1.3: PREDICATES AND QUANTIFIERS

Ex Let \( P(x) \) denote the statement "\( x < 5 \)."

\( P \) is a predicate denoting the property "is less than 5."

\( P(x) \) is not a proposition, but it becomes a prop. (T/F) once we assign a value to \( x \).

\( P(x) \) is a propositional function that depends on the variable \( x \).

Ex In the case \( x = 4 \), "\( 4 < 5 \)" is \( T \), so 
\( P(4) \) is \( T \).

Ex In the case \( x = 5 \), "\( 5 < 5 \)" is \( F \), so
\( P(5) \) is \( F \).

Ex 2 Let \( Q(x, y) \) denote "\( y = 3x + 1 \)."

Ex In the case \( (x = 2, y = 7) \), "\( 7 = 3(2) + 1 \)" is \( T \), so
\( Q(2, 7) \) is \( T \).
\( \forall \) is the "universal quantifier" (say "for all")

\[ \forall x \, P(x) \] denotes the proposition

"\( P(x) \) is \( T \) for all values of \( x \),

in the universe of discourse,"

the domain of \( P \)
(we'll say "uod")

Some common "uod"s:

\[ \mathbb{Z} \] is the set of all integers
\[ \mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \]

\[ \mathbb{Z}^+ \] is the set of all positive integers.
\[ \mathbb{Z}^+ = \{ 1, 2, 3, \ldots \} \]

\[ \mathbb{R} \] is the set of all real #s.

Special case: Finite "uod"s

If the uod is a finite set, say \( \{ x_1, x_2, \ldots, x_n \} \), then

\[ \forall x \, P(x) \iff P(x_1) \land P(x_2) \land \ldots \land P(x_n) \]

is \( T \) exactly when all these are \( T \).
Ex Let $P(x)$ be "x is an integer."

If the $\text{uo}_0 = \{1, 7, 15\}$, then $\forall x P(x)$ is $T$, because

\[ P(1) \land P(7) \land P(15) \] is $T$.

If the $\text{uo}_0 = \{1, 7, \pi\}$, then $\forall x P(x)$ is $F$, because

\[ P(1) \land P(7) \land P(\pi) \] is $F$.

one $F$

ruins it for
everybody

The $\text{uo}_0$ can be an infinite set.

Ex Let $P(x)$ be "$\exists x > x.$"

If the $\text{uo}_0$ is $\mathbb{Z}^+$, $\forall x P(x)$ is $T$.

If the $\text{uo}_0$ is $\mathbb{Z}$, $\forall x P(x)$ is $F$.

For example, $P(-1)$ is $F$. (-2 $\neq$ -1).

$x = -1$ is called a counterexample.
$\exists$ is the "existential quantifier" (say "there exists/is")

$\exists x \ P(x)$ denotes the proposition

"$P(x)$ is $T$ for at least one element (x) in the uod."

**Special case**: Finite "uod"s

If the uod is $\{x_1, x_2, ..., x_n\}$, then

$$\exists x \ P(x) \iff P(x_1) \lor P(x_2) \lor ... \lor P(x_n)$$

is $T$ exactly when at least one is $T$

**Ex Let** $P(x)$ be "$x$ is an integer."

If the uod = $\{1, \pi, e\}$, then $\exists x \ P(x)$ is $T$, because

$$P(1) \lor P(\pi) \lor P(e)$$ is $T$.

One $T$ does it!

If the uod = $\{\sqrt{2}, \pi, e\}$, then $\exists x \ P(x)$ is $F$, because

$$P(\sqrt{2}) \lor P(\pi) \lor P(e)$$ is $F$.

All are $F$. 
Ex Let $P(x)$ be "$x^2 = 16$.
Let the uod be $\mathbb{Z}$.

Then, $\exists x P(x)$ is T.
For example, $P(4)$ is T. $P(-4)$, also.

Ex Let $Q(x)$ be "$x^2 = 17$.
Let the uod be $\mathbb{Z}$.

There is no integer whose square is 17,
so $\exists x Q(x)$ is F.

Ex $P(x)$: "$x$ is even", uod = $\mathbb{Z}$

$P(4)$ "4 is even" (T)
$\forall x P(x)$ "all integers are even" (F)
$\exists x P(x)$ "there is an even integer" (T)

In these cases, $x$ is bound by a
value assignment or a quantifier.

