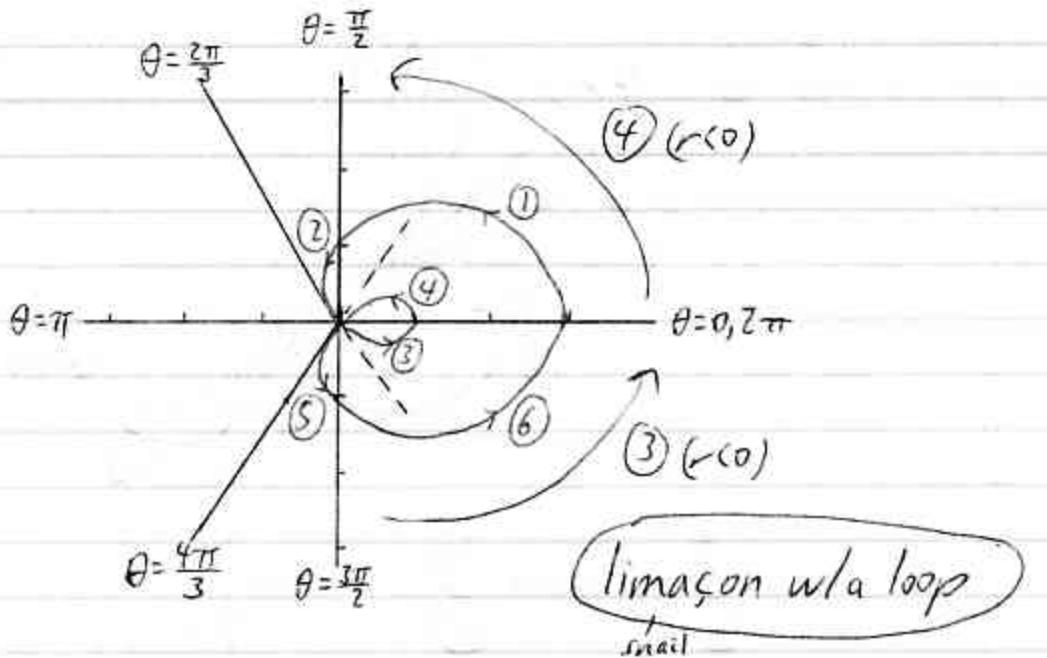
Sectors divided by:  
 quadrants  
 ↑ vs. ↓  
 + vs. -

$$\theta = 1 + 2 \cos \theta$$

$$\cos \theta = -\frac{1}{2}$$

Pictures!





Lee-muh-SOH  
Webster: locust etc.  
fritsnail  
escargot?  
prepared snail

Stewart 6.77

$$r = 1 + c \sin \theta$$

me  
 $r = 1 + a \cos \theta$   
a=1: cardioid  
a>1: limacon  
a<1: no loop  
 $\frac{1}{2}a < 1$ : dimpled

$\begin{cases} \text{cardioid} \\ \text{no dimple} \end{cases}$   
oval limacon (convex)  
 $a > 1$ , "snail"  
 $r = 1 + a \cos \theta$

$$r = 1 + \cos \theta$$



cardioid

$\approx$  "heart" (but?)



$$r = 1 + 0.7 \cos \theta$$



dimpled limacon



Circle!

