1.1: LOGIC

Gateway to precise thinking

A proposition is a statement that is either true (T) or false (F).

Ex:
Are props. Are not props.

Ex:  

All Gore is a Dem. (T)  Whazzup?
G.W. Bush is a Dem. (F)  Just do it.
The sky is blue,  
(but depends on context, opinion)

1 + 1 = 2. (T)
1 + 1 = 3. (F)

2x = 10 if x = 5. (T)  2x = 10,  
(assume we know nothing about x)

In CS, 0 = false, 1 = true.

In fuzzy logic (AI), a proposition can have a truth value between 0 and 1.
Ex: Ken is tall, 0.3

Overhead
Let \( p, q, r, \text{ etc.} \) denote propositions. We use logical operators to build compound propositions.

1. \( \neg p \) means \( \{ \) not \( p \) \\
   \{ the negation of \( p \) \\
   \{ it is not the case that \( p \) \( \}

Truth table for \( \neg p \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \neg p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( F )</td>
</tr>
<tr>
<td>( F )</td>
<td>( T )</td>
</tr>
</tbody>
</table>

If \( p \) is a \( T \) prop., then \( \neg p \) must be a \( F \) prop.

Ex: \( p: 1+1=2 \quad (T) \)
\( \neg p: 1+1\neq 2 \quad (F) \)

Ex: \( a, b \) are real \#s
\( p: a>b \)
\( \neg p: \) it is not the case that \( a>b \) \( (\text{i.e., } a\leq b) \)

Note: \( \neg \) is a unary operator.

Connectives are operators that combine 2 or more props.
\( \text{2) } p \land q \text{ means } \{ \text{p and q} \}
\text{the conjunction of p and q} \)

**Truth Table**

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>( p \land q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

\( p \land q \text{ is a T prop.} \quad \Rightarrow \quad p \text{ and q are both T.} \)

*Ex* 
\( p: \text{Gore will win. (?)} \)
\( q: 1+1=3. \quad (F) \)
\( p \land q: \text{Gore will win and } 1+1=3. \quad (F, \text{ regardless}) \)
(3) \( p \lor q \) means \{ p or q \} (inclusive or) \\
the disjunction of \( p \) and \( q \)

Truth Table

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \lor q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( T )</td>
<td>( F )</td>
<td>( T )</td>
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<td>( F )</td>
<td>( T )</td>
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<tr>
<td>( F )</td>
<td>( F )</td>
<td>( F )</td>
</tr>
</tbody>
</table>

One or both of \( p, q \) \( T \)

\( p \lor q \) is a F prop. \( \iff \)
\( p \) and \( q \) are both \( F \)

(4) \( p \oplus q \) means \{ exclusive or of \( p \) and \( q \} \)
\( p \) xor \( q \)

Truth Table

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \oplus q )</th>
</tr>
</thead>
</table>
| \( T \) | \( T \) | \( F \) \( \iff \) different from \( p \lor q \)
| \( T \) | \( F \) | \( T \) |
| \( F \) | \( T \) | \( T \) |
| \( F \) | \( F \) | \( F \) |

\( p \oplus q \) is a T prop. \( \iff \)
exactly one of \( p \) or \( q \) is \( T \) (but not both!)

(extraordinary case)
Ex Let's say:

\[ p: \text{you have soup. } (T) \]
\[ q: \text{you have salad. } (T) \]

Then, \( p \oplus q : \text{you have soup xor salad. } (F) \)
but \( p \lor q \)

Note: In math, "or" is an inclusive or (\( \lor \)), unless otherwise specified.

5. The implication \( p \rightarrow q \)

\[ \begin{array}{c}
\text{hypothesis} \\
\text{antecedent} \\
\text{premise}
\end{array} \quad \begin{array}{c}
\text{conclusion} \\
\text{consequence}
\end{array} \]

Ex: \( d: \text{Pat is a daddy,} \)
\[ m: \text{Pat is a man.} \]

There are many ways to say \( d \rightarrow m \)

- if \( d, \) [then] \( m \)
- \( d \) implies \( m \)
- \( d \) only if \( m \)
- \( d \) is sufficient for \( m \)

\[ m \text{ if } d \]
\[ m \text{ whenever } d \]
\[ m \text{ is necessary for } d \]
(p → q is a F prop) ⇐⇒ (p is T and q is F) "breaking a contract"

Truth table

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p → q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

If the hypothesis (p) is F, the prop. p → q is T, "by default".

Ex: p: 1+1=3 (F)
q: Gore will win (?)
Then, p → q: If 1+1=3, then Gore will win (T "by default")

Ex: If 1+1=2, then a lemon is yellow. (T)

p and q need not be related! (Sign: Correlation does not imply causation)

Assume Pat is a human, (or d)
This prop. must be T, because realistically there's no way p is T and q is F.

Math Ex: If an integer ends in a "3", it's an odd #.
Same idea!
Think of an "if-then" statement as a conjecture (unproven hunch). If the prop is T, we have a theorem; we can put in books. If the prop is F, no grant $!! jungle.

More

The converse of \( p \rightarrow q \) is \( q \rightarrow p \).

The contrapositive of \( p \rightarrow q \) is \( \neg q \rightarrow \neg p \).

These are "logically equivalent" (Sec 1.2): both are T or both are F.

Ex Write the converse of "Pat is a man whenever Pat is a Daddy."

Rewrite in "if-then" form:

"If Pat is a Daddy, then Pat is a man."

\[ p \rightarrow q \]

The converse is:

"If Pat is a man, then Pat is a Daddy."

\[ q \rightarrow p \]

This new prop is fundamentally different from the old one!

Ex The contrapositive is:

"If \( \neg q \), then \( \neg p \"

"If Pat is not a man, then Pat is not a Daddy."

This is fundamentally the same as the old prop!
6. The biconditional \( p \leftrightarrow q \)

means

- \( p \) if and only if \( q \)
- \( p \) is necessary and sufficient for \( q \)
- if \( p \) then \( q \), and if \( q \) then \( p \)

"conversely"

\[ (p \leftrightarrow q \text{ is } T) \iff (p, q \text{ are both } T \text{ or both } F) \]

Truth table

\[
\begin{array}{ccc}
 p & q & p \leftrightarrow q \\
 T & T & T \\
 T & F & F \\
 F & T & F \\
 F & F & T \\
\end{array}
\]

Ex. If we know this is a true prop.

"Pat is a man \( \leftrightarrow \) Pat has an Adam's apple," then both are \( T \), or both are \( F \)
HW Tips  (Relevant: pp. 1-8; ignore Boolean search, bit strings)

Ex  
\[ p: \text{I am young.} \]
\[ q: \text{I am stupid.} \]
\[ r: \text{I can attend the movie.} \]

Express \((p \lor q) \rightarrow r\) as an English sentence.

Order of ops: () 1st, ,  next, \( \lor, \lor, \rightarrow \rightarrow \)

"If I am young or stupid, then I cannot attend the movie."

