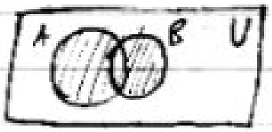
1.5: SET OPERATIONS

Let $A, B, \ldots$ be subsets of some universal set U.

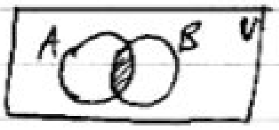
Basic Operations

① $A \cup B$ = the <u>union</u> of A and B
= the set of elements in A <u>or</u> B (inclusive "or")
= $\{x \mid x \in A \vee x \in B\}$



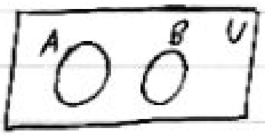
② $A \cap B$ = the <u>intersection</u> of A and B
= the set of elements in A <u>and</u> B
= $\{x \mid x \in A \wedge x \in B\}$

What do i color in?



If $A \cap B = \emptyset$, A and B are called <u>disjoint</u>.

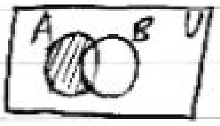
Reform, trash parties, there's Ventura



<u>Ex</u> A = Dems., B = Reps.

What do you think

③ $A - B$ = the <u>difference</u> of A and B
= the set of elements in A but not in B
$= \{x \mid x \in A \wedge x \notin B\}$



④ $\bar{A}$ or A^c = the <u>complement</u> of A (with respect to U)
$= U - A$
$= \{x \mid \underbrace{x \in U}_{\text{often assumed}} \wedge x \notin A\}$

⑤ $A \oplus B$ = the symmetric difference of A and B
= the set of elements in A xor B
(in HW)

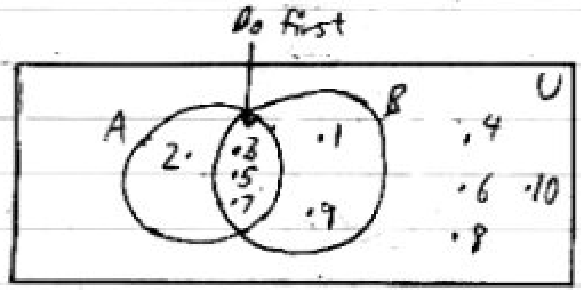
<u>Note</u> $A - B = A \cap \overline{B}$

Ex $U = \{1, 2, 3, \ldots, 10\}$
$A = \{2, 3, 5, 7\}$ (primes)
$B = \{1, 3, 5, 7, 9\}$ (odds)

Good Idea

Venn Diagram

What should we fill in 1st?



$A \cup B = \{1, 2, 3, 5, 7, 9\}$

$A \cap B = \{3, 5, 7\}$

$A - B = \{2\}$

$B - A = \{1, 9\}$

$\bar{A} = \{1, 4, 6, 8, 9, 10\}$

$\bar{B} = \{2, 4, 6, 8, 10\}$

$A \oplus B = \{1, 2, 9\}$

Basic
Set Identities (Table 1 – p. 49)

These correspond to the basic logical equivalences
(Table 5 – p. 17) Sec 1.2

		Logic	Set Theory	
Just know how $\wedge, \vee$ work	① Identity Laws	$p \wedge T \Leftrightarrow p$ $p \vee F \Leftrightarrow p$ $\underbrace{\vee F}_{\text{no impact}}$	$A \cap U = A$ $A \cup \phi = A$	
Recognize; don't memorize	② Domination Laws	$p \wedge F \Leftrightarrow F$ $p \vee T \Leftrightarrow T$ $\underbrace{\vee T}_{\text{dominate } p}$	$A \cap \phi = \phi$ $A \cup U = U$	 
	③ Idempotent Laws	$p \wedge p \Leftrightarrow p$ $p \vee p \Leftrightarrow p$ and/or-ing a prop. by itself has no impact	$A \cap A = A$ $A \cup A = A$	

[illegible] quote GII
Every time

④ Double Negation Law: $\neg(\neg p) \Leftrightarrow p$ — Complementation Law: $\overline{(\overline{A})} = A$ 

⑤, ⑥ $\wedge$ is commutative and associative

<u>Ex</u> $\underbrace{a \wedge b \wedge c \wedge \ldots \wedge z}$
can reorder and regroup, ( ) → no impact

So are $\vee, \cap, \cup$.

⑦ Distributive laws

We know $a \times (b+c) = a \times b + a \times c$,
$\times$ distributes over $+$.

Here, $\wedge$ ... $\vee$
$\vee$ ... $\wedge$
$\cap$ ... $\cup$
$\cup$ ... $\cap$

Ex $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$, etc.

[illegible]
[illegible]
— [illegible]

⑧ De Morgan's Laws

Logic	Set Theory
$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	$\overline{A \cap B} = \bar{A} \cup \bar{B}$ 
$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	$\overline{A \cup B} = \bar{A} \cap \bar{B}$ 

the negation of

Distribute $\neg$
flip connective

The complement of the $\cap$
= the $\cup$ of the complements
$\cup$ ... $\cap$

In general, $\neg(p_1 \wedge p_2 \wedge \ldots \wedge p_n)$
$\Leftrightarrow \neg p_1 \vee \neg p_2 \vee \ldots \vee \neg p_n$, etc.

