

Kant: Logic is a science of the necessary laws of thought.
Logic is the anatomy of thought. Locke
Logic is the art of going wrong with confidence. Joseph Schumpeter

L1-1
1.1

1.1: LOGIC

We'll need for 3.1.

Gateway to precise thinking

A proposition is a statement that is either true (T) or false (F).

Not tested

Ex

truth values

Are props.

Are not props.

Al Gore is a Dem. (T) Whazzup?

G.W. Bush is a Dem. (F) Just do it.

The sky is blue.

(but depends on context, opinion)

$1+1=2$. (T)

$1+1=3$. (F)

$2x=10$ if $x=5$. (T) $2x=10$.

[assume we know nothing about x]

In CS, 0=false, 1=true.

In fuzzy logic (AI), a proposition can have a truth value between 0 and 1.

Ex Ken is tall. (0.3)

Overhead

Let p, q, r , etc. denote propositions.
We use logical operators to build compound propositions.

① $\neg p$ means {not p
the negation of p
it is not the case that p }

Truth table for $\neg p$

P	$\neg P$
T	F
F	T

If p is a T_F prop., then $\neg p$ must be a F_T prop.

$$\begin{array}{l} \text{Ex } p: |+| = 2 \quad (T) \\ \neg p: |+| \neq 2 \quad (F) \end{array}$$

Ex a, b are real #5

$$p: a > b$$

$p: a > b$ ↗ one is T;
 $\neg p: \text{it is not the case that } a > b$ ↗ the other
 (i.e., $a \leq b$) is F

Note: \neg is a unary operator;

Connectives are operators that combine 2 or more props.

② $p \wedge q$ means {
 p and q
 the conjunction of p and q}

Truth table

$\neg p$	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

all possibl

$p \wedge q$ is a T prop. \longleftrightarrow exactly when
 p and q are both T.

Your grandma
 wears army
 boots.

Ex p : Gore will win. (?)

q : $1+1=3$. (F)

$p \wedge q$: Gore will win and $1+1=3$. (F, regardless)

③ $p \vee q$ means {
 p or q (inclusive or)
 the disjunction of p and q}

Truth table

<u>p</u>	<u>q</u>	<u>$p \vee q$</u>
T	T	T
T	F	T
F	T	T
<u>F</u>	<u>F</u>	<u>F</u>

} one or both
of p,q T

$p \vee q$ is a T prop. \leftrightarrow
 p and q are both F

④ $p \oplus q$ means {
 exclusive or of p and q
 p xor q }

Truth table

<u>p</u>	<u>q</u>	<u>$p \oplus q$</u>
T	T	F
T	F	T
F	T	T
F	F	F

\leftarrow different from $p \vee q$

for purposes
of circuit
design, it's
interesting
that
 $p \oplus q$ is F
 \leftrightarrow p, q have
same truth value

$p \oplus q$ is a T prop. \leftrightarrow
 exactly one of p or q is T (but not both!)
 ("soup or salad" idea!)

Ex Let's say

p : You have soup. (T)

q : You have salad. (T)

Then, $p \oplus q$: You have soup xor salad. (F)
but $p \vee q$: (T)

Note: In math, "or" is an inclusive or (③),
unless otherwise specified.

⑤ The implication $p \rightarrow q$

↑	↑
hypothesis	conclusion
antecedent	consequence
premise	

SNL sketch

Ex d : Pat is a Daddy.
 m : Pat is a man.

There are many ways to say $d \rightarrow m$

- if d , [then] m
- d implies m
- d only if m
- d is sufficient for m

- m if d
- m whenever d
- m is necessary for d

$(p \rightarrow q \text{ is a F prop}) \leftrightarrow (p \text{ is T and } q \text{ is F})$
 "Breaking a contract"

Truth table

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

} If the hypothesis (p) is F, the prop. $p \rightarrow q$ is T "by default".

Ex $p: 1+1=3$ (F)

$q: \text{Gore will win}$ (?)

Then, $p \rightarrow q: \text{If } \underbrace{1+1=3}_{\text{F}}, \text{ then } \underbrace{\text{Gore will win}}_{\text{T (by default)}} \text{ (T "by default")}$

Ex If $\underbrace{1+1=2}_{\text{F}}$, then $\underbrace{\text{a lemon is yellow}}_{\text{T (all we need)}}$. (T)

p and q need not be related! (stats: Correlation does not imply causation. \downarrow elev.)

Ex If $\underbrace{\text{Pat is a Daddy}}_{\text{p (or d)}}, \text{ then } \underbrace{\text{Pat is a man}}_{\text{q (or m)}}$.

This prop. must be T, because [realistically] there's no way p is T and q is F.

Math Ex If an integer ends in a "3", it's an odd H.

Same idea!

Key: p is T
 Math/produce

Why does this sound weird?
 No causal relship

Assume Pat
 is human.
 stronger man

Math

~~if $x^2 + y^2 = z^2$, then
x, y, z are not all prime
integers.~~

Think of an "if-then" statement as a conjecture (unproven hunch).
 If the prop. is T, we have a theorem we can put in books.
 If the prop. is F, no grant \$!! hunk!

(5) More

The converse of $p \rightarrow q$ is $q \rightarrow p$.
 The contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

not equivalent

These are "logically equivalent" (Sec 1.2).
 both are T or both are F.

Sec 1.2 #16

depending on
whether the
truth values
at p, q)

Ex Write the converse of
 "Pat is a man whenever Pat is a Daddy."

Rewrite in "if-then" form:
"If Pat is a Daddy, then Pat is a man."

p

q

The converse is:

"If Pat is a man, then Pat is a Daddy."

q

p

This new prop. is fundamentally different
 from the old one!

Ex The contrapositive is:

"If $\neg q$, then $\neg p$ "

"If Pat is not a man, then
 Pat is not a Daddy."

This is fundamentally the same as
 the old prop.

⑥ The biconditional $p \leftrightarrow q$

means $\left\{ \begin{array}{l} p \text{ if and only if } q \\ p \text{ is necessary and sufficient for } q \\ \text{if } p \text{ then } q, \text{ and if } q \text{ then } p \\ \text{"conversely"} \end{array} \right.$

$(p \leftrightarrow q \text{ is } T) \iff (p, q \text{ are both } T \text{ or both } F)$

Truth table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

We tend
to look at
the table
from right
to left.

Ex If we know this is a true prop:

"Pat is a man \leftrightarrow Pat has an Adam's apple," then
both are T, or both are F

HW Tips (Relevant: pp. 1-8; ignore boolean searches, bit strings)

Ex p : I am young.
 q : I am stupid.
 r : I can attend the movie.

Express $((p \vee q) \rightarrow \neg r)$ as an English sentence.

Order of ops: () 1st, \neg next, \vee/\wedge , $\rightarrow/\leftrightarrow$

"If I am young or stupid, then,
I cannot attend the movie."

Ex (23d) Construct a truth table for $(p \rightarrow q) \wedge (\neg p \rightarrow q)$

p	q	$(p \rightarrow q)$	$\neg p$	q	$(\neg p \rightarrow q)$	\wedge
T	T	T	F	T	T	T
T	F	F	F	F	T	F
F	T	T	T	T	T	T
F	F	T	T	F	F	F

Different approaches ok!

Concise answer:

p	q	$(p \rightarrow q) \wedge (\neg p \rightarrow q)$
T	T	T
T	F	F
F	T	F
F	F	F

19a) Converse:

If I will ski tomorrow, then it snows today.

Let's say both are T.

Then, this implication is T.

In conversation, a cause-and-effect relationship is implied.

Our sense of logic does not focus on content or cause-and-effect.

1.2: PROPOSITIONAL EQUIVALENCES

A tautology is a compound proposition that is always T, regardless of the truth values (T/fs) of its constituent props.

Ex $p \vee \neg p$ (Hamlet is or is not.)

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

(T)
T → tautology!

A contradiction is... always F....

Ex $p \wedge \neg p$ ($p, \neg p$ can't both be T.)

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

(F)
F → contradiction!

A contingency is neither.

Props. that (essentially) have the same truth tables
are logically equivalent (\Leftrightarrow or \equiv).

Ex 3, p. 16

Ex Show $p \rightarrow q \Leftrightarrow \neg p \vee q$

must be in same order!

p	q	$p \rightarrow q$
T	T	F
T	F	F
F	T	T
F	F	T

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

} can combine like Koenen

$p \rightarrow q$ F exactly when pT, qF

p	q	$p \rightarrow q$	$\neg p$	q	$\neg p \vee q$
T	□	F	F	T	T
T	F	F	F	F	F
F	□	T	T	T	T
F	F	T	T	F	T

match, so $p \rightarrow q \Leftrightarrow \neg p \vee q$

Idea: $p \rightarrow q$ is T exactly when
 $\overline{p} \equiv \overline{q}$

" $p \leftrightarrow q$ " means that, in any particular situation, both p and q are T, or both are F (i.e., $p \leftrightarrow q$ is a tautology).
think: theorem

Ex Let n be some integer.
 n is odd \iff its ones' digit is 1, 3, 5, 7, or 9.

If $n = 13$, both are T.
If $n = 14$, both are F.