Ex (23d) Construct a truth table for \((p \rightarrow q) \land (\neg p \rightarrow q)\)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>((p \rightarrow q))</th>
<th>((\neg p \rightarrow q))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

Concise answer:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>((p \rightarrow q) \land (\neg p \rightarrow q))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
19a) Converse: If I will ski tomorrow, then it snows today.

Let's say both are T.

Then, this implication is T.

In conversation, a cause-and-effect relationship is implied.

Our sense of logic does not focus on content or cause-and-effect.
1.2: PROPOSITIONAL EQUivalences

A tautology is a compound proposition that is always T, regardless of the truth values (T/Fs) of its constituent props.

Example: $p \lor \neg p$ (Hamlet is or is not.)

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\neg p$</th>
<th>$p \lor \neg p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

$\Rightarrow$ tautology!

A contradiction is always $F$.

Example: $p \land \neg p$ ($p, \neg p$ can't both be T.)

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\neg p$</th>
<th>$p \land \neg p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
</tbody>
</table>

$\Rightarrow$ contradiction!

A contingency is neither.
Props, that (essentially) have the same truth tables are logically equivalent ($\iff$ or $\equiv$).

Ex. Show $p \rightarrow q \iff \neg p \lor q$

- $p \rightarrow q$
  - $p \quad q \quad p \rightarrow q$
  - $T \quad T \quad T$
  - $T \quad F \quad F$
  - $F \quad T \quad F$
  - $F \quad F \quad T$

Can combine like Rosen

- $p \lor q$
  - $p \quad q \quad \neg p \quad \neg p \lor q$
  - $T \quad T \quad F \quad T$
  - $T \quad F \quad F \quad T$
  - $F \quad T \quad T \quad F$
  - $F \quad F \quad T \quad T$

- $p \rightarrow q$
  - $p \quad q \quad \neg p \lor q$
  - $T \quad T \quad T$
  - $T \quad F \quad T$
  - $F \quad T \quad T$
  - $F \quad F \quad T$

match, so $p \rightarrow q \iff \neg p \lor q$

Idea: $p \rightarrow q$ is $T$ exactly when $\overline{\overline{0}} = \overline{1}$
“\( p \leftrightarrow q \) means that, in any particular situation, both \( p \) and \( q \) are \( T \), or both are \( F \) (i.e., \( p \leftrightarrow q \) is a tautology).

Ex Let \( n \) be some integer.
\( n \) is odd \( \iff \) its ones' digit is \( 1, 3, 5, 7, \) or \( 9 \).

  If \( n = 13 \), both are \( T \).
  If \( n = 14 \), both are \( F \).

Note 1 Size of truth tables

\[
\begin{array}{c|c|c}
 p & q & r \\
\hline
\end{array}
\]

8 rows

In general,

\[
\begin{array}{c}
 p_1, p_2, \ldots, p_n \\
\end{array}
\]

\(2^n\) rows

Note 2 Table 5 (p.17) gives a list of basic equivalence laws, which can be used to simplify props., to show that two props. are equiv., or to show that props. are tautologies or contradictions.

Don't worry now, but we will learn their twin brothers, the set identities in 1.5.
Some laws:

\[-(\neg p) \iff p\]

\(^V^, \wedge\) are commutative and associative

\(^V\) distributes over \(^\wedge\) (like \(^\times\) over \(^+\))

\(^\wedge\) distributes over \(^V\) (\(\iff\) HW #5)

De Morgan's laws (HW #6 - one of them)

**HW Tips** (Relevant: p 14-16; scan p 17-19)

Use truth tables.

#16) Show \(p \rightarrow q \iff \neg q \rightarrow \neg p\).

What are we proving here?

An implication is logically equivalent to its contrapositive.
1.3: Predicates and Quantifiers

Ex 1 Let \( P(x) \) denote the statement \( "x < 5." \)

\( P \) is a predicate denoting the property "is less than 5."

\( P(x) \) is not a proposition, but it becomes a prop. (T/F) once we assign a value to \( x \).

\( P(x) \) is a propositional function that depends on the variable \( x \).

Ex In the case \( x = 4 \), "4 < 5" is \( T \), so \( P(4) \) is \( T \).

Ex In the case \( x = 5 \), "5 < 5" is \( F \), so \( P(5) \) is \( F \).

Ex 2 Let \( Q(x, y) \) denote "\( y = 3x + 1.\)"

Ex In the case \( (x = 2, y = 7) \), "7 = 3(2) + 1" is \( T \), so \( Q(2, 7) \) is \( T \).
∀ is the "universal quantifier" (say "for all")

∀x P(x) denotes the proposition

"P(x) is T for all values of x, in the universe of discourse,"

the domain of P
(we'll say "uod")

Some common "uod"s:

\[ \mathbb{Z} \] is the set of all integers
\[ \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\} \]

\[ \mathbb{Z}^+ \] is the set of all positive integers.
\[ \mathbb{Z}^+ = \{1, 2, 3, ...\} \]

\[ \mathbb{R} \] is the set of all real #s.

Special case: Finite "uod"s

If the uod is a finite set, say \{x_1, x_2, ..., x_n\}, then

\[ \forall x P(x) \iff P(x_1) \land P(x_2) \land ... \land P(x_n) \]

is T exactly when all these are T
Ex Let \( P(x) \) be "\( x \) is an integer."

If the uod = \( \{1, 7, 15\} \), then \( \forall x P(x) \) is \( T \), because
\[
P(1) \land P(7) \land P(15) \land T.
\]

If the uod = \( \{1, 7, \pi\} \), then \( \forall x P(x) \) is \( F \), because
\[
P(1) \land P(7) \land P(\pi) \land F.
\]

one \( F \)

ruins it for everybody

The uod can be an infinite set.

Ex Let \( P(x) \) be "\( 2x > x \)."

If the uod is \( \mathbb{Z}^+ \), \( \forall x P(x) \) is \( T \).

If the uod is \( \mathbb{Z} \), \( \forall x P(x) \) is \( F \).

For example, \( P(-1) \) is \( F \). (-2 \( \not> \)-1).
\( x=-1 \) is called a counterexample.
\( \exists \) is the "existential quantifier" (say "there exists/is")

\( \exists x \ P(x) \) denotes the proposition

"\( P(x) \) is T for at least one element (x) in the uod."

**Special case:** Finite "uod"s

If the uod is \( \{ x_1, x_2, ..., x_n \} \), then

\[
\exists x \ P(x) \iff P(x_1) \lor P(x_2) \lor ... \lor P(x_n)
\]

is T exactly when at least one is T

*Ex* Let \( P(x) \) be "x is an integer."

If the uod = \( \{ 1, \pi, e \} \), then \( \exists x \ P(x) \) is T, because

\[
P(1) \lor P(\pi) \lor P(e) \text{ is T.}
\]

If one \( \lor \) does it!

If the uod = \( \{ \sqrt{2}, \pi, e \} \), then \( \exists x \ P(x) \) is F, because

\[
P(\sqrt{2}) \lor P(\pi) \lor P(e) \text{ is F.}
\]

all are F
Ex Let \( P(x) \) be "\( x^2 = 16 \)."
Let the \text{uod} be \( \mathbb{Z} \).

