1.4: SETS

A set is a collection of objects, called elements or members.

Ex Let $S$ denote the set \{a, b, c\}.

$S$ has 3 elements: \(a, b, c\)

"\(\in\)" means "is a member of"
\[a \in S\]
\[b \in S\]
\[c \in S\]

"\(\notin\)" means "is not a member of"
\[d \notin S\]

Two sets are equal $\iff$ They have the same elements

Ex \(\{e, o, n\} = \{0, n, e\} = \{n, o, n, e\}\)

the elements are \(e, o, n\) and \(n\)
Writing sets

**Ex** (Set builder notation)

\[ \mathbb{Z} = \{ x \mid x \text{ is an integer} \} \]

= the set of all \( x \) such that...

\[ \{ ..., -2, -1, 0, 1, 2, ... \} \]

(ellipses: "follow the pattern")

**Ex** Set of all digits = \( \{ 0, 1, 2, ..., 9 \} \)

1st set the pattern

last

If a set \( S \) contains exactly \( n \) distinct elements, then
1) \( S \) is a finite set
2) the cardinality of \( S \) is \( n \) \( (|S|=n) \)

Otherwise, \( S \) is an infinite set. Ex \( \mathbb{Z}, \mathbb{R} \)

**Ex** If \( S = \{ e, 0, n \} \), \( |S|=3 \).
**Ex** If \( S = \{ n, 0, n, e \} \), \( |S|=3 \), (distinct elements)
The set with no elements is the empty set or null set, denoted by "\( \emptyset \)" or "\{\}".

\[ |\emptyset| = 0 \]

**Ex:** \( \{x \mid x \in \mathbb{R} \text{ and } x^2 = -1\} = \emptyset \)

A set can have elements that are, themselves, sets.

**Ex:** Let \( S = \{\emptyset, \{\emptyset\}, 5, \{5, 6\}, \{\{5\}, 6\}, \{\{5\}, \emptyset\}\} \)

\[ |S| = 6 \]

\[ \emptyset \in S \]
\[ \{\emptyset\} \in S \]
\[ 5 \in S \]
\[ \{5\} \notin S \]
\[ \{\{5\}\} \in S \]
SUBSETS

Assume $A$ and $B$ are sets.

$A$ is a subset of $B$ ($A \subseteq B$) $\iff$ all the elements of $A$ are also elements of $B$.

Logic: $\forall x \ (x \in A \rightarrow x \in B)$ is $T$.

Venn Diagram:

\[ U = \text{universal set} \ (\text{maybe } C = \text{set of all complex } \#s) \]

\[ \varnothing \subseteq \mathbb{R} \]

For any set $S$, 1) $\varnothing \subseteq S$ because $\forall x \ (x \in \varnothing \rightarrow x \in S)$ is $T$.

2) $S \subseteq S$.

Key Math Trick: If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

(Two sets that are subsets of each other are equal.)

$A$ is a proper subset of $B$ ($A \subset B$) $\iff$

$A \subseteq B$ and $A \neq B$.

Ex: $\mathbb{Z} \subset \mathbb{R}$.
POWER SETS

\( P(S) \), the power set of a set \( S \), is the set of all subsets of \( S \).

**Ex** If \( S = \{a, b, c\} \),

\[
P(S) = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \}
\]

\( \emptyset \subseteq S \), \( 1 - \text{e4. subsets} \), \( 2 - \text{e4. subsets} \), \( S \subseteq S \)

\(|S| = 3, \ |P(S)| = 8\)

In general, \( |P(S)| = 2^{|S|} \). Why?

1. \( S = \{x_1, x_2, ..., x_n\} \)

   \[\begin{array}{c|c|c|c|c}
   \text{IN} & \text{IN} & \text{IN} & \text{OUT} & \text{OUT} \\
   \hline
   \text{IN} & \text{IN} & \text{OUT} & \text{OUT} & \text{OUT} \\
   \text{OUT} & \text{OUT} & \text{OUT} & \text{OUT} & \text{OUT} \\
   \text{OUT} & \text{OUT} & \text{OUT} & \text{OUT} & \text{OUT} \\
   \end{array}\]

   You can enumerate all possible subsets by considering all possible IN/OUT combos.

   \[2 \times 2 \times ... \times 2 = 2^n\]

2. List the binary reps of integers from 0 to \( 2^n - 1 \).

   **Ex** \( |P(S)\{\{\emptyset, \{\emptyset\}\}\} = 2^2 = 4\)

   \( \{\emptyset, \{\emptyset\}\} \subseteq S \)

   i.e., The set \( \{\emptyset, \{\emptyset\}\} \) has 4 subsets, 
   \( \emptyset, \{\emptyset\}, \{\{\emptyset\}\}, S \)
CARTESIAN PRODUCTS

Sets are unordered.

Ex \{ \{1,2,3\}, \{2,1\}\}

same element!

In graphing, we need ordered pairs \(x, y\)

Ex \{(1,2), (2,1)\}

different

or ordered triples \((a,b,c)\)

An ordered \(n\)-tuple has the form \((x_1, x_2, \ldots, x_n)\).

\(\text{Ax}B = \text{Cartesian product of sets } A \text{ and } B\)

\(= \{ (a, b) \mid a \in A, \ b \in B \}\)

Ex \(A = \{i, s\}\)

\(B = \{n, o, t\}\)

\(AxB = \{ (i,n), (i,o), (i,t), (s,n), (s,o), (s,t) \}\)

\(BxA = \{ (n,i), (n,s), (o,i), (o,s), (t,i), (t,s) \}\)

Ex \(1\times 1\times 1\) corresponds to the standard \(x,y,z\)-plane.

\(\text{Ex } \{1,3\} \times \{1,2,3\} = \{(1,1), (1,2), (1,3), (3,1), (3,2)\}\)

\(\text{Can mark as you go along.}\)

\(\text{Ax}B = B \times A, \text{ unless } A = \emptyset \text{ or } B = \emptyset \text{ or } A = B\)

\(\text{Ex } 2 \times 2 = \emptyset\)

In general, \(A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i: 1,2, \ldots, n \}\)