Jill

Ex A woman wants a man who is tall and handsome. How can she be disappointed?

p: The man is tall.
q: The man is handsome.

$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$

The man is not tall or not handsome

Gary Coleman or Steve Buscemi

Ex p_1: Jim lives in Alabama
p_2: Jim lives in Alaska
$\vdots$
p_{50}: " , Wyoming
p_{51}: " D.C.

"Jim does not live in the U.S."

$\Leftrightarrow \neg \underbrace{(p_1 \vee p_2 \vee \dots \vee p_{50} \vee p_{51})}_{\text{Jim lives in the U.S.}}$

$\Leftrightarrow \neg p_1 \wedge \neg p_2 \wedge \dots \neg p_{51}$

~~Alabama~~ and ~~Alaska~~ and ...

Set Theory

Ex A_1 = Alabamans
A_2 = Alaskans
$\vdots$
A_{50} = Wyomingites
A_{51} = D.C.

Set of all non-Americans

$= \overline{A_1 \cup A_2 \cup \dots \cup A_{51}}$

$= \overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_{51}}$

= set of all people who are not Alabamans, not Alaskans, ..., and not D.C.

Ex 10-11 (book) pp. 49-50

Prove one of De Morgan's Laws: $\overline{A \cap B} = \bar{A} \cup \bar{B}$

① Two sets are equal $\Leftrightarrow$ they have the same elements.

Consider an arbitrary element "x" in U,

might skip some steps

$$x \in \overline{A \cap B} \Leftrightarrow x \notin A \cap B$$
$$\Leftrightarrow \neg(x \in A \cap B)$$
$$\Leftrightarrow \neg(\underbrace{x \in A}_{p} \wedge \underbrace{x \in B}_{q})$$
$$\Leftrightarrow \neg(x \in A) \vee \neg(x \in B)$$

by DeMorgan's Laws (logic)

$$\Leftrightarrow x \notin A \vee x \notin B$$
$$\Leftrightarrow x \in \bar{A} \vee x \in \bar{B}$$
$$\Leftrightarrow x \in \bar{A} \cup \bar{B}$$

x is in $\underbrace{\overline{A \cap B}}$ $\Leftrightarrow$ x is in $\underbrace{\bar{A} \cup \bar{B}}$

have the same elements, so =

Q.E.D. (end of proof)

② Another approach:

We can show $x \in \overline{A \cap B} \Rightarrow x \in \bar{A} \cup \bar{B}$

Thus, $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$?

We can show $x \in \bar{A} \cup \bar{B} \Rightarrow x \in \overline{A \cap B}$ (go backwards)

Thus, $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

Two sets that are subsets of each other are equal.

Q.E.D.

(This approach is superior if there is a step that's not reversible.)

③ Proof by "membership tables" (≈ truth tables).

Ex $\underline{A}$ $\underline{B}$ $\underline{C}$

8 rows (possible membership combos) $\begin{cases} \vdots \\ 1 \quad 1 \quad 0 \longrightarrow \text{see what happens!} \\ \text{We consider an element that's in } A \\ \text{and in } B\text{, but not in } C \\ \vdots \end{cases}$

Idea: If two sets have the same "final column", they are equal.

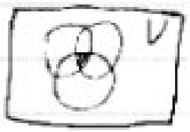
④ We can use set identities to simplify expressions or to verify harder identities. HW: My #1

<u>Generalized Unions and Intersections</u>

$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$ ← don't need ()s "well-defined" as is

$x \in \bigcup_{i=1}^{n} A_i \iff x \in \underset{\text{(one or more)}}{\underline{\text{any}}}\ A_i \quad (1 \le i \le n)$

$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$



$x \in \bigcap_{i=1}^{n} A_i \iff x \in \underline{\text{all}}$ of the $A_i \quad (1 \le i \le n)$

HW: #35, 36

HW Tips

Your proofs may look different from the book's. (Ans. in back)
The book uses set builder notation.
It's unclear how many steps you "need."
(On exams - I will understand this.)

Use our basic set identities to prove:

$$(B \cup A) \cap \overline{B} = A \cap \overline{B}$$

We can go $\rightarrow$ $\leftarrow$ $\searrow\swarrow$
Let's try to simplify the left side $\rightarrow$

(You don't have to write these.)

$(B \cup A) \cap \overline{B}$	$= \overline{B} \cap (B \cup A)$	Comm. Laws (optional) Could have distributed $\cap$.
	$= (\overline{B} \cap B) \cup (\overline{B} \cap A)$	Distributive Laws Think: $a * (b + c) = (a*b) + (a*c)$ Put in "()"s
	$= \phi \cup \underbrace{(\overline{B} \cap A)}_{\text{some set "C"}}$	$\boxed{B \ \overline{B}}$ No elements are in both
	$= \overline{B} \cap A$	$\phi \cup C = C$ (Identity)
	$= A \cap \overline{B}$	Comm. Laws

Great HW Tip: What is the HW?

My #1
De Morgan's Laws - we used them when we moved $\neg$