1.6: FUNCTIONS

A binary relation from a set $A$ to a set $B$ is a subset of $A \times B$. It's a set of ordered pairs.

Example:

$A = \{1, 2, 3\}$
$B = \{a, b, c, d\}$

The relation $R = \{(1, a), (1, b), (3, c)\}$ ($\subseteq A \times B$)

Each of the 1st components $\in A$
Each of the 2nd components $\in B$

Picture of $R$:
A function is a special type of binary relation.

Let $f$ be a function that "maps" a set $A$ to a set $B$.

$f: A \rightarrow B$

\[ f = \{(1, b), (2, a), (3, b)\} \]

Say: $f(1) = b$, $f(2) = a$, $f(3) = b$

$f$ must "point" each element of the domain ($A$) to exactly one element of the codomain ($B$).

Need

\[ \begin{array}{c}
\text{Domain} \\
\uparrow \\
\circ \\
\downarrow \\
\text{Codomain}
\end{array} \]

Forbid

\[ \begin{array}{c}
\text{Domain} \\
\uparrow \\
\circ \\
\downarrow \\
\text{Codomain}
\end{array} \]

Ex1: We have $f(1)$

Ex1: We have no ambiguities like "$f(1) = a \text{ or } b$"
\[ f(1) = b \]

The image of 1

the pre-image of b

\[ 1 \rightarrow \boxed{f} \rightarrow b \]

The range of f is the set of all the images of all the elements in \( A \). \( f(A) \)

It's a subset of the codomain.

\[ f(A) = \text{Range of } f = \{a, b\} \]

Ex 1

\[ \begin{array}{c}
\begin{array}{ccc}
1 & 2 & 3 \\
\downarrow & f & \downarrow \\
a & 4 & c \\
\end{array}
\end{array} \]

set of all elements in the codomain \( B \) that are "pointed to."
High school algebra

\[ f(x) = \frac{1}{x^2} \quad \text{Rule for } f \]

Domain of definition

Assumed domain = \( \mathbb{R} \setminus \{0\} \)

except

Write \( f: (\mathbb{R} \setminus \{0\}) \to \mathbb{R} \)

most obvious choice

Graph of \( f \)

\[ f(1) = 1 \]
\[ f(2) = \frac{1}{4} \]

etc.

\( f \) is a function, because the graph passes the Vertical Line Test (no vertical line passes through more than one point).

The Range of \( f \) is the set of all y-coords "hit" by the graph. Here, the range is \( \mathbb{R}^+ \).
Ex 2 \( f: \mathbb{Z} \rightarrow \mathbb{Z} \)

Rule: \( f(x) = x^2 \)

Range of \( f = \{0, 1, 4, 9, \ldots, 49, \ldots\} \)

\( f(\{0, 3, -4, 4\}) = \{0, 9, 16\} \)
ONE-TO-ONE FUNCTIONS

Function                 1-1 Function

Forbidden:  Also Forbidden:

\[ \forall x_1 \forall x_2 \left[ f(x_1) = f(x_2) \implies x_1 = x_2 \right] \]

A function \( f \) is one-to-one (or injective)

\[ \iff \forall x_1 \forall x_2 \left[ x_1 \neq x_2 \implies f(x_1) \neq f(x_2) \right] \]

Idea: An image has only one pre-image.

\( \exists x \) \( f(x) = e^x \), \( f : \mathbb{R} \to \mathbb{R} \)

\( f \) is a 1-1 function.

\[ f \text{ in } \mathbb{R}^2 \left( f(x) = x^2, f : \mathbb{R} \to \mathbb{R} \right) \text{ is not one-to-one} \]

\( \forall x \exists y \left[ f(x) = y \right] \text{ fails Horizontal Line Test} \]
ONTOT FUNCTIONS

Every element in the codomain must get "hit". (Codomain = Range)

A function $f : A \rightarrow B$ is onto or surjective $\iff \forall b \in B \exists a \in A \ [f(a) = b]$

Ex $f : \mathbb{R} \rightarrow \mathbb{R}$
$f(x) = 3x$

Approach 1: Graph

Range = $\mathbb{R}$

So, Codomain = Range.
So, $f$ is onto.