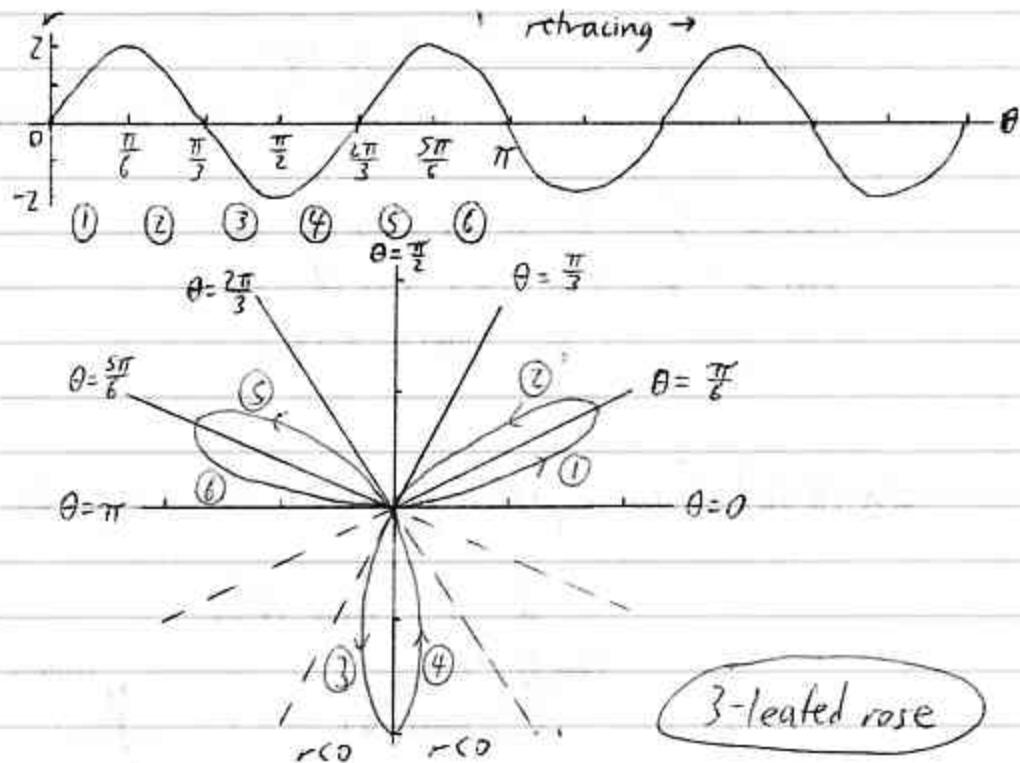
lower limb  
still limacons



$$r = 1$$

Up to 11

The evolution  
of  $r = 1 + a \cos \theta$

Ex  $r = 2 \sin(3\theta)$ 

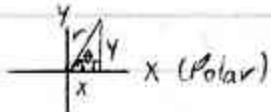
$$r = a \cos(n\theta)$$

$a \neq 0$    if  $n = 3, 5, 7, \dots \Rightarrow n$  leaves  
 if  $n = 2, 4, 6, \dots \Rightarrow 2n$  leaves

Up to 23  
except 13

3-leaved rose

⑥ Polar Eq.  $\Leftrightarrow$  Rect. Eq.



$$\begin{aligned} r^2 &= x^2 + y^2 \\ \tan \theta &= \frac{y}{x}, x \neq 0 \end{aligned}$$

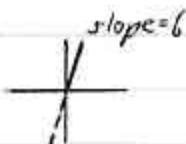
Watch quadrant?

$$\begin{aligned} \cos \theta &= \frac{x}{r} \Rightarrow x = r \cos \theta \\ \sin \theta &= \frac{y}{r} \Rightarrow y = r \sin \theta \\ &\text{even if } r < 0 \end{aligned}$$

Up to 23

Ex (#32) Find a polar eq. w/same graph as  $y = 6x$

$$\begin{aligned} \frac{y}{x} &= 6 \quad \text{Also, } (0,0) \\ \tan \theta &= 6 \\ \theta &= \tan^{-1} 6 \end{aligned}$$

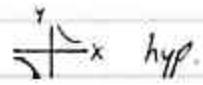


Up to 31

Ex (#42) Find a rect. eq. w/same graph as  $r^2 \sin(2\theta) = 4$ , and graph it.

$$\begin{aligned} r^2 (2 \sin \theta \cos \theta) &= 4 \\ 2 \underbrace{(r \sin \theta)}_{=y} \underbrace{(r \cos \theta)}_{=x} &= 4 \\ 2xy &= 4 \\ xy &= 2 \quad \text{or} \quad y = \frac{2}{x} \end{aligned}$$