In general, a propositional function becomes a
proposition only when all the variables are bound.
MORE THAN 1 VARIABLE

Imagine a grid:

\[ \begin{array}{c}
\forall x \forall y P(x,y) \text{ is } T & \iff & y \in \{0, \ldots, n\} \\
\forall x \forall y P(x,y) \text{ is } F & \iff & y \in \{0, \ldots, n\} \setminus \{k\}
\end{array} \]

- all combos make \( P \) true
- there is a counterexample that makes \( P \) false

Ex: \( P(x,y) \): \( xy = yx \)
    \( uod = \mathbb{R} \) for both \( x, y \)

Then, \( \forall x \forall y P(x,y) \) is \( T \).

(Multiplication is commutative for all pairs of real numbers.)

Ex: \( P(x,y) \): \( x - y = y - x \)
    \( uod = \mathbb{R} \) for both \( x, y \)

Then, \( \forall x \forall y P(x,y) \) is \( F \).

Counterexample: \( x = 1, y = 0 \)
\\
\( 1 - 0 \neq 0 - 1 \)
(2) \( \exists x \exists y \, P(x,y) \) is T \iff \( \forall x \exists y \, P(x,y) \)

there is an example combo that makes \( P \) true

\[
\begin{array}{c}
\text{is F} \iff \exists y \forall x \, \neg P(x,y)
\end{array}
\]

all combos make \( P \) false

\[\text{Ex } P(x,y): "x-y=y-x"
\]
\[\text{uod}=\mathbb{R} \text{ for both } x, y\]

Then, \( \exists x \exists y \, P(x,y) \) is T.

For example, \( (x=2, y=2) \):
\[
\frac{2-2=2-2}{2}\]

\[\text{Ex } P(x,y): "xy=\pi"
\]
\[\text{uod}=\mathbb{Z} \text{ for both } x, y\]

Then, \( \exists x \exists y \, P(x,y) \) is F.

(No two integers multiply to \( \pi \).)

\[\forall x \forall y \, P(x,y) \iff \forall y \forall x \, P(x,y)\]

\( \text{can switch} \)

\[\exists x \exists y \, P(x,y) \iff \exists y \exists x \, P(x,y)\]

\( \text{can switch} \)

\( \underline{\text{BUT order usually matters when mixing } \forall s, \exists s.} \)
3. $\forall x \exists y. P(x,y)$ is T $\iff$ y

Each x can find a y that makes P true.

y is F $\iff$ $\exists x$ who can't find a y.

Ex: $P(x,y)$: "y - x = 6"

$\text{uod} = \mathbb{R}$ for both x, y

Then, $\forall x \exists y. P(x,y)$ is T. Why?

Regardless of which real # x is...

\[
y - x = 6 \\
y = 6 + x \quad \text{(real #)}
\]

we can let y equal 6 + x, and P will be T.

Key: y can depend on x!!

Idea: $P(0,6)$ is T

$P(1,7)$ is T

$P(x,6+x)$ is T

for any real #
**Ex** $P(x, y): \"x/y = 1\"$
\[ uod = \mathbb{R} \text{ for both } x, y \]

Then, $\forall x \exists y P(x, y)$ is F. Why?

$x = 0$ can’t “find” a y (real #)

to make $P$ true.

*Note: Graph of $x/y = 1$ (like our “grid”)*

**Ex** $P(x, y): \"x/y = y\"$
\[ uod = \mathbb{Z} \text{ for both } x, y \]

Then, $\forall x \exists y P(x, y)$ is F. Why?

If $x$ is odd, it can’t find an integer $y$ to make $P$ true.