Note 1 Size of truth tables

$\begin{array}{c} p \quad q \quad r \\ \hline 8 \text{ rows} \end{array}$

In general,

$\begin{array}{c} p_1 \quad p_2 \quad \dots \quad p_n \\ \hline 2^n \text{ rows} \end{array}$

Note 2 Table 5 (p.17) gives a list of basic equivalence laws, which can be used to simplify props., to show that two props. are equiv., or to show that props. are tautologies or contradictions. as an alternative to truth tables

Don't worry now, but we will learn ^{them and} their twin brothers, the set identities in 1.5.

Just for
lectures...

Some laws:

$$\neg(\neg p) \Leftrightarrow p$$

\vee , \wedge are commutative and associative

\vee distributes over \wedge (like \times over $+$)
 \wedge ' \vee (\Leftarrow HW #5)

De Morgan's laws (HW #6-one of them)

HW Tips (Relevant: pp 14-16; scan pp 17-19)

Use truth tables.

#16) Show $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$.

What are we
proving here?

An implication is logically equivalent
to its contrapositive.

1.3: PREDICATES AND QUANTIFIERS

Ex Let $P(x)$ denote the statement " $x < 5$ ".

P is a predicate denoting the property "is less than 5."

$P(x)$ is not a proposition, but it becomes a prop. (T/F) once we assign a value to x .

$P(x)$ is a propositional function that depends on the variable x .

Ex In the case $x=4$, " $4 < 5$ " is T, so $P(4)$ is T.

Ex In the case $x=5$, " $5 < 5$ " is F, so $P(5)$ is F.

Ex 2 Let $Q(x,y)$ denote " $y = 3x + 1$ ".

Ex In the case $(x=2, y=7)$, " $7 = 3(2) + 1$ " is T, so $Q(2,7)$ is T.

\forall is the "universal quantifier" (say "for all")

$\forall x P(x)$ denotes the proposition

" $P(x)$ is T for all values of x ,
in the universe of discourse."

the domain of P
(we'll say "uod")

Some common "uod"s:

\mathbb{Z} is the set of all integers.
 $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

\mathbb{Z}^+ is the set of all positive integers.
 $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$

\mathbb{R} is the set of all real #s.

Special case: Finite "uod"s

If the uod is a finite set, say $\{x_1, x_2, \dots, x_n\}$, then

$$\forall x P(x) \Leftrightarrow P(x_1) \wedge P(x_2) \wedge \dots \wedge P(x_n)$$

$\underbrace{\quad}_{\text{is T}}$ $\underbrace{\text{exactly when}}_{\text{all these are T}}$

Ex Let $P(x)$ be "x is an integer."

If the uod = {1, 7, 15}, then $\forall x P(x)$ is T, because
 $P(1) \wedge P(7) \wedge P(15)$ is T.

If the uod = {1, 7, π }, then $\forall x P(x)$ is F, because
 $P(1) \wedge P(7) \wedge \underbrace{P(\pi)}_{\text{one } F \text{ ruins it for everybody}} \text{ is F.}$

The uod can be an infinite set.

Ex Let $P(x)$ be " $\exists x > x$."

If the uod is \mathbb{Z}^+ , $\forall x P(x)$ is T.

If the uod is \mathbb{Z} , $\forall x P(x)$ is F.

For example, $P(-1)$ is F. (-2 \nmid -1).
 $x = -1$ is called a counterexample.

$$\begin{array}{c|cc} x & \frac{x}{2} \\ \hline 1 & 1 \\ 2 & 1 \\ 3 & \\ 4 & 2 \\ \vdots & \vdots \end{array}$$

\exists is the "existential quantifier" (say "there exists/is")

$\exists x P(x)$ denotes the proposition

" $P(x)$ is T for at least one element (x) in the uod."

Special case: Finite "uod's"

If the uod is $\{x_1, x_2, \dots, x_n\}$, then

$$\exists x P(x) \Leftrightarrow \underbrace{P(x_1) \vee P(x_2) \vee \dots \vee P(x_n)}_{\substack{\text{is T} \\ \text{exactly} \\ \text{when}}} \quad \underbrace{\checkmark}_{\substack{\text{at least one is T}}}$$

Ex Let $P(x)$ be " x is an integer."

If the uod = $\{1, \pi, e\}$, then $\exists x P(x)$ is T, because

$$P(1) \vee P(\pi) \vee P(e) \text{ is T.}$$

one T
does it!

If the uod = $\{\sqrt{2}, \pi, e\}$, then $\exists x P(x)$ is F, because

$$P(\sqrt{2}) \vee P(\pi) \vee P(e) \text{ is F.}$$

all are F

Ex Let $P(x)$ be " $x^2 = 16$ ".
Let the uod be \mathbb{Z} .

Then, $\exists x P(x)$ is T.
for example, $P(4)$ is T. $P(-4)$, also.

Ex Let $Q(x)$ be " $x^2 = 17$ ".
Let the uod be \mathbb{Z} .

There is no integer whose square is 17,
so $\exists x Q(x)$ is F.

Ex $P(x)$: "x is even", uod = \mathbb{Z}

$P(4)$	"4 is even"	(T)	} these are propositions
$\forall x P(x)$	"all integers are even"	(F)	
$\exists x P(x)$	"there is an even integer"	(T)	

In these cases, x is bound by a
value assignment or a quantifier.

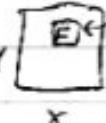
In general, a propositional function becomes a proposition only when all the variables are bound.

MORE THAN 1 VARIABLE

p.31-a great
cure for
insomnia!

Imagine a grid: $\{ \begin{array}{|c|c|c|} \hline & \text{uod for } y \\ \hline \text{uod for } x & \diagdown \\ \hline \end{array} \}$

① $\forall x \forall y P(x,y)$ is T \Leftrightarrow  all combos make P true

is F \Leftrightarrow  there is a counterexample that makes P false

Ex $P(x,y)$: " $xy = yx$ "
 $\text{uod} = \mathbb{R}$ for both x, y

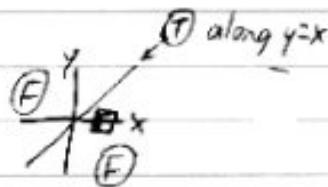
Then, $\forall x \forall y P(x,y)$ is T.

(Multiplication is commutative for all pairs of real #s.)

Ex $P(x,y)$: " $x-y = y-x$ "
 $\text{uod} = \mathbb{R}$ for both x, y

Then, $\forall x \forall y P(x,y)$ is F.

Counterexample: $x=1, y=0$
 $1-0 \neq 0-1$



② $\exists x \exists y P(x, y)$ is T \Leftrightarrow $\exists \boxed{?} \forall x$ there is an example combo that makes P true

is F $\Leftrightarrow \exists \boxed{F} \forall x$ all combos make P false

Ex $P(x, y)$: " $x - y = y - x$ "
 $\text{uod} = \mathbb{R}$ for both x, y

Then, $\exists x \exists y P(x, y)$ is T.

for example, $(x=2, y=2)$:
 $2 - 2 = 2 - 2$

Ex $P(x, y)$: " $xy = \pi$ "
 $\text{uod} = \mathbb{Z}$ for both x, y

Then, $\exists x \exists y P(x, y)$ is F.

(No two integers multiply to π .)

$\forall x \forall y P(x, y) \Leftrightarrow \forall y \forall x P(x, y)$
can switch

$\exists x \exists y P(x, y) \Leftrightarrow \exists y \exists x P(x, y)$
can switch

BUT order usually matters when mixing \forall s, \exists s.

③ $\forall x \exists y P(x, y)$ is T \Leftrightarrow  Each x can "find" a y that makes P true.

is F \Leftrightarrow  There is an x who can't find a y.

Ex $P(x, y)$: " $y - x = 6$ "
 $x, y \in \mathbb{R}$ for both x, y

Then, $\forall x \exists y P(x, y)$ is T. Why?

Regardless of which real # x is...

$$\begin{aligned} y - x &= 6 \\ y &= 6 + x \quad (\text{real #}) \end{aligned}$$

we can let y equal $6 + x$,
and P will be T.

Key: y can depend on x !!

Idea: $P(0, 6)$ is T

$P(1, 7)$ is T

$P(x, 6+x)$ is T
 \vdots
any real #

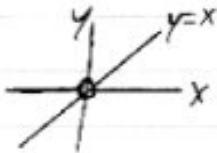
X

Ex $P(x, y)$: " $\frac{x}{y} = 1$ "
 $\text{uod} = \mathbb{R}$ for both x, y

Then, $\forall x \exists y P(x, y)$ is F. Why?

$x=0$ can't "find" a y (real #) to make P true.

Note: Graph of $\frac{x}{y} = 1$ (like our "grid")



Ex $P(x, y)$: " $\frac{x}{2} = y$ "
 $\text{uod} = \mathbb{Z}$ for both x, y

Then, $\forall x \exists y P(x, y)$ is F. Why?

If x is odd, it can't find an integer y to make P true.

Key: Watch your "uod"s!!

If $\text{uod} = \mathbb{R}$ for both x, y , then T:
any real x can pick y to be $\frac{x}{2}$ (real).