Up to 47



Ex (#6 again)  $r = -2 \sin \theta$

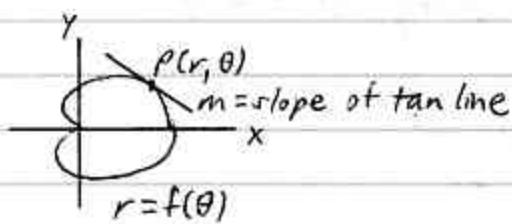
$$\begin{aligned} r^2 &= -2r \sin \theta \\ x^2 + y^2 &= -2y \\ x^2 + y^2 + 2y &= 0 \\ x^2 + (y^2 + 2y + 1) &= 1 \\ x^2 + (y+1)^2 &= 1 \end{aligned}$$

Circle w/center:  $(0, -1)$   
radius = 1



## D) Tangent Lines

Rule:  
 find  $\theta$  s.t.  $r=0$   
 $m = \tan \theta$   
 if  $dr/d\theta \neq 0$



$$\begin{aligned} m &= \frac{dy}{dx} \\ &= \frac{dy/d\theta}{dx/d\theta} \end{aligned}$$

$$= \frac{\frac{d}{d\theta}(r \sin \theta)}{\frac{d}{d\theta}(r \cos \theta)} \quad \begin{matrix} \text{if } r=f(\theta), \text{ so use} \\ \text{Product Rule} \end{matrix}$$

$$= \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta}$$

$$m = \frac{dy/d\theta}{dx/d\theta} \quad \begin{matrix} \text{Tan line is} \\ \frac{H}{V} \end{matrix}$$

$\frac{0}{\neq 0}$	$\frac{\neq 0}{0}$
--------------------	--------------------

Stewart L'H?

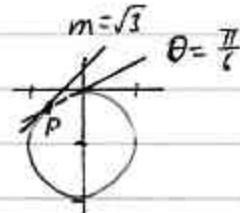
(?) if  $\frac{0}{0}$

Ex (#52) Find slope of tan line to  $r = -2\sin\theta$  (#0)  
at the pt. corresp. to  $\theta = \frac{\pi}{6}$ .

$$\begin{aligned}\theta &= \frac{\pi}{6} \Rightarrow r = -2\sin\left(\frac{\pi}{6}\right) \\ &= -2\left(\frac{1}{2}\right) \\ &= -1\end{aligned}$$

$$\begin{aligned}r &= -2\sin\theta \\ \frac{dr}{d\theta} &= -2\cos\theta \\ \left[\frac{dr}{d\theta}\right]_{\theta=\frac{\pi}{6}} &= -2\cos\left(\frac{\pi}{6}\right) \\ &= -2\left(\frac{\sqrt{3}}{2}\right) \\ &= -\sqrt{3}\end{aligned}$$

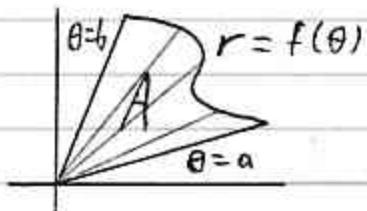
$$\begin{aligned}m &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \\ &= \frac{(-\sqrt{3})\sin\frac{\pi}{6} + (-1)\cos\frac{\pi}{6}}{(-\sqrt{3})\cos\frac{\pi}{6} - (-1)\sin\frac{\pi}{6}} \\ &= \frac{(-\sqrt{3})\left(\frac{1}{2}\right) + (-1)\left(\frac{\sqrt{3}}{2}\right)}{(-\sqrt{3})\left(\frac{\sqrt{3}}{2}\right) - (-1)\left(\frac{1}{2}\right)} \\ &= \frac{-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}}{-\frac{3}{2} + \frac{1}{2}} \\ &= \frac{-\sqrt{3}}{-1} \\ &= \boxed{\sqrt{3}}\end{aligned}$$