*Key: Watch your “uod”s !!.*

If $uod = \mathbb{R}$ for both $x, y$, then $T$:
any real $x$ can pick $y$ to be $x$ (real).
How color? \[ \exists x \forall y \; P(x, y) \text{ is } T \iff \exists y \quad \text{(Each } x \text{ can find )} \]

There is a "magic" \( x \) that will make \( P \) true, regardless of \( y \) (in \( y \)'s \( \text{mod} \)), \( x \) \( \iff \) there is no such \( \exists y \text{ that makes } P \text{ false.} \)

\[ \exists x \; P(x, y) : \left( \frac{\ln x}{y} = 0 \right) \quad \text{uod} = IR^+ \text{ for both } x, y \]

Then, \( \exists x \forall y \; P(x, y) \text{ is } T. \text{ Why?} \]

Exclude \( y = 0 \text{ from consideration } \)

\( x = 1 \) is a "magic" real \( \# \) that always makes \( P \) true.

\[ \exists x \; P(x, y) : \left( x + y = 3 \right) \quad \text{uod} = IR \text{ for both } x, y \]

Then, \( \exists x \forall y \; P(x, y) \text{ is } F. \text{ Why? } \]

There is no "magic \( x \)" that works with all real "\( y \)'s to make \( P \) true.
Order doesn't matter: $(x=1), (y=2)$

Order usually matters!

$(\text{Prop}_A) \forall x \exists y \ P(x, y)$

This is $T \iff y \begin{array}{|c|}
\hline
x \hline
\end{array}$

$(\text{Prop}_B) \exists y \forall x \ P(x, y)$

This is $T \iff y \begin{array}{|c|}
\hline
x \hline
\end{array}$

There is a magic $y$...

Note

If $(\text{Prop}_A)$ is $T$, then $(\text{Prop}_B)$ is not necessarily $T$.

If $(\text{Prop}_B)$ is $T$, then $(\text{Prop}_A)$ must be $T$.

Shorthand: $A \rightarrow B$

$B \rightarrow A$
**Exercise (3 vars)**

\[ P(x, y, z); \quad "xy = z" \]
\[ \text{unod } = \text{IR for } x, y, \text{ and } z \]

Then, \( \forall x \forall y \exists z \ P(x, y, z) \) is T. Why?

Each pair of real #s has a real product.

\[ \exists x \exists y \in \text{R} \ (x \neq y, x > 0) \]

Also, \( \exists z \forall x \forall y \ P(x, y, z) \) is F. Why?

There is no "magic" real # \( z \) that is the product for every pair of real #s.

(See also Ex 24 on p. 32)

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**Negations**

\[ \neg \forall x \ P(x) \iff \exists x \ \neg P(x) \]

\( \text{truth: if } \neg P(x) \text{ is true, then } P(x) \text{ is false.} \)

\[ \neg \exists x \ P(x) \iff \forall x \ \neg P(x) \]

\( \text{truth: if } \forall x \ P(x) \text{ is true, then } P(x) \text{ is true for all } x. \)

**Trick:** if \( \neg \) switches sides

\[ \exists x \iff \forall x \ \neg \]
HW Tips

In 1.3 (book), skip Exs. 12, 13, 16-21.

#10) Let \( Q(x, y) \) be "x has been a contestant on y" where for x = the set of all students at your school and for y = the set of all quiz shows on TV. Express in terms of \( Q \), quantifiers, and connectives:

a) There is a student at your school who has been a contestant on a TV quiz show.

\[ \exists x \exists y \ Q(x, y) \]

b) No student at your school has ever been a contestant on a TV quiz show.

\[ \forall x \forall y \neg Q(x, y) \]

\[ \forall x \exists y \neg Q(x, y) \]

\[ \exists x \exists y \neg Q(x, y) \]

c) There is a student at your school who has been on Jeopardy and Whose Fortune.

\[ \exists x (Q(x, \text{Jeopardy}) \land Q(x, \text{W.O.F.})) \]
d) Every TV quiz show has had a student from your school...

\[ \forall y \exists x \ Q(x, y) \]

#11) \( L(x, y) \): "x loves y" 

\[ y \text{ loved } x \]
\[ \text{(x, y have same uod)} \]
\[ \text{all people in the world} \]

d) - f): different possible answers (7, \( \square \), since same uod)

For me (give vals)

1) Nobody loves everybody

Book: \( \forall x \forall y \neg L(x, y) \)

Me: \( \neg \exists x \forall y \ L(x, y) \)

e) There is somebody whom Lydia does not love.

Book: \( \exists x \neg L(\text{lydia}, x) \)

Me: \( \exists y \neg L(\text{lydia}, y) \)

f) There is somebody whom no one loves.

Book: \( \exists x \forall y \neg L(y, x) \)

Me: \( \exists y \forall x \neg L(x, y) \)

\[ \exists y \forall x \neg L(x, y) \]