How about? (4) $\exists x \forall y P(x, y)$ is T \Leftrightarrow $y \begin{array}{|c|c|} \hline & | \\ \hline x & | \\ \hline \end{array}$

There is a "magic" x that will make P true, regardless of y (in y 's uod)

is F \Leftrightarrow there is no such column (Each x can find a y that makes P false.)

Ex $P(x, y)$: " $\frac{\ln x}{y} = 0$ "
 $\text{uod} = \mathbb{R}^+$ for both x, y

Then, $\exists x \forall y \neq 0 P(x, y)$ is T. Why?

exclude
 $y=0$
from
consideration

$x=1$ is a "magic" real # that always makes P true.

Ex $P(x, y)$: " $x+y=3$ "
 $\text{uod} = \mathbb{R}$ for both x, y

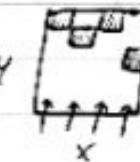
Then, $\exists x \forall y P(x, y)$ is F. Why?

There is no "magic x " that works with all real " $y's to make P true.$

Order doesn't matter:
 $(x=3) \wedge (y=4)$

Order usually matters!

$\text{Prop. A} \quad \forall x \exists y P(x, y)$

This is T \Leftrightarrow 

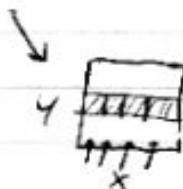
$\text{Prop. B} \quad \exists y \forall x P(x, y)$

This is T \Leftrightarrow  There is a magic y...

Note

If Prop. A is T, then Prop. B is not necessarily T.

If Prop. B is T, then Prop. A must be T.



Shorthand: $A \rightarrow B$
 $B \rightarrow A$

Ex (3 vars)

$P(x, y, z)$: " $xy = z$ "
 $\text{dom} = \mathbb{R}$ for x, y , and z

Then, $\forall x \forall y \exists z P(x, y, z)$ is T. Why?

Each pair of real #'s has a real product.

$\exists x \exists y \exists z$
is T ($x=0, y=0$)

Also, $\exists z \forall x \forall y P(x, y, z)$ is F. Why?

There is no "magic" real # z that is
the product for every pair of real #'s.

(See also Ex 24 on p. 32)

NEGATIONS

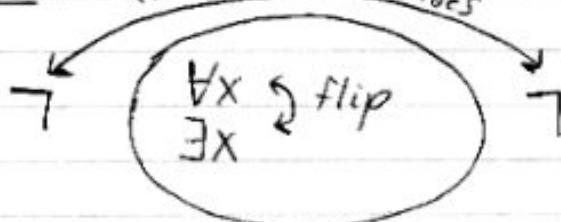
$$\neg \forall x P(x) \Leftrightarrow \exists x \underbrace{\neg P(x)}_{\text{is } T \Leftrightarrow P(x) \text{ is } F}$$

\boxed{F} There is an x that makes P false

$$\neg \exists x P(x) \Leftrightarrow \forall x \neg P(x)$$

$\boxed{FF \dots FF}$ All x 's make P false

Trick: if \neg switches sides



HW Tips

In 1.3 (book), skip Exs. 12, 13, 16-21.

#10) Let $Q(x, y)$ be " x has been a contestant on y "
 and for $x =$ the set of all students at your school.
 and for $y =$ the set of all quiz shows on TV.
 Express in terms of Q , quantifiers, and connectives:

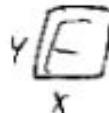
- a) There is a student at your school who has been a contestant on a TV quiz show.

$$\exists x \exists y Q(x, y)$$



- b) No student at your school has ever been a contestant on a TV quiz show.

$$\forall x \forall y \neg Q(x, y)$$



$$\neg \exists x \exists y \overset{\text{or}}{Q}(x, y)$$

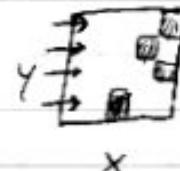
- c) There is a student at your school who has been ... on Jeopardy and W.O.F.

$$\exists x (Q(x, \text{Jeopardy}) \wedge Q(x, \text{W.O.F.}))$$



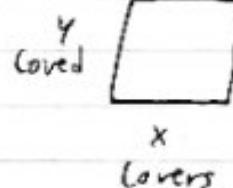
d) Every TV quiz show has had a student from your school...

$$\rightarrow \forall y \exists x Q(x, y)$$



x

#11) $L(x, y)$: "x loves y"



(x, y have same udd)
all people in
the world

d)-f): different possible answers (\top , $\frac{\square}{\square}$ since same udd)

for me (Give ovals)

d) Nobody loves everybody



Book: $\forall x \exists y \neg L(x, y)$

Me: $\neg \exists x \forall y \neg L(x, y)$

e) There is somebody whom Lydia does not love.



Book: $\exists x \neg L(Lydia, x)$

Me: $\exists y \neg L(Lydia, y)$

f) There is somebody whom no one loves.



Book: $\exists x \forall y \neg L(y, x)$

Me: $\exists y \forall x \neg L(x, y)$

$\neg \forall y \exists x L(x, y)$

#12) TFTFTFTFTFFT

1.4: SETS

The Joy of Sets

A set is a collection of objects, called elements or members.

Ex Let S denote the set $\{a, b, c\}$.

S has 3 elements: a
 b
 c

" \in " means "is a member of"

$$\begin{array}{l} a \in S \\ b \in S \\ c \in S \end{array}$$

" \notin " means "is not a member of"
 $d \notin S$

Two sets are equal \Leftrightarrow They have the same elements

Ex $\{e, o, n\} = \{o, n, e\} = \{n, o, e\}$

the elements are e, o , and n

Writing sets

Ex (Set builder notation) →

$$\mathbb{Z} = \{x \mid x \text{ is an integer}\}$$

= the set of all x such that ...

$$= \{ \underbrace{\dots, -2, -1}_\text{ellipses; "follow the pattern"}, 0, 1, 2, \dots \}$$

"follow the pattern"

Ex Set of all digits = $\{0, 1, 2, \dots, 9\}$

\uparrow 1st set the pattern \uparrow last

If a set S contains exactly n distinct elements, then

1) S is a finite set

2) the cardinality of S is n ($|S|=n$)

Otherwise, S is an infinite set. Ex \mathbb{Z}, \mathbb{R}

Ex If $S = \{e, o, n\}$, $|S|=3$.

Ex If $S = \{n, o, n, e\}$, $|S|=3$. (distinct elements)

The set with no elements is the empty set or null set, denoted by " ϕ " or " $\{\}$ ".

$$|\phi| = 0$$

$$\text{Ex } \{x | x \in \mathbb{R} \text{ and } x^2 = -1\} = \phi$$

A set can have elements that are, themselves, sets.

$$\text{Ex } S = \underbrace{\{\phi\}}_1, \underbrace{\{\{\phi\}\}}_2, \underbrace{5}_3, \underbrace{\{S, 6\}}_4, \underbrace{\{\{S\}, 6\}}_5, \underbrace{\{\{\{6\}\}\}}_6$$

$$|S| = 6$$

$$\phi \in S$$

$$\{\phi\} \in S$$

$$5 \in S$$

$$\{S\} \notin S$$

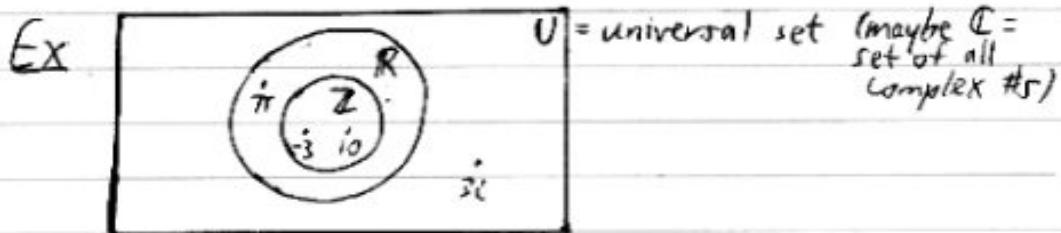
SUBSETS

Assume A and B are sets.

A is a subset of B ($A \subseteq B$) \Leftrightarrow
all the elements of A are also elements of B

Logic: $\forall x (x \in A \rightarrow x \in B)$ is T

Venn Diagram:



$$\mathbb{Z} \subseteq \mathbb{R}$$

For any set S ,

- 1) $\emptyset \subseteq S$ because $\forall x (x \in \emptyset \rightarrow x \in S)$ is T
- 2) $S \subseteq S$ always F

Key Math Trick: If $A \subseteq B$ and $B \subseteq A$, then $A = B$
(Two sets that are subsets of each other are equal.)

A is a proper subset of B ($A \subset B$) \Leftrightarrow
 $A \subseteq B$ and $A \neq B$

Ex $\mathbb{Z} \subset \mathbb{R}$

POWER SETS

$P(S)$, the power set of a set S , is the set of all subsets of S .

Ex If $S = \{a, b, c\}$,

$$P(S) = \left\{ \underbrace{\emptyset}_{\emptyset \subseteq S}, \underbrace{\{a\}}_{1\text{-elt. subsets}}, \underbrace{\{b\}}_{1\text{-elt. subsets}}, \underbrace{\{c\}}_{1\text{-elt. subsets}}, \underbrace{\{a, b\}}_{2\text{-elt. subsets}}, \underbrace{\{a, c\}}_{2\text{-elt. subsets}}, \underbrace{\{b, c\}}_{2\text{-elt. subsets}}, \underbrace{\{a, b, c\}}_{S \subseteq S} \right\}$$

$$|S| = 3, |P(S)| = 8$$

In general, $|P(S)| = 2^{|S|}$. Why?