13.4:  $\int_s$  in PCs

## (A) Areas

Sweeping out



$$A = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta$$

or  $\int_a^b \frac{1}{2} r^2 d\theta$

if  $f$  cont.  
 $r \geq 0$  for  $\theta$  in  $[a, b]$

wheel  
crash

Stewart 680  
 sketch:  
 $0 < a < b \leq \pi$   
 too  
 restrictive

No

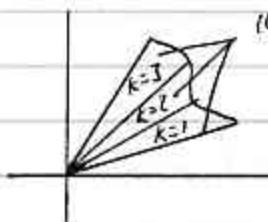


$$0 < b - a \leq 2\pi$$

No



Idea



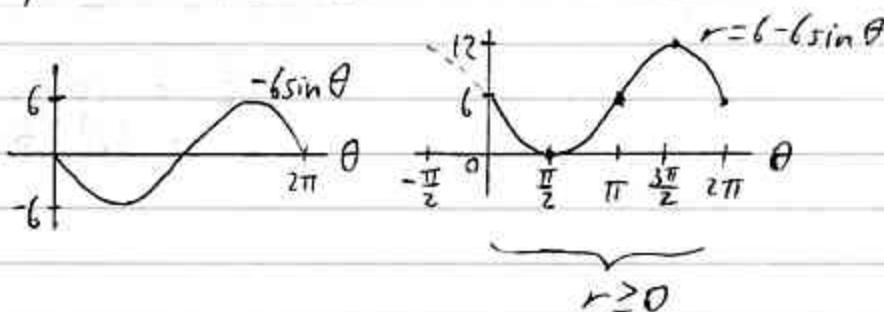
"like" a sector of a circle

$$\text{Area} = \frac{1}{2} r_k^2 (\Delta\theta_k)$$

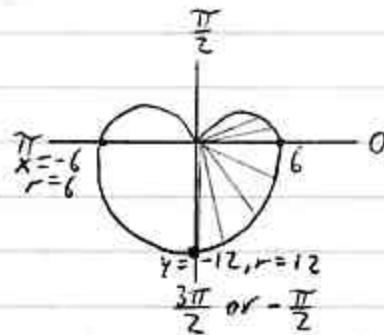
depends on  
which pieceFiner subdivisions  $\rightarrow \int_a^b \frac{1}{2} r^2 d\theta$

Ex (#4) Find the area [of the region]  
bounded by [the graph of]  
 $r = 6 - 6 \sin \theta$   
 or  $r = 6(1 - \sin \theta)$

① Graph  $r$  vs.  $\theta$  (or Table).



② Graph in PCs



Same beh from  
 $-\frac{\pi}{2}$  to 0 as  
 from  $\frac{3\pi}{2}$  to  $2\pi$

③ Exploit symmetry, and pick a, b.

$$A = 2 \cdot \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (6 - 6 \sin \theta)^2 d\theta \quad (\text{Turns out to be easier than } \int_0^{2\pi} \dots)$$

check  
 How big is our butt?  
 Swoosh can't  
 be < 0, ignore  
 him - he's  
 dead, anyway.

Could do  $\int_{\pi/2}^{3\pi/2}$   
 $\int_{\pi/2}^{\pi}$   
 but can't do quick  
 trick we'll be  
 using (boring  
 further analysis!)

(4)  $\int$ 

$$A = 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [6(1-\sin \theta)]^2 d\theta$$

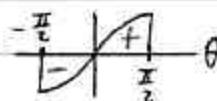
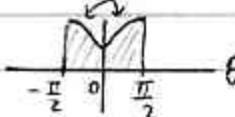
Now some  
MISO tricks

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 36(1 - 2\sin \theta + \sin^2 \theta) d\theta$$

↑ odd      ↓ even

$$= 36 \left[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \sin^2 \theta) d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\sin \theta d\theta \right]$$

= 0

Don't care  
what graph  
looks like

$$= 2 \cdot 1$$

$$= 36; 2 \int_0^{\frac{\pi}{2}} (1 + \sin^2 \theta) d\theta$$

~~We PR1~~

$$= 72 \int_0^{\frac{\pi}{2}} \left( 1 + \frac{1 - \cos(2\theta)}{2} \right) d\theta$$

$$= 72 \int_0^{\frac{\pi}{2}} \left( 1 + \frac{1}{2} - \frac{1}{2} \cos(2\theta) \right) d\theta$$

$$= 72 \int_0^{\frac{\pi}{2}} \left[ \frac{3}{2} - \frac{1}{2} \cos(2\theta) \right] d\theta$$

$$= 36 \int_0^{\frac{\pi}{2}} [3 - \cos(2\theta)] d\theta$$

$$= 36 \left[ 3\theta - \frac{1}{2} \sin(2\theta) \right]_0^{\frac{\pi}{2}}$$

Guess-and-✓

If  $\cos \theta \neq 0$ !!