Then, \( \exists x \ P(x) \) is \( T \).
For example, \( P(4) \) is \( T \). \( P(-4) \), also.

Ex Let \( Q(x) \) be "\( x^2 = 17 \)."
Let the \text{uod} be \( \mathbb{Z} \).

There is no integer whose square is 17,
so \( \exists x \ Q(x) \) is \( F \).

Ex \( P(x) \): "\( x \) is even", \text{uod} = \( \mathbb{Z} \)

\[
P(4) \quad \text{"4 is even" (T)}
\V x \ P(x) \quad \text{"all integers are even" (F)} \] these
\E x \ P(x) \quad \text{"there is an even integer" (T)} \] propositions

In these cases, \( x \) is \textbf{bound} by a
value assignment or a quantifier.

In general, a \textbf{propositional function} becomes a
proposition only when all the variables are bound.
MORE THAN 1 VARIABLE

Imagine a grid:

\[ \forall x \forall y \ P(x, y) \text{ is } T \iff \text{ all combos make } P \text{ true} \]

\[ \text{ is } F \iff \text{ there is a counterexample that makes } P \text{ false} \]

Ex \[ P(x, y): \ "xy = yx" \]
\[ \text{ uod } = \mathbb{R} \text{ for both } x, y \]

Then, \[ \forall x \forall y \ P(x, y) \text{ is } T. \]
(Multiplication is commutative for all pairs of real #s.)

Ex \[ P(x, y): \ "x - y = y - x" \]
\[ \text{ uod } = \mathbb{R} \text{ for both } x, y \]

Then, \[ \forall x \forall y \ P(x, y) \text{ is } F. \]
Countereexample: \[ x = 1, y = 0 \]
\[ 1 - 0 = 0 - 1 \]

\[ \text{ along } y = x \]

\[ \theta \]
2. \( \exists x \exists y \ P(x,y) \) is true if and only if there is an example combo that makes \( P \) true.

\[
\text{is } F \iff \exists y \ F
\]

all combos make \( P \) false

\[\begin{align*}
\text{Ex} & \quad P(x,y): "x-y=y-x" \\
uod &= \mathbb{R} \text{ for both } x,y
\end{align*}\]

Then, \( \exists x \exists y \ P(x,y) \) is true.

For example, \( (x=2,y=2) \):

\[
\frac{2-2}{2-2} = 2-2
\]

\[\begin{align*}
\text{Ex} & \quad P(x,y): "xy = \pi" \\
uod &= \mathbb{Z} \text{ for both } x,y
\end{align*}\]

Then, \( \exists x \exists y \ P(x,y) \) is false.

(NO two integers multiply to \( \pi \)).

\[\begin{align*}
\forall x \forall y \ P(x,y) & \iff \forall y \forall x \ P(x,y) \\
\text{can switch}
\end{align*}\]

\[\begin{align*}
\exists x \exists y \ P(x,y) & \iff \exists y \exists x \ P(x,y) \\
\text{can switch}
\end{align*}\]

BUT order usually matters when mixing \( \forall s \), \( \exists s \).
\( \forall x \exists y \ P(x, y) \text{ is } T \iff y \)

Each \( x \) can "find" a \( y \) that makes \( P \) true.

\( \neg \exists x \forall y \ P(x, y) \text{ is } F \iff y \)

There is an \( x \) who can't find a \( y \).

Ex: \( P(x, y) \): "\( y - x = 6 \)"

\( \text{uod} = \mathbb{R} \) for both \( x, y \)

Then, \( \forall x \exists y \ P(x, y) \) is \( T \). Why?

Regardless of which real \# \( x \) is...

\[
\begin{align*}
y - x &= 6 \\
y &= 6 + x \quad \text{(real \#)}
\end{align*}
\]

we can let \( y \) equal \( 6 + x \),
and \( P \) will be \( T \).

Key: \( y \) can depend on \( x \)!!

Idea: \( P(0, 6) \) is \( T \)
\( P(1, 7) \) is \( T \)

\( P(x, 6 + x) \) is \( T \)

\( \forall \) \( \text{real \#} \).
\textbf{Ex} \ P(x,y): \ "\frac{x}{y} = 1" \quad \text{\textit{uod} = IR for both } x, y \quad \text{\textit{uod} = IR for both } x, y

Then, \ \forall x \exists y \ P(x,y) \text{ is F. Why?}

\begin{align*}
x = 0 & \text{ can't "find" a } y \text{ (real #)} \\
& \text{to make } P \text{ true.}
\end{align*}

\textit{Note:} Graph of \ \frac{x}{y} = 1 \text{ (like our "grid")}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{grid.png}
\caption{Graph showing \ \frac{x}{y} = 1}
\end{figure}

\textbf{Ex} \ P(x,y): \ "\frac{x}{y} = y" \quad \text{\textit{uod} = Z for both } x, y

Then, \ \forall x \exists y \ P(x,y) \text{ is F. Why?}

\begin{align*}
\text{If } x & \text{ is odd, it can't find an integer } y \text{ to make } P \text{ true.}
\end{align*}

\textit{Key:} Watch your "\textit{uod}"s !!

\textbf{If} \ \textit{uod} = IR \text{ for both } x, y, \text{ then } T:\n\begin{align*}
\text{any real } x \text{ can pick } y \text{ to be } \frac{x}{y} \text{ (real).}
\end{align*}
How color? ④ \( \exists x \forall y \ P(x, y) \) is T \( \iff \) there is no such column (Each x can find y that makes \( P \) false.)

\[ \exists x \forall y \ P(x, y) \text{ : "} \frac{\ln x}{y} = 0 \text{"}\]
\[ u_{od} = IR^+ \text{ for both } x, y \]

Then, \( \exists x \forall y \ P(x, y) \) is T. Why?

\[ x = 1 \text{ is a "magic" real # that always makes } P \text{ true.} \]

\[ \exists x \forall y \ P(x, y) \text{ : "} x + y = 3 \text{"}\]
\[ u_{od} = IR \text{ for both } x, y \]

Then, \( \exists x \forall y \ P(x, y) \) is F. Why?

There is no "magic x" that works with all real "y's" to make \( P \) true.
Order usually matters!

\[ \forall x \exists y \, P(x, y) \]

This is \( T \iff \)

Note

If \( \forall x \exists y \, P(x, y) \) is \( T \), then \( \exists y \forall x \, P(x, y) \) is not necessarily \( T \).

If \( \exists y \forall x \, P(x, y) \) is \( T \), then \( \forall x \exists y \, P(x, y) \) must be \( T \).

Shorthand: \( A \rightarrow B \)

\( B \rightarrow A \)
Ex (3 vars)

\[ P(x, y, z); \quad xy = z \]
\[ \text{uod} = IR \quad \text{for} \ x, y, \text{and} \ z \]

Then, \( \forall x \forall y \exists z \ P(x, y, z) \) is T. Why?

Each pair of real #s has a real product.

Also, \( \exists z \forall x \forall y \ P(x, y, z) \) is F. Why?

There is no "magic" real # \( z \) that is the product for every pair of real #s.

