

1.3: PREDICATES AND QUANTIFIERS

Ex Let $P(x)$ denote the statement " $x < 5$."

P is a predicate denoting the property "is less than 5."

$P(x)$ is not a proposition, but it becomes a prop. (T/F) once we assign a value to x .

$P(x)$ is a propositional function that depends on the variable x .

Ex In the case $x=4$, " $4 < 5$ " is T, so $P(4)$ is T.

Ex In the case $x=5$, " $5 < 5$ " is F, so $P(5)$ is F.

Ex 2 Let $Q(x,y)$ denote " $y = 3x + 1$."

Ex In the case $(x=2, y=7)$, " $7 = 3(2) + 1$ " is T, so $Q(2,7)$ is T.

\forall is the "universal quantifier" (say "for all")

$\forall x P(x)$ denotes the proposition

" $P(x)$ is T for all values of x ,
in the universe of discourse."

the domain of P
(we'll say "uod")

Some common "uod"s:

\mathbb{Z} is the set of all integers.
 $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

\mathbb{Z}^+ is the set of all positive integers.
 $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$

\mathbb{R} is the set of all real #s.

Special case: Finite "uod"s

If the uod is a finite set, say $\{x_1, x_2, \dots, x_n\}$, then

$$\underbrace{\forall x P(x)}_{\text{is T}} \iff \underbrace{P(x_1) \wedge P(x_2) \wedge \dots \wedge P(x_n)}_{\text{all these are T}}$$

exactly when

Ex Let $P(x)$ be "x is an integer."

If the uod = $\{1, 7, 15\}$, then $\forall x P(x)$ is T, because
 $P(1) \wedge P(7) \wedge P(15)$ is T.

If the uod = $\{1, 7, \pi\}$, then $\forall x P(x)$ is F, because
 $P(1) \wedge P(7) \wedge P(\pi)$ is F.

one F
ruins it for
everybody

The uod can be an infinite set.

Ex Let $P(x)$ be " $2x > x$."

If the uod is \mathbb{Z}^+ , $\forall x P(x)$ is T.

x	$2x$
1	2
2	4
3	6
...	...

If the uod is \mathbb{Z} , $\forall x P(x)$ is F.

For example, $P(-1)$ is F. ($-2 \not> -1$).
 $x = -1$ is called a counterexample.

\exists is the "existential quantifier" (say "there exists/is")

$\exists x. P(x)$ denotes the proposition

" $P(x)$ is T for at least one element x in the uod."

Special case: Finite "uod"s

If the uod is $\{x_1, x_2, \dots, x_n\}$, then

$$\underbrace{\exists x P(x)}_{\text{is T}} \iff \underbrace{P(x_1) \vee P(x_2) \vee \dots \vee P(x_n)}_{\text{at least one is T}}$$

exactly when

Ex Let $P(x)$ be "x is an integer."

If the uod = $\{1, \pi, e\}$, then $\exists x P(x)$ is T, because

$$\underbrace{P(1) \vee P(\pi) \vee P(e)}_{\substack{\text{one T} \\ \text{does it!}}} \text{ is T.}$$

If the uod = $\{\sqrt{2}, \pi, e\}$, then $\exists x P(x)$ is F, because

$$\underbrace{P(\sqrt{2}) \vee P(\pi) \vee P(e)}_{\text{all are F}} \text{ is F.}$$

Ex Let $P(x)$ be " $x^2 = 16$."
Let the uod be \mathbb{Z} .

Then, $\exists x P(x)$ is T.
For example, $P(4)$ is T. $P(-4)$, also.

Ex Let $Q(x)$ be " $x^2 = 17$."
Let the uod be \mathbb{Z} .

There is no integer whose square is 17,
so $\exists x Q(x)$ is F.

Ex $P(x)$: " x is even", uod = \mathbb{Z}

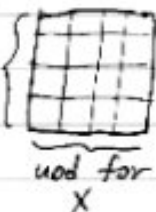
$P(4)$	"4 is even"	(T)	} these are propositions
$\forall x P(x)$	"all integers are even"	(F)	
$\exists x P(x)$	"there is an even integer"	(T)	


In these cases, x is bound by a
value assignment or a quantifier.


In general, a propositional function becomes a
proposition only when all the variables are bound.

MORE THAN 1 VARIABLE

p. 31 - a great
cure for
insomnia!