5.1

$$\textcircled{1} \quad S = \{x_1, x_2, \dots, x_n\}$$

\uparrow \uparrow \uparrow
 IN or OUT IN or OUT ... IN or OUT

You can enumerate all possible subsets by considering all possible IN/OUT combos.

2.4

$$2 \times 2 \times \dots \times 2 = 2^n$$

\textcircled{2} List the binary reps of integers from 0 to $2^n - 1$

$$\text{Ex } |P(\underbrace{\{\emptyset, \{\emptyset\}\}}_{|S|=2})| = 2^2 = 4$$

$\begin{array}{ccccccc} \emptyset & x_1 & x_2 & \dots & x_n \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{array}$
 0: elmt in subset

i.e., The set $\{\emptyset, \{\emptyset\}\}$ has 4 subsets,
 $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, S$

CARTESIAN PRODUCTS

Sets are unordered.

Ex $\{\{1, 2\}, \{2, 1\}\}$
same element!

In graphing, we need ordered pairs $\begin{matrix} \uparrow \\ \downarrow \\ x \end{matrix}$

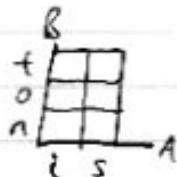
Ex $\{(1, 2), (2, 1)\}$ $\begin{matrix} \uparrow \\ \downarrow \\ \text{different} \\ x \end{matrix}$

or ordered triples (a, b, c) $\begin{matrix} \uparrow \\ \downarrow \\ x \\ y \end{matrix}$

An ordered n-tuple has the form (x_1, x_2, \dots, x_n) .

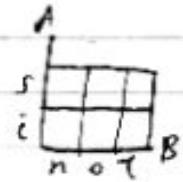
$A \times B =$ Cartesian product of sets A and B
 $= \{(a, b) \mid a \in A, b \in B\}$

Ex $A = \{i, s\}$
 $B = \{n, o, t\}$



$$A \times B = \{(i, n), (i, o), (i, t), (s, n), (s, o), (s, t)\}$$

$$B \times A = \{(n, i), (n, s), (o, i), (o, s), (t, i), (t, s)\}$$



Ex $\mathbb{R} \times \mathbb{R}$
corresponds
to the
standard
xy-plane

Ex $\{1, 3\} \times \{1, 2\} = \{(1, 1), (1, 2), (3, 1), (3, 2)\}$ $\begin{matrix} \uparrow \\ \downarrow \\ \text{can mark} \\ \text{anywhere along} \\ \text{y-axis} \end{matrix}$

Ex $\mathbb{Z} \times \mathbb{Z}$ (lattice points)

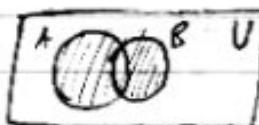
$$\text{In general, } A_1 \times A_2 \times \dots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A_i \text{ for } i=1, 2, \dots, n\}$$

1.5: SET OPERATIONS

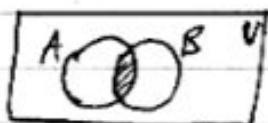
Let A, B, \dots be subsets of some universal set U .

Basic Operations

- ① $A \cup B =$ the union of A and B
 = the set of elements in A or B (inclusive "or")
 = $\{x | x \in A \vee x \in B\}$

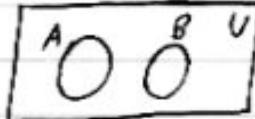


- ② $A \cap B =$ the intersection of A and B
 = the set of elements in A and B
 = $\{x | x \in A \wedge x \in B\}$



If $A \cap B = \emptyset$, A and B are called disjoint.

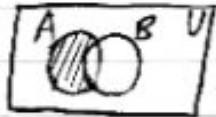
Reform, free party,
there's Ventura



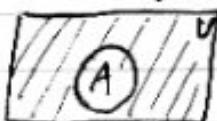
Ex $A = \text{Dems}$, $B = \text{Reps}$.

What do you think?

- ③ $A - B =$ the difference of A and B
= the set of elements in A but not in B
= $\{x \mid x \in A \wedge x \notin B\}$



- ④ \bar{A} or $A^c =$ the complement of A (with respect to U)
= $U - A$
= $\{x \mid x \in U \wedge x \notin A\}$
often assumed



- ⑤ $A \oplus B =$ the symmetric difference of A and B
= the set of elements in A xor B
(in HW)

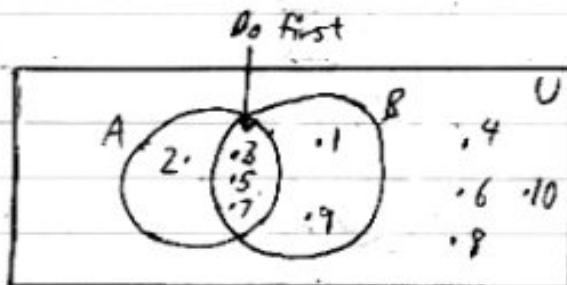
Note $A - B = A \cap \bar{B}$

Ex $U = \{1, 2, 3, \dots, 10\}$
 $A = \{2, 3, 5, 7\}$ (primes)
 $B = \{1, 3, 5, 7, 9\}$ (odds)

Good Idea

Venn Diagram

What should we fill in 1st?



$$A \cup B = \{1, 2, 3, 5, 7, 9\}$$

$$A \cap B = \{3, 5, 7\}$$

$$A - B = \{2\}$$

$$B - A = \{1, 9\}$$

$$\bar{A} = \{1, 4, 6, 8, 9, 10\}$$

$$\bar{B} = \{2, 4, 6, 8, 10\}$$

$$A \oplus B = \{1, 2, 9\}$$

Basic

Set Identities (Table 1 - p. 49)

These correspond to the basic logical equivalences
(Table 5 - p. 17),
Sec 1.2

Just
know

how
1, v work
Recognize;
don't
memorize

{ ① Identity Laws

$$\begin{array}{l} \text{Logic} \\ p \wedge T \Leftrightarrow p \\ p \vee F \Leftrightarrow p \\ \text{no impact} \end{array}$$

$$\begin{array}{l} \text{Set Theory} \\ A \cap U = A \\ A \cup \emptyset = A \end{array}$$



② Domination Laws

$$\begin{array}{l} p \wedge F \Leftrightarrow F \\ p \vee T \Leftrightarrow T \\ \text{dominate } p \end{array}$$

$$\begin{array}{l} A \cap \emptyset = \emptyset \\ A \cup U = U \end{array}$$

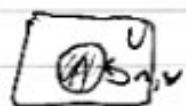


③ Idempotent Laws

$$\begin{array}{l} p \wedge p \Leftrightarrow p \\ p \vee p \Leftrightarrow p \end{array}$$

and/or-ing a prop.
by itself has no impact

$$\begin{array}{l} A \cap A = A \\ A \cup A = A \end{array}$$



④

Double Negation Law

$$\neg(\neg p) \Leftrightarrow p$$

Complementation Law

$$(\bar{A}) = A$$



⑤, ⑥

\wedge is commutative and associative

Ex $a \wedge b \wedge c \dots \wedge z$

can reorder and regroup, () → no impact

So are \vee , \neg , \wedge .

Alain quote GIC
Every time

⑦ Distributive Laws

We know $a \times (b+c) = ab + ac$,
 × distributes over +.

Here, \wedge \vee
 v ^
 ^ v
 n v
 v n

Ex $p \neg(q \vee r) = (p \neg q) \vee (p \neg r)$, etc.

cheat sheet
not until mature
without child bearing

⑧ De Morgan's Laws

Logic

$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

$$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$$

Distribute \neg
 flip connective

$$\text{In general, } \neg(p_1 \wedge p_2 \wedge \dots \wedge p_n) \Leftrightarrow \neg p_1 \vee \neg p_2 \vee \dots \vee \neg p_n, \text{ etc.}$$

Set Theory

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$



The complement of the \cap of the complements
 = the \cup of the complements

Jill
Ex A woman wants a man who is tall and handsome.
How can she be disappointed?

p : The man is tall.

q : The man is handsome.

$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$$

The man is not tall
or not handsome

Gary Coleman
or Steve Buscemi

Ex p_1 : Jim lives in Alabama

p_2 : Jim lives in Alaska

p_{50} : Jim lives in Wyoming

p_{51} : Jim lives in D.C.

"Jim does not live in the U.S."
 $\Leftrightarrow \neg(p_1 \vee p_2 \vee \dots \vee p_{50} \vee p_{51})$

Jim lives in the U.S.

$$\Leftrightarrow \neg p_1 \wedge \neg p_2 \wedge \dots \wedge \neg p_{51}$$

Alabamans and Alaskans and...

Set Theory

Ex A_1 = Alabamans

A_2 = Alaskans

A_{50} = Wyomingites

A_{51} = D.C.