(See also Ex 24 on p. 32)

**NEGATIONS**

\[ \neg \forall x \ P(x) \iff \exists x \neg P(x) \]
\[ \text{is T} \iff \text{P(x) is F} \]

\[ \neg \exists x \ P(x) \iff \forall x \neg P(x) \]
\[ \text{ All x's make P false.} \]

**Trick:** if \( \neg \) switches sides, flip

\[ \neg \]

\[ \neg \]

\[ \forall x \, \exists x \]
HW Tips

In 1.3 (book), skip Exs. 12, 13, 16-21.

10) Let $Q(x, y)$ be "$x$ has been a contestant on $y$" and for $x =$ the set of all students at your school and for $y =$ the set of all quiz shows on TV. Express in terms of $Q$, quantifiers, and connectives:

a) There is a student at your school who has been a contestant on a TV quiz show.

$$\exists x \exists y \ Q(x, y)$$

b) No student at your school has ever been a contestant on a TV quiz show.

$$\forall x \forall y \neg Q(x, y)$$

or

$$\neg \exists x \exists y \ Q(x, y)$$

c) There is a student at your school who has been on Jeopardy and W.O.F.

$$\exists x \ (Q(x, \text{Jeopardy}) \land Q(x, \text{W.O.F.}))$$
d) Every TV quiz show has had a student from your school...

\[ \forall y \exists x \ Q(x, y) \]

#11) \( L(x, y) \): "x loves y" (x, y have same age) all people in the world

\[ \text{loved} \quad \framebox{x} \quad \framebox{y} \]

\( (y, x) \) since same age

\( \text{lovers} \)

d)–f) different possible answers (7; since same age)

For me (Give ours)

1) Nobody loves everybody

Book: \( \forall x \exists y \neg L(x, y) \)

Me: \( \neg \exists x \forall y \ L(x, y) \)

e) There is somebody whom Lydia does not love.

Book: \( \exists x \neg L(\text{Lydia}, x) \)

Me: \( \exists y \ L(\text{Lydia}, y) \)

f) There is somebody whom no one loves.

Book: \( \exists x \forall y \neg L(y, x) \)

Me: \( \forall y \exists x \ L(x, y) \)

\#22) T T T F I F T F T F F T F
A set is a collection of objects, called elements or members.

Ex Let $S$ denote the set \{a, b, c\}.

$S$ has 3 elements: $a$, $b$, $c$.

"$\in$" means "is a member of"

\begin{align*}
a & \in S \\
b & \in S \\
c & \in S
\end{align*}

"$\notin$" means "is not a member of"

\begin{align*}
d & \notin S
\end{align*}

Two sets are equal $\iff$ They have the same elements

Ex \{a, b, c\} = \{b, c, a\} = \{n, o, e\}

the elements are $e$, $o$, and $n$.
Writing sets

**Ex** (Set builder notation)

\[ \mathbb{Z} = \{ x \mid x \text{ is an integer} \} \]

= the set of all \( x \) such that ...

= \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \\

ellipses: 
"follow the pattern"

**Ex** Set of all digits = \{0, 1, 2, \ldots, 9\}

\[ ^{1st \ set} \text{ the } \frac{\text{pattern}}{\text{last}} \]

If a set \( S \) contains exactly \( n \) distinct elements, then

1) \( S \) is a finite set

2) the cardinality of \( S \) is \( n \) \( (|S| = n) \)

Otherwise, \( S \) is an infinite set. **Ex** \( \mathbb{Z}, \mathbb{R} \)

**Ex** If \( S = \{e, 0, n\} \), \( |S| = 3 \)

**Ex** If \( S = \{n, 0, e\}, |S| = 3 \). (distinct elements)
The set with no elements is the empty set or null set, denoted by "∅" or "\{\}".

\[ |∅| = 0 \]

\[ \text{Ex} \ {x \in \mathbb{R} \text{ and } x^2 = -1} = ∅ \]

A set can have elements that are, themselves, sets.

\[ \text{Ex} \ S = \{∅, \{∅\}, 5, \{5,6\}, \{5,6\}, \{5,6\}\} \]

\[ |S| = 6 \]

∅ ∈ S

{∅} ∈ S

5 ∈ S

{5} ∈ S

{5} ∉ S
SUBSETS

Assume $A$ and $B$ are sets.

$A$ is a subset of $B$ $(A \subseteq B) \iff$
all the elements of $A$ are also elements of $B$

Logic: $\forall x \ (x \in A \rightarrow x \in B)$ is $T$

Venn Diagram:

\[ \text{Ex} \]

\[ U = \text{universal set (maybe } U = \text{set of all complex numbers)} \]

$Z \subseteq R$

For any set $S$, 1) $\emptyset \subseteq S$ because $\forall x \ (x \in \emptyset \rightarrow x \in S)$ is $T$
2) $S \subseteq S$

Key Math Trick: If $A \subseteq B$ and $B \subseteq A$, then $A = B$
(Two sets that are subsets of each other are equal.)

$A$ is a proper subset of $B$ $(A \subset B) \iff$
$A \subseteq B$ and $A \neq B$

Ex: $Z \subset R$
**POWER SETS**

$P(S)$, the power set of a set $S$, is the set of all subsets of $S$.

**Ex** If $S = \{a, b, c\}$,

$$P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

$$|S| = 3, |P(S)| = 8$$

In general, $|P(S)| = 2^{|S|}$. Why?

1. $S = \{x_1, x_2, \ldots, x_n\}$

   You can enumerate all possible subsets by considering all possible IN/OUT combos.

2. $2 \times 2 \times \ldots \times 2 = 2^n$

   **Ex** $|P(\{\emptyset, \{\emptyset\}\})| = 2^2 = 4$

   i.e., The set $\{\emptyset, \{\emptyset\}\}$ has 4 subsets,

   $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, S$
**Cartesian Products**

Sets are unordered.

\[ \{\{1, 2\}, \{2, 1\}\} \]

Same element!

In graphing, we need ordered pairs \( y \times x \)

\[ \{(1, 2), (2, 1)\} \]

Different

Or ordered tuples \((a, b, c)\)

An ordered \(n\)-tuple has the form \((x_1, x_2, \ldots, x_n)\).

\[ A \times B = \text{Cartesian product of sets } A \text{ and } B = \{(a, b) \mid a \in A, b \in B\} \]

\[ A = \{i, s\}, \quad B = \{n, o, t\} \]

\[ A \times B = \{(i, n), (i, o), (i, t), (s, n), (s, o), (s, t)\} \]

\[ B \times A = \{(n, i), (n, s), (o, i), (o, s), (t, i), (t, s)\} \]

\[ \text{Ex} \{1, 2\} \times \{1, 2\} = \{(1, 1), (1, 2), (2, 1), (2, 2)\} \]

Can mark as you go along

\[ A \times B = B \times A, \text{ unless } A = \emptyset \text{ or } B = \emptyset \text{ or } A = B \]

Ex \( \mathbb{R} \times \mathbb{R} \) corresponds to the standard xy-plane

Ex \( 2 \times 2 \) \(\text{ (affine point)}\)

In general, \( A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\} \)
1.5: SET OPERATIONS

Let $A, B, \ldots$ be subsets of some universal set $U$.