Imagine a grid: $\underbrace{\quad}_{\text{uod for } x}$ $\left\{ \begin{array}{l} \text{uod} \\ \text{for} \\ y \end{array} \right.$ 

① $\forall x \forall y P(x,y)$ is T \Leftrightarrow  all combos
make P true

is F \Leftrightarrow  there is a
counterexample
that makes P false

Ex $P(x,y)$: " $xy = yx$ "
uod = \mathbb{R} for both x, y

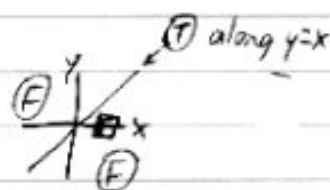
Then, $\forall x \forall y P(x,y)$ is T.

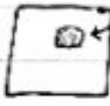
(Multiplication is commutative for all
pairs of real #s.)

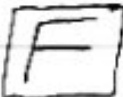
Ex $P(x,y)$: " $x - y = y - x$ "
uod = \mathbb{R} for both x, y

Then, $\forall x \forall y P(x,y)$ is F.

Counter-example: $x=1, y=0$
 $1-0 \neq 0-1$



② $\exists x \exists y P(x,y)$ is T \Leftrightarrow  there is an example combo that makes P true

is F \Leftrightarrow  all combos make P false

Ex $P(x,y)$: " $x-y=y-x$ "
 $uod = \mathbb{R}$ for both x,y

Then, $\exists x \exists y P(x,y)$ is T.

For example, $(x=2, y=2)$:
 $2-2=2-2$

Ex $P(x,y)$: " $xy = \pi$ "
 $uod = \mathbb{Z}$ for both x,y


Then, $\exists x \exists y P(x,y)$ is F.

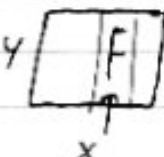
(No two integers multiply to π .)

$\forall x \forall y P(x,y) \Leftrightarrow \forall y \forall x P(x,y)$
 can switch

$\exists x \exists y P(x,y) \Leftrightarrow \exists y \exists x P(x,y)$
 can switch

BUT order usually matters when mixing \forall s, \exists s.

③ $\forall x \exists y P(x,y)$ is T \Leftrightarrow  Each x can "find" a y that makes P true.

is F \Leftrightarrow  There is an x who can't find a y.

Ex $P(x,y)$: " $y-x=6$ "
 $uod = \mathbb{R}$ for both x,y

Then, $\forall x \exists y P(x,y)$ is T. Why?

Regardless of which real # x is...

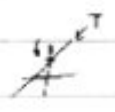
$$y-x=6$$

$$y=6+x \text{ (real \#)}$$

we can let y equal $6+x$,
 and P will be T.

Key: y can depend on x !!

Idea: $P(0,6)$ is T
 $P(1,7)$ is T
 \vdots
 $P(x,6+x)$ is T
 \uparrow
 any real #

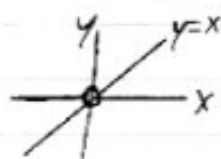


Ex $P(x,y): \frac{x}{y} = 1$
 $\text{uod} = \mathbb{R}$ for both x,y

Then, $\forall x \exists y P(x,y)$ is F. Why?

$x=0$ can't "find" a y (real #)
 to make P true.

Note: Graph of $\frac{x}{y} = 1$ (like our "grid")



Ex $P(x,y): \frac{x}{2} = y$
 $\text{uod} = \mathbb{Z}$ for both x,y

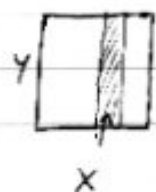
Then, $\forall x \exists y P(x,y)$ is F. Why?

If x is odd, it can't find
 an integer y to make P true.

Key: Watch your "uod"s !!

If $\text{uod} = \mathbb{R}$ for both x,y , then T:
 any real x can pick y to be $\frac{x}{2}$ (real).

How color?

④ $\exists x \forall y P(x,y)$ is T \Leftrightarrow  There is a "magic" x that will make P true, regardless of y (in y 's uod)

is F \Leftrightarrow there is no such column (Each x can find a y that makes P false.)

Ex $P(x,y)$: " $\frac{\ln x}{y} = 0$ "
uod = \mathbb{R}^+ for both x, y

Then, $\exists x \forall y \neq 0 P(x,y)$ is T. Why?

exclude
 $y=0$
from
consideration

$x=1$ is a "magic" real #
that always makes P true.

Ex $P(x,y)$: " $x+y=3$ "
uod = \mathbb{R} for both x, y

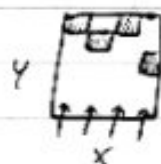
Then, $\exists x \forall y P(x,y)$ is F. Why?

There is no "magic x " that works with all real " y 's to make P true.