Set of all

non-Americans

= $A_1 \cup A_2 \cup \dots \cup A_{51}$

= $\overline{A_1 \cap A_2 \cap \dots \cap A_{51}}$

= set of all people
who are not

Alabamans, not

Alaskans, ...

and not D.C.

Ex 10-11 (book) pp. 49-50

Prove one of De Morgan's Laws: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

① Two sets are equal \Leftrightarrow they have the same elements.

Consider an arbitrary element "x" in U .

might step some steps

$$\begin{aligned}
 x \in \overline{A \cap B} &\Leftrightarrow x \notin A \cap B \\
 &\Leftrightarrow \neg(x \in A \cap B) \\
 &\Leftrightarrow \neg(\underbrace{x \in A}_{P} \wedge \underbrace{x \in B}_{Q}) \\
 &\Leftrightarrow \neg(x \in A) \vee \neg(x \in B) \\
 &\quad \text{by DeMorgan's Laws (logic)} \\
 &\Leftrightarrow x \in \overline{A} \vee x \in \overline{B} \\
 &\Leftrightarrow x \in \overline{A} \cup \overline{B} \\
 &\Leftrightarrow x \in \overline{A \cup B}
 \end{aligned}$$

x is in $\overline{A \cap B} \Leftrightarrow x$ is in $\overline{A \cup B}$
 have the same elements, so =
 Q.E.D. (end of proof)

② Another approach:

We can show $x \in \overline{A \cap B} \Rightarrow x \in \overline{A} \cup \overline{B}$

Thus, $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$?

We can show $x \in \overline{A} \cup \overline{B} \Rightarrow x \in \overline{A \cap B}$ (go backwards)

Thus, $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Two sets that are subsets of each other are equal.

Q.E.D.

(This approach is superior if there is a step that's not reversible.)

③ Proof by "membership tables" (\approx truth tables).

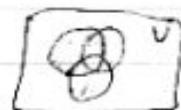
Ex	A	B	C	
	1	1	0	→ see what happens!
8 rows (possible membership combs)				We consider an element ^{that's} in A and in B, but not in C

Idea: If two sets have the same "final column", they are equal.

④ We can use set identities to simplify expressions or to verify harder identities. HW: My #1

Generalized Unions and Intersections

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n \quad \leftarrow \begin{array}{l} \text{don't need } ()_r \\ \text{"well-defined" as is} \end{array}$$



$$x \in \bigcup_{i=1}^n A_i \iff x \in \text{any } A_i \quad (1 \leq i \leq n) \quad \text{(one or more)}$$

$$\bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \dots \cap A_n$$



$$x \in \bigcap_{i=1}^n A_i \iff x \in \text{all of the } A_i \quad (1 \leq i \leq n)$$

HW: #35, 36

HW Tips

Your proofs may look different from the book's. (Ans. in back)

The book uses set builder notation.

It's unclear how many steps you "need."
(On exams - I will understand this.)

Use our basic set identities to prove:

$$(B \cup A) \cap \bar{B} = A \cap \bar{B}$$

We can go \curvearrowleft \curvearrowright
Let's try to simplify the left side \curvearrowright

$$(B \cup A) \cap \bar{B} = \bar{B} \cap (B \cup A)$$

(You don't have to write there.)
Comm. Laws (optional)
Could have distributed ∩.

$$= (\bar{B} \cap B) \cup (\bar{B} \cap A)$$

Distributive Laws
Think: $a + (b+c) = (a+b) + (a+c)$
Put in "()"s

$$= \emptyset \cup \underbrace{(\bar{B} \cap A)}_{\text{some set "C"}}$$

⊗ No elements are in both

$$= \bar{B} \cap A$$

$\emptyset \cup C = C$ (Identity)

$$= A \cap \bar{B}$$

Comm. Laws

My #1

& Morgan's Laws - we used them when we moved →

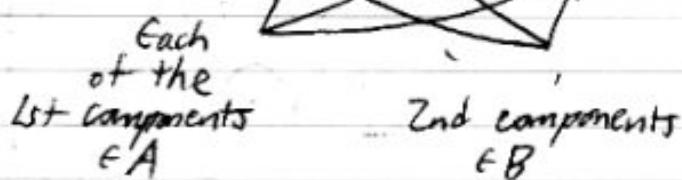
Great HW Tip:
What is the
HW?

1.6: FUNCTIONS

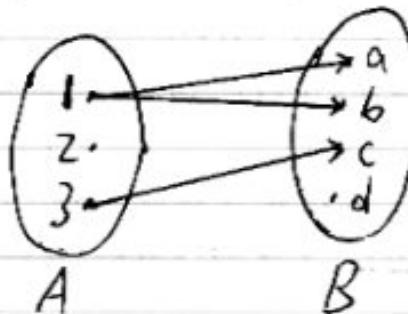
A binary relation from a set A to a set B
is a subset of $A \times B$. It's a set of ordered pairs. ($\binom{A}{B}$)

Ex $A = \{1, 2, 3\}$
 $B = \{a, b, c, d\}$

The relation $R = \{(1, a), (1, b), (3, c)\} \subseteq A \times B$

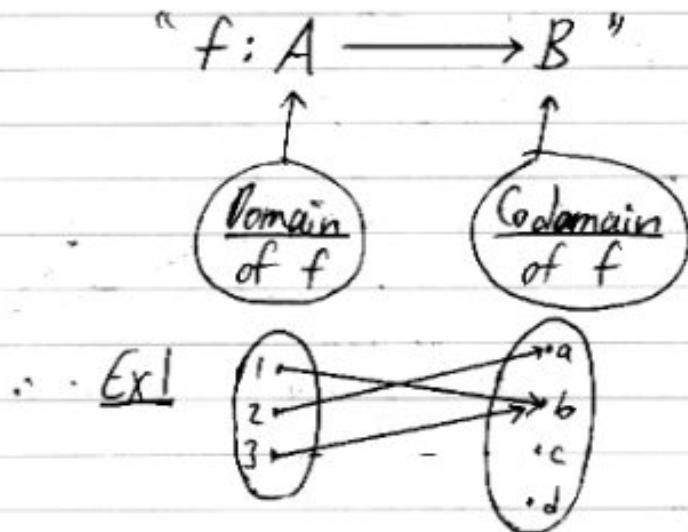


Picture of R :



A function is a special type of binary relation.

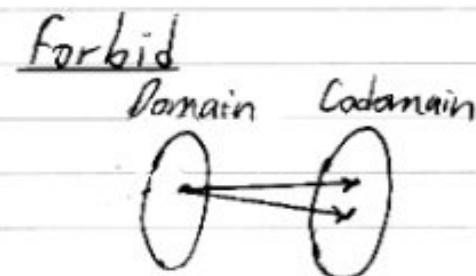
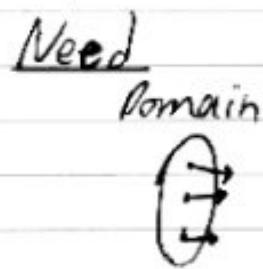
Let f be a function that "maps" a set A to a set B .



$$f = \{(1, b), (2, a), (3, b)\}$$

Say: $f(1) = b$, $f(2) = a$, $f(3) = b$

f must "point" each element of the domain (A) to exactly one element of the codomain (B).



Ex! We have
 $f(1)$
 $f(2)$
 $f(3)$

Ex! We have no
ambiguities like
" $f(1) = a$ or b "

$f(1) = b$

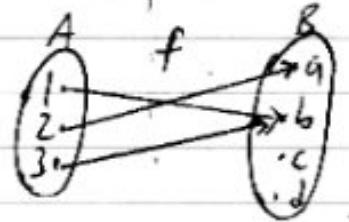
↑
the image of 1

the pre-image of b

$$1 \rightarrow [f] \rightarrow b$$

The range of f is the set of all the images of all the elements in A . " $f(A)$ "
It's a subset of the codomain.

Ex 1



$$f(A) = \text{Range of } f = \{a, b\}$$

= set of all elements
in the codomain (B)
that are "pointed to."

High school algebra

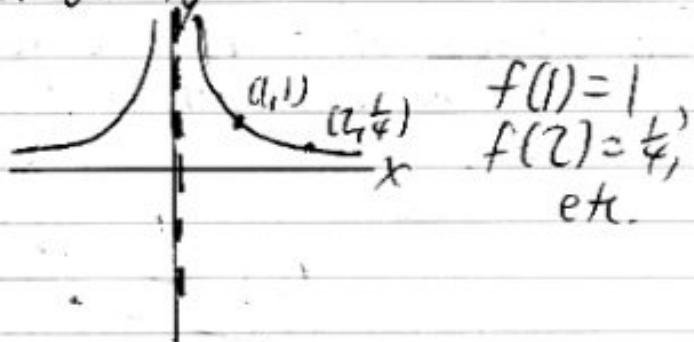
$$f(x) = \frac{1}{x^2} \quad \leftarrow \text{Rule for } f$$

Domain of
definition
(#59)

Assumed domain = $\mathbb{R} \setminus \{0\}$
except

Write " $f: (\mathbb{R} \setminus \{0\}) \rightarrow \mathbb{R}$ "
i. most obvious choice

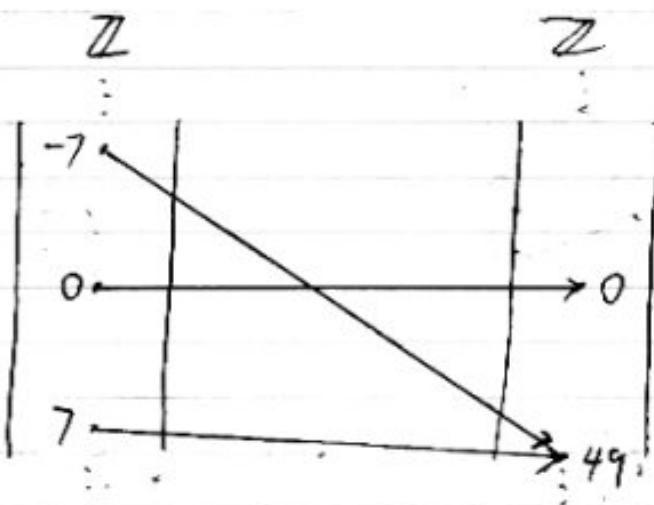
Graph of f



f is a function, because the graph passes the Vertical Line Test (no vertical line passes through more than one point).