**Basic Operations**

1. $A \cup B =$ the union of $A$ and $B$  
   = the set of elements in $A$ or $B$ (inclusive "or")  
   = \{ $x | x \in A \lor x \in B$ \}

2. $A \cap B =$ the intersection of $A$ and $B$  
   = the set of elements in $A$ and $B$  
   = \{ $x | x \in A \land x \in B$ \}

What do I color in?

If $A \cap B = \emptyset$, $A$ and $B$ are called disjoint.

Ex: $A =$ Dems, $B =$ Reps.
3. $A - B = \text{the difference of } A \text{ and } B$
   \[ = \text{the set of elements in } A \text{ but not in } B \]
   \[ = \{x \mid x \in A \land x \notin B\} \]

4. $A^c$ or $A^c = \text{the complement of } A \text{ (with respect to } U)$
   \[ = U - A \]
   \[ = \{x \mid x \in U \land x \notin A\} \]

5. $A \oplus B = \text{the symmetric difference of } A \text{ and } B$
   \[ = \text{the set of elements in } A \text{ xor } B \]
   \[ \text{(in HW)} \]

Note $A - B = A \cap B$
Ex \[ U = \{1, 2, 3, ..., 10\} \]
\[ A = \{2, 3, 5, 7\} \text{ (primes)} \]
\[ B = \{1, 3, 5, 7, 9\} \text{ (odds)} \]

Venn Diagram

\[ A \cup B = \{1, 2, 3, 5, 7, 9\} \]
\[ A \cap B = \{3, 5, 7\} \]
\[ A - B = \{2\} \]
\[ B - A = \{1, 9\} \]
\[ \overline{A} = \{1, 4, 6, 8, 9, 10\} \]
\[ \overline{B} = \{2, 4, 6, 8, 10\} \]
\[ A \oplus B = \{1, 2, 9\} \]
Basic Set Identities (Table 1 - p. 49)
These correspond to the basic logical equivalences (Table 5 - p. 17)

1. Identity Laws
   \[ p \land \top \iff p \]
   \[ p \lor \bot \iff p \]
   \[ p \land \top \iff p \]
   \[ p \lor \bot \iff p \]
   \[ p \land \top \iff p \]
   \[ p \lor \bot \iff p \]

2. Domination Laws
   \[ p \land F \iff F \]
   \[ p \lor T \iff T \]
   \[ p \land \top \iff p \]
   \[ p \lor \bot \iff p \]

3. Idempotent Laws
   \[ p \land p \iff p \]
   \[ p \lor p \iff p \]
   \[ p \land p \iff p \]
   \[ p \lor p \iff p \]

4. Double Negation Law
   \[ \neg \neg p \iff p \]

5, 6. \[ \land \text{ is commutative and associative} \]
   \[ \forall a \land b \land c \ldots \land z \]
   \[ \text{can reorder and regroup, ( ) no impact} \]
   So are \( \lor, \land, \lor \).
7 Distributive laws

We know \( a \times (b + c) = a \times b + a \times c \).

\( \times \) distributes over +.

Here, \( V \) distributes over \( ^\wedge \).

\[ \wedge \quad \wedge \quad \wedge \]

Example: \( p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r) \), etc.

8 De Morgan’s Laws

<table>
<thead>
<tr>
<th>Logic</th>
<th>Set Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \neg (p \wedge q) \iff \neg p \vee \neg q )</td>
<td>( A \cap B = A \cup \overline{B} )</td>
</tr>
<tr>
<td>( \neg (p \vee q) \iff \neg p \wedge \neg q )</td>
<td>( A \cup B = \overline{A} \cap \overline{B} )</td>
</tr>
</tbody>
</table>

Distribute \( \neg \), flip connective.

In general, \( \neg (p_1 \wedge p_2 \wedge \ldots \wedge p_n) \iff \neg p_1 \vee \neg p_2 \vee \ldots \vee \neg p_n \), etc.
Jill

Ex A woman wants a man who is tall and handsome. How can she be disappointed?

\[ p: \text{The man is tall,} \]
\[ q: \text{The man is handsome.} \]

\[ \neg(p \land q) \iff \neg p \lor \neg q \]

The man is not tall or not handsome.

---

Gary Coleman
or Shig Sasaki

Ex  \[ p_1: \text{Jim lives in Alabama} \]
\[ p_2: \text{Jim lives in Alaska} \]
\[ \neg p_1 \land \neg p_2 \land \neg p_0 \land p_1 \]

\[ \land \text{Jim lives in the U.S.} \]

\[ \neg(p_1 \lor p_2 \lor \ldots \lor p_0 \lor p_1) \]

\[ \neg p_1 \land \neg p_2 \land \ldots \land p_1 \]

Alabamans and Alaskans and...

---

Set Theory

Ex \[ A_1: \text{Alabamans} \]
\[ A_2: \text{Alaskans} \]
\[ A_3: \text{Wyomingites} \]
\[ A_5: \text{D.C.} \]

Set of all non-Americans

\[ = A_1 \cup A_2 \cup \ldots \cup A_5 \]

\[ = A_1 \cap A_2 \cap \ldots \cap A_5 \]

\[ = \text{set of all people who are not Alabamans, not Alaskans, and not D.C.} \]
Ex 10-11 (book) pp 49-50

Prove one of De Morgan's Laws: $A \cap B = \overline{A \cup B}$

1. Two sets are equal $\iff$ they have the same elements.

Consider an arbitrary element "x" in $U$.

$$x \in \overline{A \cap B}$$
$$\iff x \notin A \cap B$$
$$\iff \neg (x \in A \cap B)$$
$$\iff \neg (x \in A \land x \in B)$$
$$\iff \neg (x \in A) \lor \neg (x \in B)$$

by De Morgan's Laws (logic)

$$\iff x \in \overline{A} \lor x \in \overline{B}$$
$$\iff x \in \overline{A} \lor \overline{x} \in B$$
$$\iff x \in \overline{A} \cup \overline{B}$$

$x$ is in $\overline{A \cap B}$ $\iff x$ is in $\overline{A} \cup \overline{B}$

\text{have the same elements, so =}

Q.E.D. (end of proof)

2. Another approach:

We can show $x \in \overline{A \cap B} \Rightarrow x \in \overline{A \cup B}$

Thus, $\overline{A \cap B} \subseteq \overline{A \cup B}$

We can show $x \in \overline{A \cup B} \Rightarrow x \in \overline{A \cap B}$ (go backwards)

Thus, $\overline{A \cup B} \subseteq \overline{A \cap B}$

Two sets that are subsets of each other are equal.

Q.E.D.

(This approach is superior if there is a step that's not reducible.)
3) Proof by "membership tables" (≈ truth tables).

\[
\begin{array}{ccc}
\text{Ex} & A & C \\
\hline
1 & 1 & 0
\end{array}
\]

We consider an element in \(A\) and in \(B\), but not in \(C\).

Idea: If two sets have the same "final column", they are equal.