Approach 2: Algebra
Let $b$ be any element in the codomain ($\mathbb{R}$).
Can we always find a pre-image for $b$? \textcolor{red}{YES}

$b = 3x$

$\Rightarrow x = \frac{b}{3}$ (pre-image)

So, $f$ is onto.
**BIJECTIONS**

A bijection (or a one-to-one correspondence) is a function that is both one-to-one and onto.

\[
\begin{array}{ccc}
A & B & A & B \\
\longrightarrow & \text{or} & \longrightarrow & \text{could switch, anyway}
\end{array}
\]

(You can get this picture, maybe by moving dots.)

**Special Case**

\[ f: A \to A, \text{ where } A \text{ is a finite set} \]

\[ \begin{array}{ccc}
A & A \\
\end{array} \quad \text{Key: same-size sets} \]

\[ f \text{ is onto } \iff f \text{ is 1-1. } (\Rightarrow \text{ would be a waste!}) \]

If \( f \) is 1-1 or onto \( \Rightarrow \) \( f \) is a bijection
Review

Not functions

If \( f \) is a function from \( A \) to \( B \),
graph \( f \) by plotting the points \( \{(a, f(a)) | a \in A\} \subseteq A \times B \).
A function that is both 1-1 and onto is called a bijection (or a one-to-one correspondence).

**Special Case**
If $S$ is a finite set, then for $f: S \rightarrow S$,

\[
\begin{cases} 
1-1 \\
onto \\
\text{bijective}
\end{cases} \quad \text{if any one holds, then all 3 hold.}
\]
INVERSE FUNCTIONS

Let \( f: A \rightarrow B \) be a bijection.

Then, \( f \) has an inverse function \( f^{-1}: B \rightarrow A \), which is a bijection that reverses \( f \).

How is \( f^{-1} \) defined?

\[ f(a) = b \iff f^{-1}(b) = a \]

\[ \begin{array}{c}
\text{Note: } (f^{-1})^{-1} = f \\
\end{array} \]

\( f \) is a bijection \( \iff f \) is invertible

Ex

\[ \begin{array}{c}
A \quad \text{\ } \quad \text{B} \\
\downarrow \quad \quad \quad \quad \downarrow \\
1 \quad \quad \quad \quad m \\
2 \quad \quad \quad \quad n \\
\end{array} \]

\( f \) is a bijection, so \( f \) is invertible.

\[ \begin{array}{c}
f(1) = n, \text{ so } f^{-1}(n) = 1 \\
f(2) = m, \text{ so } f^{-1}(m) = 2 \\
\end{array} \]

Compositions combine functions (pp. 62-4; not tested): \( f(g(x)) \)
Two key functions that map $\mathbb{R} \rightarrow \mathbb{Z}$

Let $x \in \mathbb{R}$.

The **floor function** (or greatest integer function) rounds $x$ down.

$$f(x) = \lfloor x \rfloor \text{ or } \lfloor x \rfloor = (\text{closest integer that's } \leq x)$$

The **ceiling function** rounds $x$ up.

$$f(x) = \lceil x \rceil = (\text{closest integer that's } \geq x)$$

**Exs**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\lfloor x \rfloor$</th>
<th>$\lceil x \rceil$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>9.1</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>-10.7</td>
<td>-11</td>
<td>-10</td>
</tr>
</tbody>
</table>

$\lfloor x \rfloor \leq x \leq \lceil x \rceil$ always
Ex We have 50 balls.
A bag carries 12 balls.
How many bags do we need?

To be safe,

$$\left\lfloor \frac{50}{12} \right\rfloor = \left\lfloor 4.16 \right\rfloor = 5 \text{ bags}$$

How many bags can we fill up?

$$\left\lceil \frac{50}{12} \right\rceil = \left\lceil 4.16 \right\rceil = 4 \text{ bags}$$

$\lfloor x \rfloor$, $\lceil x \rceil$ are step functions

\[ y = \lfloor x \rfloor \quad \quad y = \lceil x \rceil \]

\[ \begin{array}{c|c|c|c|c}
| x | & -2 & -1 & 0 & 1 & 2 \\
\hline
| y = \lfloor x \rfloor | & \circ & \circ & \circ & \circ & \circ \\
\hline
| y = \lceil x \rceil | & \circ & \circ & \circ & \circ & \circ \\
\end{array} \]

HW Tip
In roster, $\mathbb{N} = \{0, 1, 2, 3, \ldots \}$
some books exclude