The Range of f is the set of all y -coords "hit" by the graph.
Here, the range is \mathbb{R}^+ .

Ex 2 $f: \mathbb{Z} \rightarrow \mathbb{Z}$
Rule: $f(x) = x^2$



Range of $f = \{0, 1, 4, 9, \dots, 49, \dots\}$

$$f(\{0, 3, -4, 4\}) = \{0, 9, 16\}$$

ONE-TO-ONE FUNCTIONS

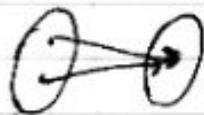
Function

1-1 function

Forbidden:

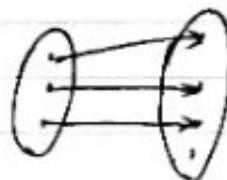


Also forbidden:



A function f is one-to-one (or injective)
 $\Leftrightarrow \forall x_1 \forall x_2 [f(x_1) = f(x_2) \rightarrow x_1 = x_2]$ (A)
in domain of f
 $\Leftrightarrow \forall x_1 \forall x_2 [x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)]$

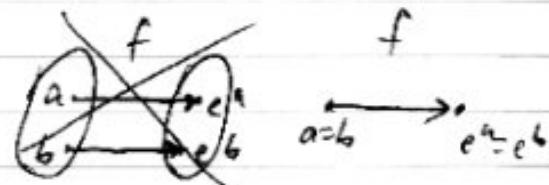
Idea: An image has only one pre-image.



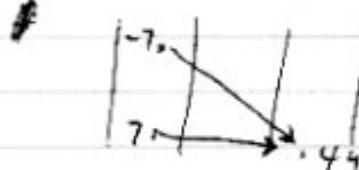
(A) Ex $f(x) = e^x$, $f: \mathbb{R} \rightarrow \mathbb{R}$
 f is a 1-1 function.

$\nexists x$ The graph passes
the Horizontal Line Test

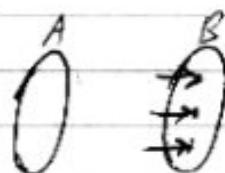
If $e^a = e^b$, then
 $a = b$



f in Ex 2 ($f(x) = x^2$, $f: \mathbb{Z} \rightarrow \mathbb{Z}$) is not one-to-one



\therefore fails Horizontal Line Test
 $x^2 = 49 \leftarrow 2 \text{ pre-images}$

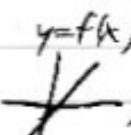
ONTO FUNCTIONS

Every element in the codomain must get "hit".
(Codomain = Range)

A function $f: A \rightarrow B$ is onto or surjective \Leftrightarrow
 $\forall b \in B \exists a \in A [f(a) = b]$
 wodt für b wodt für A

f in Ex 2 ($f(x) = x^2$, $f: \mathbb{Z} \rightarrow \mathbb{Z}$) is not onto,
 because not every element in the codomain (\mathbb{Z}) is in the range ($\{0, 1, 4, 9, \dots\}$).

Ex $f: \mathbb{R} \rightarrow \mathbb{R}$
 $f(x) = 3x$

Approach 1: Graph $y=f(x)$  Range = \mathbb{R}

So, Codomain = Range.
 So, f is onto.

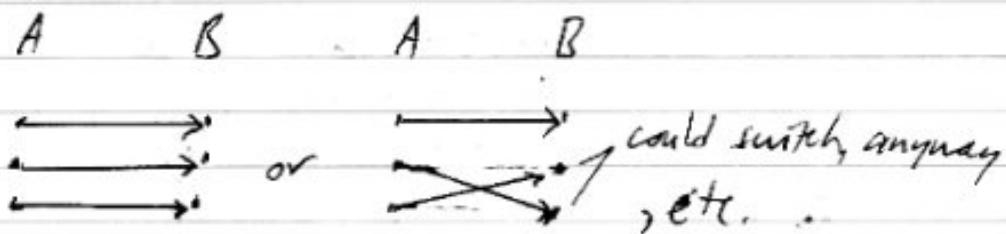
Approach 2:
 Let b be any element in the codomain (\mathbb{R}).
 Can we always find a pre-image for b ? YES

$$b = 3x \\ \Rightarrow x = \frac{b}{3} \text{ (pre-image)} \\ \text{So, } f \text{ is onto.} \quad ? \rightarrow b$$

(in domain ✓)

BIJECTIONS

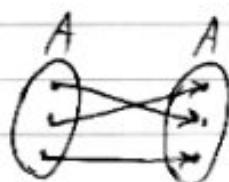
A bijection (or a one-to-one correspondence)
is a function that is both one-to-one and onto.



(You can get this picture,
maybe by moving dots.)

Special Case

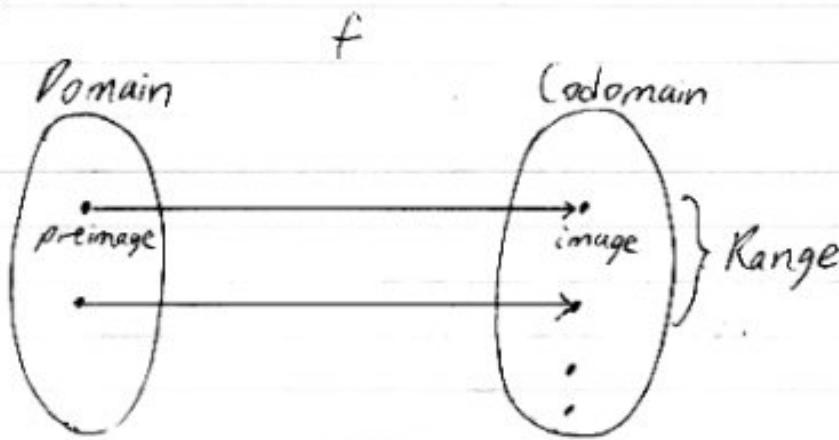
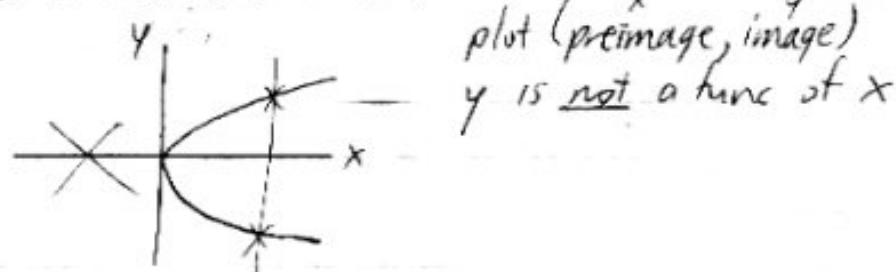
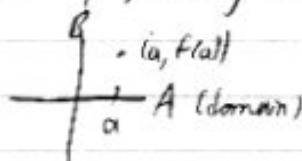
$f: A \rightarrow A$, where A is a finite set



Key: same-size sets

f is onto $\Leftrightarrow f$ is 1-1. (\supseteq would be a waste!)

If f is 1-1 or onto $\Rightarrow f$ is a bijection

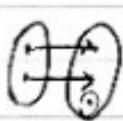
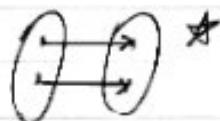
ReviewNot functionsf is not a func from \mathbb{R} to \mathbb{R} :2 reasons: negative \mathbb{R} s don't get mapped
graph fails VLTIf f is a function from A to B,
graph f by plotting the points $\{(a, f(a)) | a \in A\} \subseteq A \times B$ 

Onto?

Y

N

Y

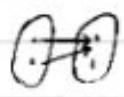


$f: \mathbb{R} \rightarrow \mathbb{R}$

$$\begin{aligned} &\text{passes HLT} & f(x) \\ & \cancel{\exists x} e^x & = e^x \\ & & \rightarrow a = b \end{aligned}$$

1-1?