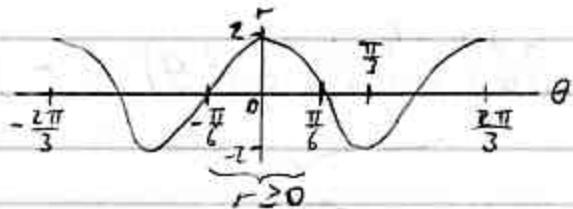
$$= 36 \left( \left[ \frac{3\pi}{2} - \frac{1}{2} \underbrace{\sin \pi}_{=0} \right] - [0] \right)$$

Up to 7

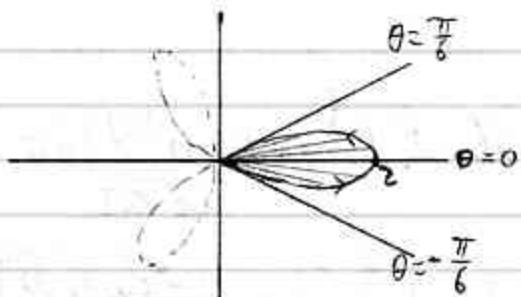
$$= [54\pi]$$

Ex (#10) Find the area bounded by one loop of  $r = 2 \cos 3\theta$ . (3-leaved rose)

① Graph  $r$  vs.  $\theta$



② Partial graph in PC



We can use sym  
to ensure we're  
only dealing w/  
 $\theta$ -values where  
 $r \geq 0$ . I may  
work for nice curves  
but allowing  $r < 0$   
opens Pandora's  
box.

③ Sym.

$$A = 2 \cdot \int_{0}^{\pi/6} \frac{1}{2} [2 \cos(3\theta)]^2 d\theta$$

④ ∫

$$A = 2 \int_0^{\pi/6} \frac{1}{2} [2 \cos(3\theta)]^2 d\theta$$

: Work it out!  $= 4 \cos^2(3\theta) = 4 \cdot \frac{1 + \cos(6\theta)}{2}$

$$= \boxed{\frac{\pi}{3}}$$

(Area of entire rose is  $\pi$ .)

Last time:  $\sin^2$

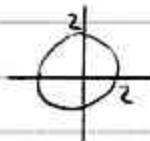
What's area of whole rose?  $\pi$   
Up to 9

$$A = \int_a^b \frac{1}{2} [f(\theta)]^2 d\theta - \int_a^b \frac{1}{2} [g(\theta)]^2 d\theta$$

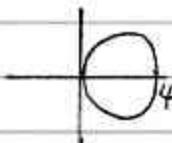
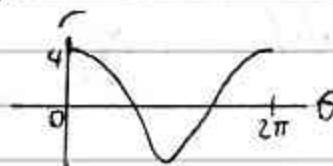
$$= \int_a^b \frac{1}{2} r_{\text{out}}^2 d\theta - \underbrace{\int_a^b \frac{1}{2} r_{\text{in}}^2 d\theta}_{\text{missing } \textcircled{2}}$$

Ex (#20) Find the area outside  $r=2$  and inside  $r=4 \cos \theta$ .

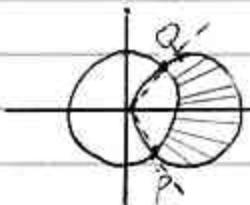
$$(r=2)$$



$$(r=4 \cos \theta)$$



Superimpose

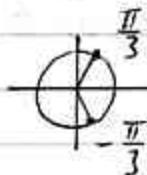


Find  $\theta$ s corr. to P, Q

To find intersec. pts., solve system.

$$\begin{cases} r = 2 \\ r = 4 \cos \theta \end{cases}$$

$$2 = 4 \cos \theta$$
$$\cos \theta = \frac{1}{2}$$



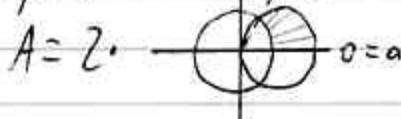
Note Bec. a pt. can have diff. PC reps.  
there may be other intersec. pts. (Pole?)  
Graph!