4) We can use set identities to simplify expressions or to verify harder identities. HW: My #1

**Generalized Unions and Intersections**

\[
\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n \quad \text{<don't need (\(\emptyset\)) is "well-defined" as is}
\]

\[
x \in \bigcup_{i=1}^{n} A_i \iff x \in \text{any } A_i \quad (1 \subseteq n)
\]

\[
\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n \\
x \in \bigcap_{i=1}^{n} A_i \iff x \in \text{all of the } A_i \quad (1 \subseteq n)
\]

HW: #35, 36
HW Tips

Your proofs may look different from the book's. (Ans. in back)
The book uses set builder notation.
It's unclear how many steps you "need."
(On exams - I will understand this.)

Use our basic set identities to prove:

\[(B \cup A) \cap \overline{B} = A \cap \overline{B}\]

We can go...

Let's try to simplify the left side...

\[(B \cup A) \cap \overline{B} = \overline{B} \cap (B \cup A) \text{ Comm. Laws (optional)}
\]

Could have distributed \[\cap\].

\[= (B \cap \overline{B}) \cup (\overline{B} \cap A) \text{ Distributive Laws}
\]

Think: \[a + (b + c) = (a + b) + (a + c)\]
Put in "\(c\)"'s

\[= \emptyset \cup (\overline{B} \cap A) \text{ } \emptyset \text{ No elements are in both}
\]

\[= \overline{B} \cap A \text{ } \emptyset \cup C = C \text{ (Identity)}
\]

\[= A \cap \overline{B} \text{ Comm. Laws}
\]

Great HW Tip: What's the HW?

My #: 1

De Morgan's Laws - we used them when we moved.
1.6: FUNCTIONS

A binary relation from a set $A$ to a set $B$ is a subset of $A \times B$. It's a set of ordered pairs. 

$\text{Ex} \quad A = \{1, 2, 3\}$
$B = \{a, b, c, d\}$

The relation $R = \{(1, a), (1, b), (3, c)\} \subseteq A \times B$

Each of the 1st components $\in A$ 2nd components $\in B$

Picture of $R$:

```
1  a
\downarrow
2  b
\downarrow
3  c
\downarrow
\downarrow
\downarrow
A \quad B
```
A function is a special type of binary relation.

Let \( f \) be a function that "maps" a set \( A \) to a set \( B \).

\[
\begin{align*}
  f : A & \rightarrow B \\
  \text{Domain of } f & \qquad \text{Codomain of } f
\end{align*}
\]

\[
\begin{array}{c}
\text{Ex}  \\
\begin{array}{ccc}
1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow \\
a & b & c \\
\end{array}
\end{array}
\]

\[
f = \{ (1, b), (2, a), (3, b) \}
\]

Say: \( f(1) = b, f(2) = a, f(3) = b \)

\( f \) must "point" each element of the domain \( (A) \) to exactly one element of the codomain \( (B) \).

\[
\text{Need} \quad \text{Forbid}
\]

\[
\begin{array}{cc}
\text{Domain} & \text{Codomain} \\
\begin{array}{c}
\begin{array}{c}
1 \\
\end{array}
\end{array} & \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{Ex! We have} \\
f(1) \quad f(2) \quad f(3)
\end{array}
\]

\[
\begin{array}{c}
\text{Ex! We have no} \\
\text{ambiguities like} \\
"f(1) = a \text{ or } b"
\end{array}
\]
\[ f(1) = b \]

the image of 1

the pre-image of b

\[ 1 \rightarrow f \rightarrow b \]

The range of f is the set of all the images of all the elements in A. \( \{f(A)\} \)

It's a subset of the codomain.

Ex 1

\[ f(A) = \text{Range of } f = \{a, b\} \]

= set of all elements in the codomain \( B \) that are "pointed to."
High school algebra

\[ f(x) = \frac{1}{x^2} \quad \text{← Rule for } f \]

**Domain of definition**

Assumed domain = \( \mathbb{R} \setminus \{0\} \)

\[ \text{Write } f : (\mathbb{R} \setminus \{0\}) \rightarrow \mathbb{R} \]

\[ \text{Graph of } f \]

\( f(1) = 1, \quad f(2) = \frac{1}{4}, \text{ etc.} \)

\[ f \text{ is a function, because the graph passes the Vertical Line Test} \]

(No vertical line passes through more than one point).

The Range of \( f \) is the set of all y-coords "hit" by the graph. Here, the range is \( \mathbb{R}^+ \).
Ex 2 \( f : \mathbb{Z} \to \mathbb{Z} \)

Rule: \( f(x) = x^2 \)

Range of \( f = \{0, 1, 4, 9, \ldots, 49, \ldots\} \)

\[ f(\{0, 3, -4, 4\}) = \{0, 9, 16\} \]
ONE-TO-ONE FUNCTIONS

Function  |  1-1 Function
---|---
Forbidden:  |  Also Forbidden:

A function $f$ is one-to-one (or injective) if
\[
\forall x_1, x_2 \in \text{domain of } f \quad (x_1 \neq x_2 \implies f(x_1) \neq f(x_2)).
\]

Idea: An image has only one pre-image.

\[\text{(A)} \quad \text{Ex. } f(x) = e^x, f: \mathbb{R} \rightarrow \mathbb{R} \quad \text{The graph passes the Horizontal Line Test}\]

If $e^a = e^b$ then $a = b$.

\[f \text{ in Ex.2 } (f(x) = x^2, f: \mathbb{R} \rightarrow \mathbb{R}) \text{ is not one-to-one} \]

\[\text{fails Horizontal Line Test} \quad \exists x \in \mathbb{R} \quad f(x) = x^2 = 4 \quad \text{has two pre-images} x = 2, x = -2.\]
ONTO FUNCTIONS

Every element in the codomain must get "hit". (Codomain = Range)

A function \( f : A \rightarrow B \) is onto or surjective \( \iff \forall b \in B \exists a \in A \) \( f(a) = b \)

\( f \) in \( \mathbb{R} \times \mathbb{Z} \) \( (f(x) = x^2, f : \mathbb{Z} \rightarrow \mathbb{Z}) \) is not onto, because not every element in the codomain (\( \mathbb{Z} \)) is in the range (\( \{0, 1, 4, 9, \ldots\} \)).

\( \text{Ex } f : \mathbb{R} \rightarrow \mathbb{R} \)
\( f(x) = 3x \)

**Approach 1:** Graph \( \not\exists x \) Range = \( \mathbb{R} \)

So, Codomain = Range.
So, \( f \) is onto.

**Approach 2:**
Let \( b \) be any element in the codomain (\( \mathbb{R} \)). Can we always find a pre-image for \( b \)? \( \text{YES} \)

\( b = 3x \)
\[ x = \frac{b}{3} \] \( \text{pre-image} \)
\( \text{in domain} \)

So, \( f \) is onto.
BIJECTIONS

A bijection (or a one-to-one correspondence) is a function that is both one-to-one and onto.

\[ A \rightarrow B \quad \text{or} \quad B \rightarrow A \]

could switch, anyway

(You can get this picture, maybe by moving dots.)