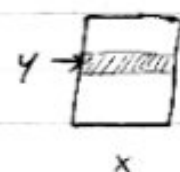
Order doesn't
matter:
(x=3) & (y=4)

Order usually matters!

$$\text{Prop. A } \forall x \exists y P(x, y)$$

This is T \Leftrightarrow 

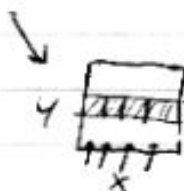
$$\text{Prop. B } \exists y \forall x P(x, y)$$

This is T \Leftrightarrow  There is a magic y...

Note

If Prop. A is T, then Prop. B is not necessarily T.

If Prop. B is T, then Prop. A must be T.



Shorthand: $A \rightarrow B$
 $B \rightarrow A$

Ex (3 vars)

$P(x, y, z)$: " $xy = z$ "
uod = \mathbb{R} for x, y , and z

Then, $\forall x \forall y \exists z P(x, y, z)$ is T. Why?

Each pair of real #s has a real product.

Also, $\exists z \forall x \forall y P(x, y, z)$ is F. Why?

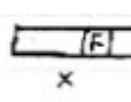
There is no "magic" real # z that is the product for every pair of real #s.

(See also Ex 24 on p. 32)

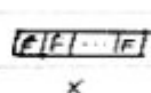
NEGATIONS

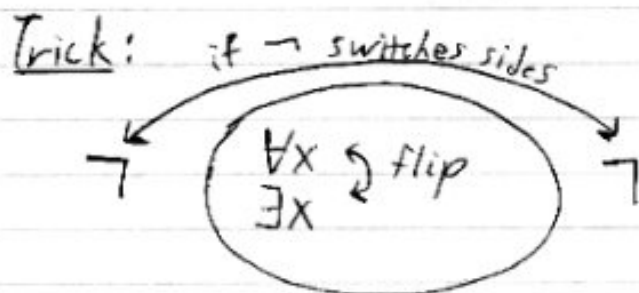
$$\neg \forall x P(x) \iff \exists x \neg P(x)$$

is T \iff $P(x)$ is F

 There is an x that makes P false

$$\neg \exists x P(x) \iff \forall x \neg P(x)$$

 All x 's make P false.



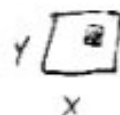
HW Tips

In 1.3 (book), skip Exs. 12, 13, 16-21.

#10) Let $Q(x, y)$ be "x has been a contestant on y"
 uod for x = the set of all students at your school.
 uod for y = the set of all quiz shows on TV.
 Express in terms of Q , quantifiers, and connectives:

a) There is a student at your school who has been a contestant on a TV quiz show.

$$\exists x \exists y Q(x, y)$$

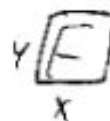


b) No student at your school has ever been a contestant on a TV quiz show.

$$\forall x \forall y \neg Q(x, y)$$

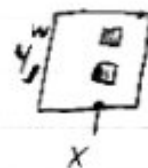
or

$$\neg \exists x \exists y Q(x, y)$$



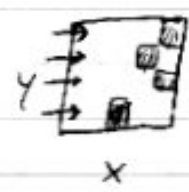
c) There is a student at your school who has been ... on Jeopardy and W.o.F.

$$\exists x (Q(x, \text{Jeopardy}) \wedge Q(x, \text{W.o.F.}))$$

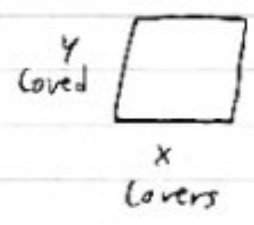


d) Every TV quiz show has had a student from your school...

$\forall y \exists x Q(x, y)$



#11) $L(x, y)$: "x loves y"



(x, y have same uod)
all people in the world

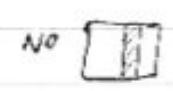
d)-f): different possible answers (\neg , \forall , \exists since same uod)

For me (Give ovals)

d) Nobody loves everybody

Book: $\forall x \exists y \neg L(x, y)$

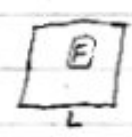
Me: $\neg \exists x \forall y L(x, y)$



e) There is somebody whom Lydia does not love.

Book: $\exists x \neg L(\text{Lydia}, x)$

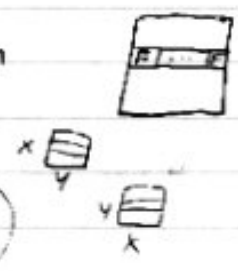
Me: $\exists y \neg L(\text{Lydia}, y)$



f) There is somebody whom no one loves.

Book: $\exists x \forall y \neg L(y, x)$

Me: $\exists y \forall x \neg L(x, y)$ or $\neg \forall y \exists x L(x, y)$



#22) T F T F T F T F T