N



$$\begin{aligned} &\text{fails HLT} & \exists x f: \mathbb{R} \rightarrow \mathbb{R} \\ & \cancel{\exists x} f(x) = x^2 & f(x) = x^2 \\ & & a^2 = b^2 \\ & & \rightarrow a = b \end{aligned}$$

Range =
Codomain

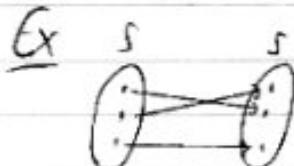
* A function that is both 1-1 and onto
is called a bijection (or a one-to-one correspondence)

Special Case

If S is a finite set, then for $f: S \rightarrow S$,

$\left\{ \begin{array}{l} \text{1-1} \\ \text{onto} \\ \text{bijective} \end{array} \right.$

 $\left\{ \begin{array}{l} \text{if any one} \\ \text{holds, then} \\ \text{all 3 hold.} \end{array} \right.$



A 1-1 $f: S \rightarrow S$
must be onto,
and vice versa

INVERSE FUNCTIONS

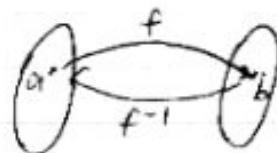
Let $f: A \rightarrow B$ be a bijection.

Then, f has an inverse function

$f^{-1}: B \rightarrow A$, which is a bijection that reverses f .

How is f^{-1} defined?

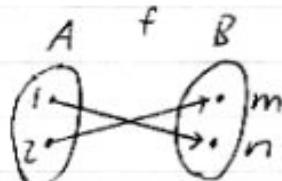
$$f(a) = b \Leftrightarrow f^{-1}(b) = a$$



$$\text{Note: } (f^{-1})^{-1} = f$$

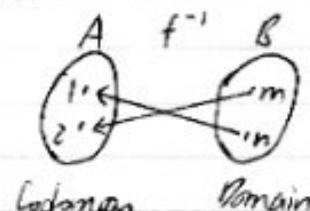
f is a bijection $\Leftrightarrow f$ is invertible

Ex



f is a bijection, so f is invertible.

$$\begin{aligned} f(1) &= n, \text{ so } f^{-1}(n) = 1 \\ f(2) &= m, \text{ so } f^{-1}(m) = 2 \end{aligned}$$



Compositions combine functions (pp. 62-4; not tested): $f(g(x))$

$\lfloor x \rfloor, \lceil x \rceil$ Two key functions that map $\mathbb{R} \rightarrow \mathbb{Z}$ Let $x \in \mathbb{R}$.The floor function (or greatest integer function) rounds x down.

$$\begin{aligned} f(x) &= \lfloor x \rfloor \text{ or } [x] \\ &= (\text{closest integer that's } \leq x) \end{aligned}$$

The ceiling function rounds x up.

$$\begin{aligned} f(x) &= \lceil x \rceil \quad \leftarrow \text{"goes" on ceiling} \\ &= (\text{closest integer that's } \geq x) \end{aligned}$$

Exs

$$\begin{array}{ccc} x & \lfloor x \rfloor & \lceil x \rceil \end{array}$$

$$\begin{array}{ccc} 5 & 5 & 5 \\ 9.1 & 9 & 10 \\ -10.7 & -11 & -10 \end{array}$$

$$\begin{array}{c} \lfloor x \rfloor \leq x \leq \lceil x \rceil \text{ always} \\ \hline \leftarrow \rightarrow \mathbb{R} \end{array}$$

Ex We have 50 balls.

A bag carries 12 balls.

How many bags do we need?

To be safe,

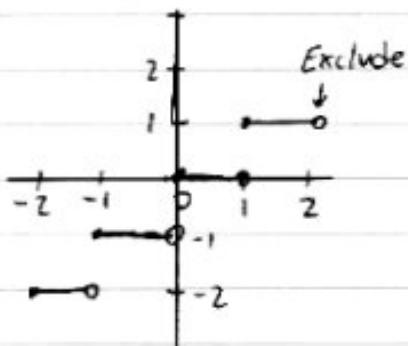
$$\left\lceil \frac{50}{12} \right\rceil = \left\lceil 4.\bar{1} \right\rceil = 5 \text{ bags.}$$

How many bags can we fill up?

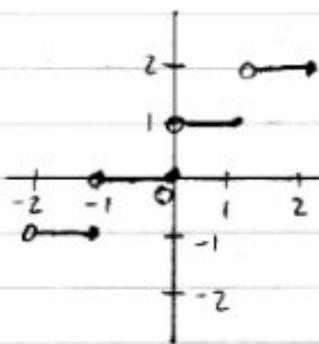
$$\left\lfloor \frac{50}{12} \right\rfloor = \left\lfloor 4.\bar{1} \right\rfloor = 4 \text{ bags}$$

$\lfloor x \rfloor$, $\lceil x \rceil$ are step functions

$$y = \lfloor x \rfloor$$



$$y = \lceil x \rceil$$



HW Tip

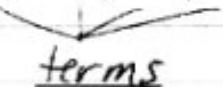
In Rosen $\mathbb{N} = \{0, 1, 2, 3, \dots\}$

some books exclude

1.7: SEQUENCES AND SUMMATIONS

A sequence represents an ordered list.

Ex The sequence $\{a_n\}$ ("a-sub-n")
usually denotes

$a_1, a_2, a_3, a_4, \dots$

terms

We sometimes start with a_0 , not a_1 :

$a_0, a_1, a_2, a_3, \dots$

(typical in calculus)

A string is a finite sequence.

Ex $1, 0, 0, 1$ $\{a_n\}_{n=1}^4$
 $a_1 \quad a_2 \quad a_3 \quad a_4$

The general term (a_n) in a sequence may be given by a formula or rule.

Ex $a_n = (-1)^n 3n$, Start with $n=1$
(assume unless told otherwise)

$$n=1: a_1 = (-1)^1(3)(1) = (-1)(3) = -3$$

$$n=2: a_2 = (-1)^2(3)(2) = (1)(6) = 6$$

$$n=3: a_3 = (-1)^3(3)(3) = (-1)(9) = -9$$

$$n=4: a_4 = (-1)^4(3)(4) = (1)(12) = 12$$

Sequence:

$$-3, 6, -9, 12, \dots \leftarrow \text{an alternating sequence}$$

(signs alternate)

In I.Q. tests, you're asked to find the most "obvious" pattern:

Ex $3, 7, 11, 15, 19, \dots$

$\underbrace{\quad}_{\infty \text{ possibl},}$
but $23, 27, 31, \dots$
seem most
"obvious"
(successively
adding 4)

Web p.72

can input 6+ few terms
puzzle seqs.

Special Sequence Types

① Arithmetic progressions

- successively add some common difference "d"

Ex $3, 7, 11, 15, 19, \dots$

$$a_1 = 3, d = 4$$

$$a_n = a_1 + (n-1)d = 3 + (n-1)(4)$$

$\# \text{steps} \quad \text{step size}$

② Geometric progressions

- successively multiply some common ratio "r"

Ex $5, 15, 45, 135, \dots$

$$a_1 = 5, r = 3$$

$$a_n = a_1 \cdot r^{\underbrace{n-1}_{\text{# steps}}} = 5(3)^{n-1}$$

$\underbrace{\quad}_{\text{multiplic}} \quad$

SUMMATIONS

$$\text{Ex } \sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n \quad (n \in \mathbb{Z}^+)$$

(Sum of 1st n terms)

$$\text{Ex } \sum_{i=m}^n a_i = a_m + a_{m+1} + \dots + a_n \quad (m, n \in \mathbb{Z})$$

i is the index of summation
(could use j, k, \dots)

The index sweeps through all the integers from the lower limit (m) to the upper limit (n).

$$\begin{aligned}\text{Ex } \sum_{k=3}^5 \underbrace{(k \cdot 2^k - 1)}_{a_k} &= a_3 + a_4 + a_5 \\ &= [3 \cdot 2^3 - 1] \\ &\quad + [4 \cdot 2^4 - 1] \\ &\quad + [5 \cdot 2^5 - 1] \\ &= 23 + 63 + 159 \\ &= 245\end{aligned}$$

$$\text{Ex } \sum_{i \in S} a_i \quad \text{where } S = \{1, 4, 6\}$$

\leftarrow indices sweep through S

$$= a_1 + a_4 + a_6$$

Ex (Double summation)

Arise from nested loops in programs.

$$\sum_{i=1}^3 \sum_{j=1}^i \frac{i}{j} = (\text{Case } i=1) + (\text{Case } i=2) + (\text{Case } i=3)$$

$$= \begin{cases} i=1 & \sum_{j=1}^1 \frac{1}{j} \\ i=2 & + \sum_{j=1}^2 \frac{2}{j} \\ i=3 & + \sum_{j=1}^3 \frac{3}{j} \end{cases}$$

$$= \frac{1}{1}$$

$$+ \frac{2}{1} + \frac{2}{2}$$

$$+ \frac{3}{1} + \frac{3}{2} + \frac{3}{3}$$

$$= \left(9\frac{1}{2} \right)$$

Table 2 (p. 76) has special Σ formulas,

HW Tip #17) $\sum_{i=\#}^{\#} \sum_{j=\#}^{\#} f(i,j)$

may be faster
to work this out first

(Not tested) SPECIAL SUMS

$$\text{Ex1 } \sum_{i=1}^{100} i = 1 + 2 + 3 + \dots + 100$$

Gauss - when he was 10, his teacher wanted to occupy the class. Slapped him - really scared.