Special Case

f: A → A, where A is a finite set

\[
\begin{array}{ccc}
A & A \\
\downarrow & \downarrow \\
\quad & \quad
\end{array}
\]

Key: same-size sets

f is onto ⇔ f is 1-1. (⇒ would be a waste!)

If f is 1-1 or onto ⇒ f is a bijection
Review

\[
\begin{array}{c}
\text{Domain} \\
\text{preimage} \\
\end{array} \quad f \quad \begin{array}{c}
\text{Codomain} \\
\text{image} \quad \text{Range}
\end{array}
\]

Not functions

\[
\begin{array}{c}
\begin{array}{c}
\text{Not a function from } IR \to IR:
\end{array} \\
\begin{array}{c}
\text{plot (preimage, image)} \\
y \text{ is not a func of } x
\end{array}
\end{array}
\]

2 reasons: negative #s don't get mapped
graph fails VLT

If \( f \) is a function from \( A \to B \),
\[ \text{graph } f \text{ by plotting the points } \{(a, f(a)) \mid a \in A\} \subseteq A \times B \]
\[
\begin{array}{c}
\text{f(a)} \\
\text{A (domain)}
\end{array}
\]
A function that is both 1-1 and onto is called a bijection (or a one-to-one correspondence).

**Special Case**
If $S$ is a finite set, then for $f:S \to S$,

\[
\begin{cases}
1-1 \\
onto \\
bijective
\end{cases}
\]

if any one holds, then all 3 hold.

Ex $s_1 \neq s_2$

A 1-1 $f:S \to S$
must be onto, and vice versa.
**Inverse Functions**

Let $f: A \rightarrow B$ be a bijection.

Then, $f$ has an inverse function

$\quad f^{-1}: B \rightarrow A$, which is a bijection that reverses $f$.

How is $f^{-1}$ defined?

$f(a) = b \iff f^{-1}(b) = a$

\[\begin{array}{c}
\xymatrix{
a \ar[r]^f & b \\
\downarrow & \\
f^{-1} & a}
\end{array}\]

Note: $(f^{-1})^{-1} = f$

$f$ is a bijection $\iff f$ is invertible

**Ex**

\[\begin{array}{c}
\xymatrix{1 & \ar[r]^f & m \\
2 & \ar[r] & n}
\end{array}\]

$f$ is a bijection, so $f$ is invertible.

$f(1) = n$, so $f^{-1}(n) = 1$

$f(2) = m$, so $f^{-1}(m) = 2$

Compositions combine functions (pp. 62-4, not tested): $f(g(x))$
Two key functions that map IR $\rightarrow \mathbb{Z}$

Let $x \in \mathbb{R}$.

The **floor function** (or **greatest integer function**) rounds $x$ down.

$$f(x) = \lfloor x \rfloor \text{ or } \lceil x \rceil = \text{(closest integer that's } \leq x \text{)}$$

The **ceiling function** rounds $x$ up.

$$f(x) = \lceil x \rceil \Rightarrow \text{"hooks" on ceiling}$$

$$= \text{(closest integer that's } \geq x \text{)}$$

**Exs**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\lfloor x \rfloor$</th>
<th>$\lceil x \rceil$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>9.1</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>-10.7</td>
<td>-11</td>
<td>-10</td>
</tr>
</tbody>
</table>

$\lfloor x \rfloor \leq x \leq \lceil x \rceil$ always

$\iff$
Ex We have 50 balls.
A bag carries 12 balls.
How many bags do we need?

To be safe,

\[ \left\lfloor \frac{50}{12} \right\rfloor = \left\lfloor 4.167 \right\rfloor = 5 \text{ bags} \]

How many bags can we fill up?

\[ \left\lfloor \frac{50}{12} \right\rfloor = \left\lfloor 4.16 \right\rfloor = 4 \text{ bags} \]

\( \lfloor x \rfloor, \lceil x \rceil \) are step functions

\[ y = \lfloor x \rfloor \quad y = \lceil x \rceil \]

Start at 0

HW Tip
In \( \mathbb{Z} \) \( \mathbb{N} = \{0, 1, 2, 3, \ldots \} \)
some books exclude
1.7: SEQUENCES AND SUMMATIONS

A sequence represents an ordered list.

Ex. The sequence \( \{a_n\} \) (not "set")
usually denotes

\[ a_1, a_2, a_3, a_4, \ldots \]

terms

We sometimes start with \( a_0 \), not \( a_1 \):

\[ a_0, a_1, a_2, a_3, \ldots \]

(typical in calculus)

A string is a finite sequence.

Ex. \( 1, 0, 0, 1 \) \( \{a_n\}_{n=1}^4 \)
\[ a_1, a_2, a_3, a_4 \]

The general term \( (a_n) \) in a sequence may be
given by a formula or rule.

Ex. \( a_n = (-1)^n 3n \), Start with \( n = 1 \)
(assume unless told otherwise)

\[ a_1 = (-1)^1 \cdot 3(1) = (-1)(3) = -3 \]
\[ a_2 = (-1)^2 \cdot 3(2) = (1)(6) = 6 \]
\[ a_3 = (-1)^3 \cdot 3(3) = (-1)(9) = -9 \]
\[ a_4 = (-1)^4 \cdot 3(4) = (1)(12) = 12 \]
Sequence:

\[-3, 6, -9, 12, \ldots\]

(an alternating sequence
(signs alternate)

In I.Q. tests, you're asked to find the most "obvious" pattern:

\[
\text{Ex 3, 7, 11, 15, 19, \ldots}\quad \text{Web p. 72}
\]

(\text{\small can input list few terms puzzle seqs.})

Several possibilities, but 23, 27, 31, etc. seem most "obvious" (successively adding 4)

Special Sequence Types

1. Arithmetic progressions
   - successively add some common difference "d"

\[
\text{Ex 3, 7, 11, 15, 19, \ldots}
\]

\[
a_n = a_1 + (n-1)d = 3 + (n-1)(4)
\]

2. Geometric progressions
   - successively multiply some common ratio "r"

\[
\text{Ex 5, 15, 45, 135, \ldots}
\]

\[
a_n = a_1 \cdot r^{n-1} = 5 \cdot 3^{n-1}
\]
SUMMATIONS

\[ \sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n \quad (n \in \mathbb{Z}^+) \]

(Sum of 1st \( n \) terms)

\[ \sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_n \quad (m, n \in \mathbb{Z}) \]

\( i \) is the index of summation

Could use \( j, k, \ldots \)

The index sweeps through all the integers from the lower limit \( m \) to the upper limit \( n \).

\[ \sum_{k=3}^{5} (k \cdot 2^k - 1) = a_3 + a_4 + a_5 \]

\[ \sum_{k=3}^{5} (k \cdot 2^k - 1) = \left[ 3 \cdot 2^3 - 1 \right] + \left[ 4 \cdot 2^4 - 1 \right] + \left[ 5 \cdot 2^5 - 1 \right] \]

\[ = 23 + 63 + 159 \]

\[ = 245 \]

\[ \sum_{i \in s} a_i \quad \text{where} \; S = \{1, 4, 6\} \]

\[ = a_1 + a_4 + a_6 \]
Ex (Double summation)

Arise from nested loops in programs.

\[
\sum_{i=1}^{3} \sum_{j=1}^{n} \frac{1}{i} \times \frac{1}{j} = (\text{Case } i=1) + (\text{Case } i=2) + (\text{Case } i=3)
\]

\[
\begin{align*}
(i=1) & \quad \sum_{j=1}^{n} \frac{1}{j} \\
(i=2) & \quad + \sum_{j=1}^{n} \frac{2}{j} \\
(i=3) & \quad + \sum_{j=1}^{n} \frac{3}{j}
\end{align*}
\]

\[
= \frac{1}{1} + \frac{2}{1} + \frac{3}{2} + \frac{3}{2} + \frac{3}{3} + \frac{3}{3}
\]

\[
= \frac{9}{2}
\]

Table 2 (p. 76) has special \(\Sigma\) formulas.