$\theta$  can be anything

Larson

but paths still intersect  
Maybe 1 hit +  
at  $\frac{\pi}{3} + 2\pi n$   
Other at  $\frac{4\pi}{3} + 2\pi n$

Sym



I

On test, maybe  
just set up.

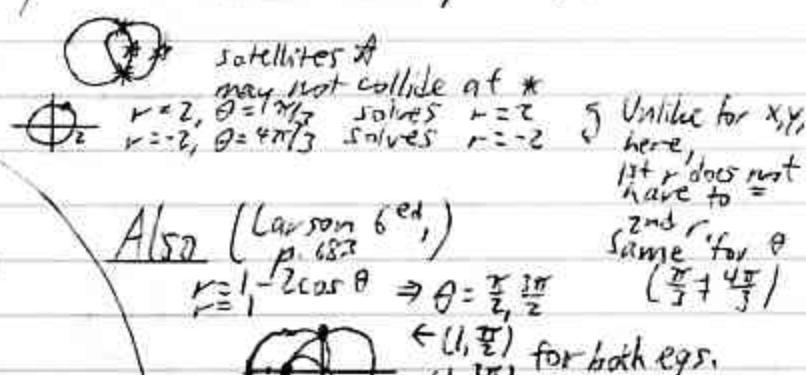
$$A = 2 \cdot \left[ \int_0^{\frac{\pi}{3}} \frac{1}{2} [4 \cos \theta]^2 d\theta - \int_0^{\frac{\pi}{3}} \frac{1}{2} [2]^2 d\theta \right]$$

"Set-up"

$$= \frac{4\pi}{3} + 2\sqrt{3}$$

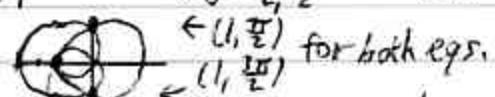
$$\approx 7.65$$

Up to 23



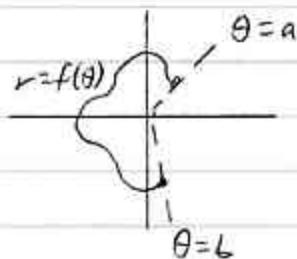
Also (Larson 6<sup>ed</sup>, p. 182)  
 $r = 1 - 2 \cos \theta \Rightarrow \theta = \frac{\pi}{2}, \frac{4\pi}{3}$

$$r = 1 - 2 \cos \theta \Rightarrow \theta = \frac{\pi}{2}, \frac{4\pi}{3}$$



Misst.: occurs w/ coords  $(1, \pi)$  on 1<sup>st</sup> graph  
 $(-1, 0)$  on 2<sup>nd</sup> graph

### ⑧ Arc Length "L"



$$L = \int_a^b \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} d\theta$$

or  $\int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$

if  $0 < b - a \leq 2\pi$   
 $f'$  cont. on  $[a, b]$ ,  
 $r \geq 0$

Ex Find the length of  $r = 3\sec\theta$  from  
 $\theta = 0$  to  $\theta = \frac{\pi}{3}$ .

$\cos\theta \neq 0$  here  
 $\Rightarrow \sec\theta$  never  
 zero

$$\cos\theta > 0 \text{ on } [0, \frac{\pi}{3}],$$

$\Rightarrow \sec\theta$

$$L = \int_0^{\pi/3} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$\begin{aligned} r &= 3\sec\theta \\ \frac{dr}{d\theta} &= 3\sec\theta \tan\theta \quad \text{cont. on } [0, \frac{\pi}{3}] \end{aligned}$$