Trick:

$$\begin{aligned} S &= (1 + 2) + (3) + \dots + (100) \\ S &= (100 + 99) + (98) + \dots + (1) \end{aligned}$$

↓ reverse

$$2S = 101 + 101 + 101 + \dots + 101$$

100 copies

$$2S = 101(100)$$

$$S = \frac{101(100)}{2} = 5050$$

$$\underline{\text{Ex 2}} \text{ In general, } \sum_{i=1}^n i = 1 + 2 + 3 + \dots + n \quad (n \in \mathbb{Z}^+)$$

Same trick!

etc.

$$\begin{aligned} S &= (1) + (2) + \dots + (n) \\ S &= (n) + (n-1) + \dots + (1) \\ 2S &= (n+1) + (n+1) + \dots + (n+1) \\ &\qquad\qquad\qquad \underbrace{\hspace{10em}}_{n \text{ copies}} \\ S &= \frac{n(n+1)}{2} \end{aligned}$$

More generally, let's consider the sum of any n consecutive terms in an arithmetic sequence.

Let's say $a_1 + a_2 + \dots + a_n$

$$\text{Sum} = \underbrace{\left(\frac{a_1 + a_n}{2} \right)}_{\substack{\text{Average} \\ \text{of 1st,} \\ \text{last terms}}} (n) \quad \# \text{ terms}$$

$$\begin{aligned} S &= a_1 + \overbrace{a_2 + \dots + a_n}^{a_1 + d} \\ S &= a_n + \overbrace{a_{n-1} + \dots + a_1}^{a_n - d} \\ &\downarrow \qquad \qquad \qquad \text{same} \\ (a_1 + a_n) + (\text{same}) + \dots + (a_1 + a_n) & \qquad \qquad \qquad \text{n copies} \\ 2S &= (a_1 + a_n)(n) \\ S &= \frac{(a_1 + a_n)}{2}(n) \end{aligned}$$

Why does
this make
sense?
Interpret
this formula

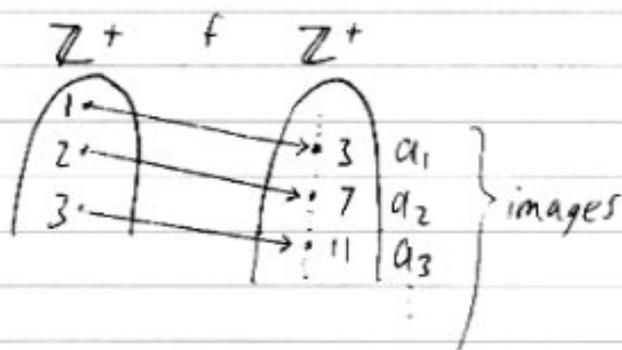


$$\begin{aligned} \text{Ex } 7 + 8 + 9 + 10 + 11 &= 9 + 9 + 9 + 9 + 9 = (9)(5) = 45 \\ \text{Communism } \text{Ex } \underbrace{3 + 7 + 11 + \dots + 83}_{21 \text{ terms}} &= \left(\frac{3 + 83}{2} \right)(21) = 903 \end{aligned}$$

CARDINALITY

A sequence a_1, a_2, a_3, \dots (each $a_i \in S$)
 is a function $f: \mathbb{Z}^+ \rightarrow S$

Ex. $3, 7, 11, 15, \dots$ ← each $\in \mathbb{Z}^+$



A set is countable \Leftrightarrow

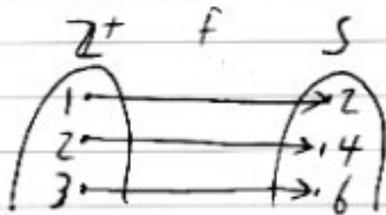
- ① The set is finite.
- or ② All the elements can be listed as
 a_1, a_2, a_3, \dots

card.
algebraic
 \aleph_0

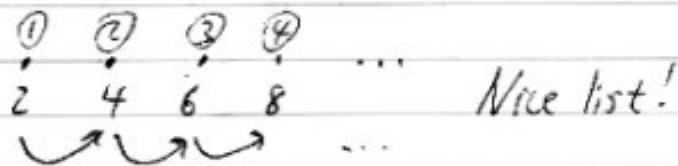
(i.e., There is a bijection (1-1 corrsp.)
 between the set and \mathbb{Z}^+)

Ex Let S = set of all positive even integers
 $= \{2, 4, 6, 8, \dots\}$

S is countable.



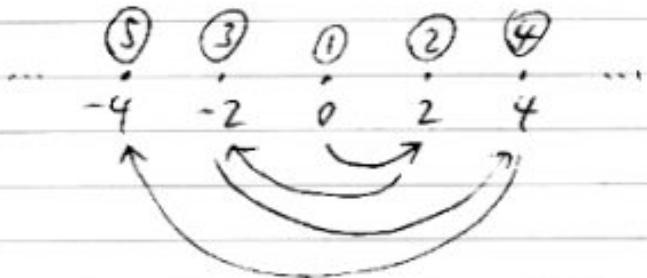
f defined by $f(n) = 2n$ is a bijection $\mathbb{Z}^+ \rightarrow S$



Ex Let S = set of all even integers
 $= \{\dots, -4, -2, 0, 2, 4, \dots\}$

S is countable.

How can we make a nice list?

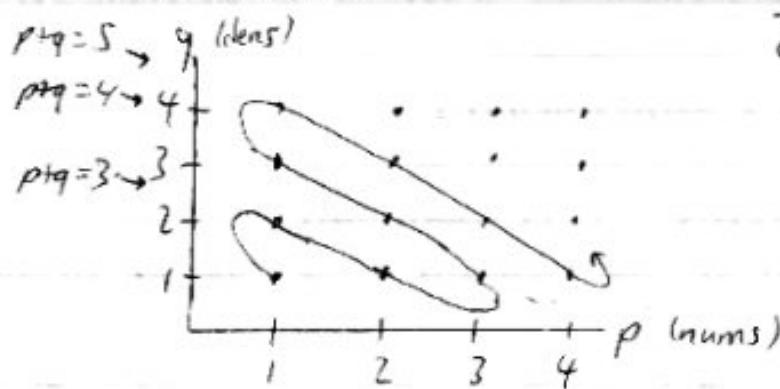


Sequence: 0, 2, -2, 4, -4, ...

Ex \mathbb{Q}^+ = set of all positive rational #'s

$$= \left\{ x \mid x \text{ can be written as } \frac{p}{q}, \text{ where } p \in \mathbb{Z}^+, q \in \mathbb{Z}^+ \right\}$$

is countable.



$\mathbb{Z}^+ \times \mathbb{Z}^+$ lattice

How can we make a nice list?

- ① $\frac{1}{1} = 1$
- ② $\frac{1}{2}, \frac{2}{1} = 2$
- ③ $\frac{2}{2} = 1$ (repeat!)
- ④ $\frac{3}{1} = 3, \frac{2}{3}, \frac{3}{2}, \dots$
- ⑤ $\frac{1}{3}$

$$\begin{aligned} p+q &= 2 \\ p+q &= 3 \\ p+q &= 4 \end{aligned}$$

List: $1, \frac{1}{2}, 2, \frac{2}{1}, \frac{1}{3}, \dots$

Ex $\mathbb{Q} = \text{all rational #'s}$ is countable.

List: $0, 1, -1, \frac{1}{2}, -\frac{1}{2}, 2, -2, \dots$
need negative partners

Ex Show \mathbb{R} is uncountable.

Let $S = \{x \mid x \in \mathbb{R}, 0 < x < 1\}$

It is sufficient to show S is uncountable.

Proof by Contradiction

Assume S is countable.

Then, all the elements of S can be listed:

$x_1: 0.\underline{d_{11}}d_{12}d_{13}\dots$ (all d_{ij} are digits)

$x_2: 0.d_{21}\underline{d_{22}}d_{23}\dots$

$x_3: 0.d_{31}d_{32}\underline{d_{33}}\dots$

:

We can construct a new # in S that is not on the list.

new # = $0.\underset{\text{P}}{d_1}\underset{\text{Q}}{d_2}d_3\dots$

$\begin{array}{ll} 5 \text{ if } d_{11}=4 & 5 \text{ if } d_{22}=4 \\ 4 \text{ if } d_{11} \neq 4 & 4 \text{ if } d_{22} \neq 4 \end{array} \dots$

Ex

$$\begin{aligned}x_1 &= 0.\underline{7}12\dots && \text{Diagonalization argument} \\x_2 &= 0.3\underline{4}6\dots \\x_3 &= 0.55\underline{9}\dots \\&\quad \downarrow\downarrow\downarrow \text{etc.}\end{aligned}$$

$$\text{New \#} = 0.454\dots$$

Idea: The new # will differ from each listed # by at least 1 digit.

In particular, the new # and x_i will differ in the i^{th} decimal place.

BUT we assumed that all the elements in S were listed!

Our list will never be "good enough!"

So, our assumption that S was countable was wrong.

card of
the continuum

$\therefore S$ is uncountable,
 $\therefore \mathbb{R}$ is uncountable.

See Ex 17 on pp 77-8.