*HW Tip*

*(#17)* \[ \sum_{i=1}^{n} \sum_{j=1}^{n} f(i,j) \]

*maybe faster to work this out first*
(Not tested) SPECIAL SUMS

\[ \sum_{i=1}^{100} i = 1 + 2 + 3 + \ldots + 100 \]

Gauss - when he was 10, his teacher wanted to occupy the class. Slapped him - really stared.

Trick:

\[
\begin{align*}
S &= \frac{1 + 2 + 3 + \ldots + 100}{2} \\
S &= \frac{100 + 99 + 98 + \ldots + 1}{2} \\
2S &= 101 + 101 + 101 + \ldots + 101 \\
\text{100 copies}
\end{align*}
\]

\[ 2S = 101 \cdot (100) \]

\[ S = \frac{101 \cdot (100)}{2} = 5050 \]
Ex 2/ In general, \[ \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n \quad (n \in \mathbb{Z}^+) \]

Same trick:

\[ S = \frac{(1+2+\ldots+n)}{n} \]

\[ 2S = \frac{(n+1)(n+1) + \ldots + (n+1)}{n} \]

\[ n \text{ copies} \]

\[ S = \frac{n(n+1)}{2} \]

More generally, let's consider the sum of any \( n \) consecutive terms in an arithmetic sequence.

Let's say \( a_1 + a_2 + \ldots + a_n \)

\[ S = \frac{a_1 + a_2 + \ldots + a_n}{n} \]

\[ S = \frac{a_n + a_{n-1} + \ldots + a_1}{n} \]

\[ \frac{1}{n} \text{ terms} \]

\[ \frac{(a_1 + a_n)}{2} \text{ of 1st, last terms} \]

Ex 2. \[ 7 + 8 + 9 + 10 + 11 = 9 + 9 + 9 + 9 = 9(5) = 45 \]

Communist Ex 1. \[ 3 + 7 + 11 + \ldots + 83 = \frac{(3 + 83)}{2} \times 21 = 903 \]
CARDINALITY

A sequence \( a_1, a_2, a_3, \ldots \) (each \( a_i \in S \))

is a function \( f: \mathbb{Z}^+ \to S \)

**Ex.** \( 3, 7, 11, 15, \ldots \) \( \in \) each \( \in \mathbb{Z}^+ \)

\[
\begin{array}{ccc}
\mathbb{Z}^+ & f & \mathbb{Z}^+ \\
1 \rightarrow & 3 & a_1 \\
2 \rightarrow & 7 & a_2 \\
3 \rightarrow & 11 & a_3 \\
\end{array}
\]

\{ images \}

A set is countable \( \iff \)

1. The set is finite.
2. All the elements can be listed as \( a_1, a_2, a_3, \ldots \)

(i.e., There is a bijection (1-1 correspondence) between the set and \( \mathbb{Z}^+ \))
Ex Let $S$ = set of all positive even integers

$$S = \{2, 4, 6, 8, \ldots \}$$

$S$ is countable.

$$\mathbb{Z}^+ \overset{f}{\rightarrow} S$$

$f$ defined by $f(n) = 2n$ is a bijection $\mathbb{Z}^+ \rightarrow S$

Nice list!

---

Ex Let $S$ = set of all even integers

$$S = \{-4, -2, 0, 2, 4, \ldots \}$$

$S$ is countable.

How can we make a nice list?

Sequence: 0, 2, -2, 4, -4, ...
Ex. \( \mathbb{Q}^+ \) = set of all positive rational \#s

\[
\left\{ x \mid x \text{ can be written as } \frac{p}{q}, \text{ where } p \in \mathbb{Z}^+, q \in \mathbb{Z}^+ \right\}
\]

is countable.

\( \mathbb{Z}^+ \times \mathbb{Z}^+ \) lattice

\[
\begin{array}{c|c|c}
 p \qquad q \\
\hline
 1 \qquad 1 \quad 2 \\
 2 \qquad 2 \quad 4 \\
 3 \qquad 3 \\n\end{array}
\]

\( 0 + 1 = 1 \)

\( \frac{2}{3}, \frac{3}{2} = 3 \)

\( \frac{4}{3} = 3, \frac{3}{2} = \frac{3}{2} \) (repeat!)

\( \frac{1}{3} \)

List: \( 1, \frac{1}{2}, 2, 3, \frac{1}{3}, ... \)

Ex. \( \mathbb{Q} \) = all rational \#s = is countable.

List: \( 0, 1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{2}{2}, -2, ... \)

need negative partners
Ex Show $\mathbb{R}$ is uncountable.

Let $S = \{x \mid x \in \mathbb{R}, 0 < x < 1\}$

It is sufficient to show $S$ is uncountable.

Proof by Contradiction

Assume $S$ is countable. Then, all the elements of $S$ can be listed:

$x_1: 0.d_{11}d_{12}d_{13}...$ (all $d_{ij}$ are digits)

$x_2: 0.d_{21}d_{22}d_{23}...$

$x_3: 0.d_{31}d_{32}d_{33}...$

$...$

We can construct a new # in $S$ that is not on the list.

new # = $0, d_1d_2d_3...$

\[
P \begin{array}{c}
S \text{ if } d_1 = 4 \\
S \text{ if } d_2 = 4 \\
4 \text{ if } d_1 \neq 4 \\
4 \text{ if } d_2 \neq 4 \\
\end{array}
\]
Ex

\[ x_1 = 0.\overline{712} \ldots \quad \text{Diagonalization argument} \]
\[ x_2 = 0.3\overline{6} \ldots \]
\[ x_3 = 0.55\overline{9} \ldots \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \text{etc.} \]

New \# = 0.454 \ldots

Idea: The new \# will differ from each listed \# by at least 1 digit.

In particular, the new \# and \( x_i \)

will differ in the \( i \)th decimal place.

BUT we assumed that all the elements in \( S \)

were listed!

Our list will never be "good enough!"

So, our assumption that \( S \) was countable was wrong.

\[ \therefore \] \( S \) is uncountable,

\[ \therefore \] \( \mathbb{R} \) is uncountable.

See Ex 17 on pp. 77-8.