$$= \int_0^{\pi/3} \sqrt{(3\sec\theta)^2 + (3\sec\theta\tan\theta)^2} d\theta$$

$$= \int_0^{\pi/3} \sqrt{9\sec^2\theta + 9\sec^2\theta\tan^2\theta} d\theta$$

$$= \int_0^{\pi/3} \sqrt{9\sec^2\theta \underbrace{(1 + \tan^2\theta)}_{=\sec^2\theta}} d\theta$$

$$= \int_0^{\pi/3} \sqrt{9\sec^4\theta} d\theta$$

$$= \int_0^{\pi/3} \underbrace{3 \sec^2 \theta}_{\geq 0} d\theta$$

$$= [3 \tan \theta]_0^{\pi/3}$$

$$= 3(\tan \frac{\pi}{3} - \tan 0)$$

$$= 3(\sqrt{3} - 0)$$

$$= \boxed{3\sqrt{3}}$$

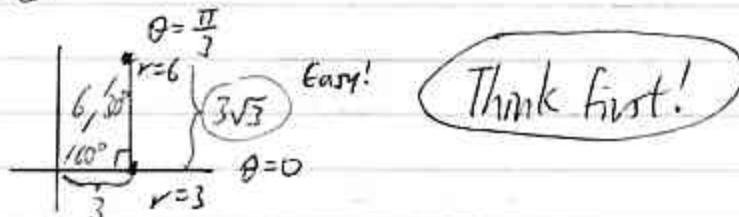
Wait a minute....

$$r = 3 \sec \theta$$

$$r = \frac{3}{\cos \theta}$$

$$r \cos \theta = 3$$

$$x = 3$$



CH. 13 REVIEW(13.1) Plane Curves

Graph C:  $x = f(t)$   
 $y = g(t)$   
 $t \text{ in } I$



May help to get rect. eq.

- ① ETP
- ② Directly relate  $x, y$

$$\text{Ex } \begin{cases} x = 3t \\ y = 9t^2 \end{cases} \Rightarrow y = x^2$$

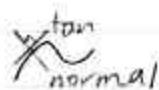
- ③ Use ID's

$$\text{Ex } \begin{cases} x = \cos t \\ y = \sin t \end{cases} \Rightarrow x^2 + y^2 = 1$$

Restrict graph; Orientation Tools:

- ① Point - plotting ( $\checkmark t=0$ , endpoints of interval; shouldn't be only tool)
- ② Sign of  $\frac{dx}{dt}, \frac{dy}{dt}$
- ③ ETP, ④ Use ID's, other relationships
- ④ Restrictions on  $x, y$  (Sign, extreme values)
- ⑤ As  $t \nearrow$ , does  $x \nearrow, \searrow?$   $y \nearrow, \searrow?$  When?
- ⑥ As  $t \rightarrow \infty$  or  $\rightarrow -\infty \dots$
- ⑦ Graph  $x$  and/or  $y$  against  $t$ .
- ⑧ Templates (circles, ellipses, etc.)

(13.2)



$$y' = \frac{dy}{dx} = \frac{\frac{dy/dt}{dx/dt}}{\text{whole thing}}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

$$\sim L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

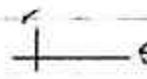
Hor, vert. tan lines

dt

whole thing

(13.3) PCs

Graph  $r$  or  $r^2 = f(\theta)$

- ① 
- ②  $r < 0$  
- ③ When does,  $r \rightarrow 0$ ?
- ④  $r = 0$ ? (Pole)

Basic shapes

O, limacons,  $\heartsuit$ ,  $\infty$ , roses

Polar Eq.  $\Leftrightarrow$  Rect. Eq.

$$r^2 = x^2 + y^2$$

$$\tan \theta = \frac{y}{x}$$

Watch Q!

$$x = r \cos \theta$$

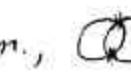
$$y = r \sin \theta$$

Tangent lines

$$m = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}(r \sin \theta)}{\frac{d}{d\theta}(r \cos \theta)}$$

(13.4) S<sub>5</sub>

$$A = \int_a^b \frac{1}{2} r^2 d\theta$$

Sym., ,  $r_{\text{out}}$  vs.  $r_{\text{in}}$ , "-" areas

$$\text{L} = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$