1.6: FUNCTIONS

A binary relation from a set A to a set B is a subset of $A \times B$. It's a set of ordered pairs. (a, b)

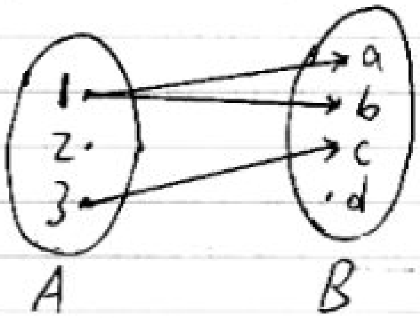
Ex $A = \{1, 2, 3\}$
$B = \{a, b, c, d\}$

The relation $R = \{(1,a), (1,b), (3,c)\}$ $(\subseteq A \times B)$

Each of the 1st components $\in A$

2nd components $\in B$

Picture of R:

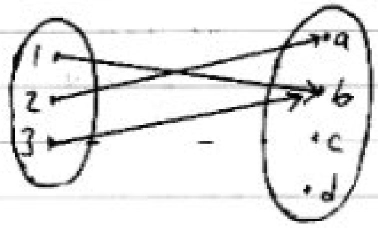


A <u>function</u> is a special type of binary relation.

Let f be a function that "maps" a set A to a set B.

$$\text{“ } f : \underset{\substack{\uparrow \\ \boxed{\text{Domain of } f}}}{A} \longrightarrow \underset{\substack{\uparrow \\ \boxed{\text{Codomain of } f}}}{B} \text{ ”}$$

<u>Ex1</u>



$f = \{(1,b), (2,a), (3,b)\}$

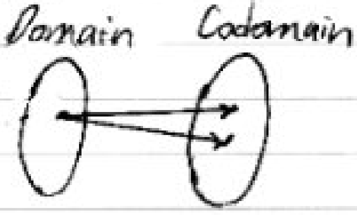
Say: $f(1)=b$, $f(2)=a$, $f(3)=b$

f must "point" each element of the domain (A) to <u>exactly one</u> element of the codomain (B).

<u>Need</u>

<u>Ex1</u> We have
f(1)
f(2)
f(3)

<u>Forbid</u>



<u>Ex1</u> We have no ambiguities like "f(1) = a or b"

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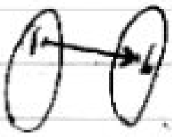
$f(1) = b$

the *image* of 1

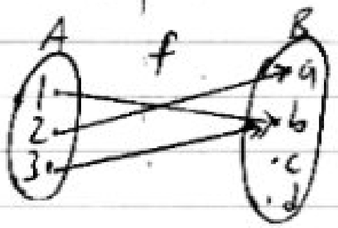
the *pre-image* of b



$$1 \longrightarrow \boxed{f} \longrightarrow b$$

The *range* of f is the set of *all the images* of all the elements in A. "f(A)"
It's a subset of the codomain.

Ex 1



$f(A)$ = Range of $f = \{a, b\}$
= set of all elements in the codomain (B) that are "pointed to."

High school algebra

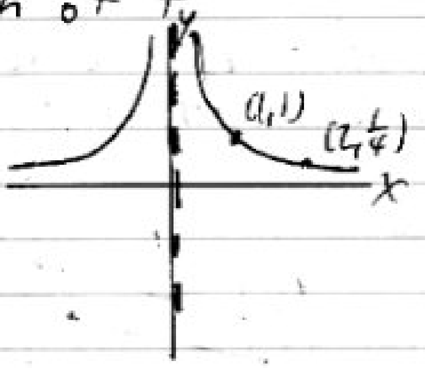
$f(x) = \frac{1}{x^2}$ ← Rule for f

Domain of definition (#59)

Assumed domain $= \mathbb{R} \setminus \{0\}$
$\uparrow$ except

Write "$f: (\mathbb{R} \setminus \{0\}) \longrightarrow \mathbb{R}$"
↑ most obvious choice

Graph of f



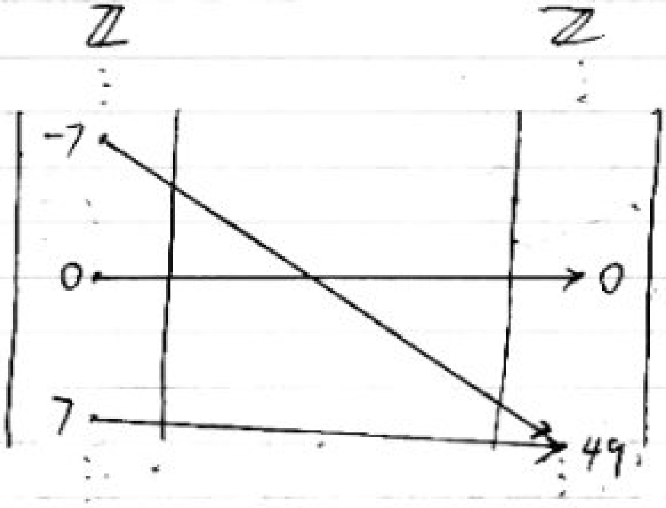
$f(1) = 1$
$f(2) = \frac{1}{4}$,
etc.

f is a function, because the graph passes the Vertical Line Test (no vertical line passes through more than one point).

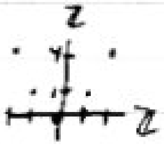
The Range of f is the set of all y-coords "hit" by the graph. Here, the range is $\mathbb{R}^+$.

<u>Ex 2</u> $f: \mathbb{Z} \to \mathbb{Z}$

Rule: $f(x)=x^2$



Range of $f = \{0,1,4,9,\ldots,49,\ldots\}$

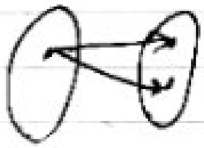


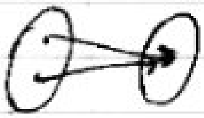
$f(\{0,3,-4,4\}) = \{0,9,16\}$

ONE-TO-ONE FUNCTIONS

Function — 1-1 Function

Forbidden: — Also Forbidden:



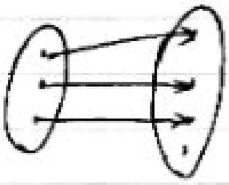


A function f is <u>one-to-one</u> (or <u>injective</u>)

$$\Leftrightarrow \underbrace{\forall x_1 \forall x_2}_{\text{in domain of } f} [f(x_1) = f(x_2) \longrightarrow x_1 = x_2] \quad (☆)$$

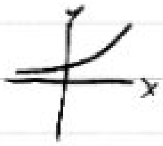
$$\Leftrightarrow \forall x_1 \forall x_2 [x_1 \neq x_2 \longrightarrow f(x_1) \neq f(x_2)]$$

Idea: An image has only one pre-image.



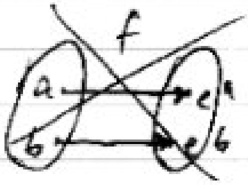
(☆) <u>Ex</u> $f(x) = e^x$, $f: \mathbb{R} \to \mathbb{R}$

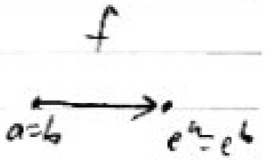
f is a 1-1 function.



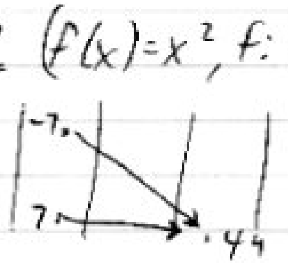
The graph passes the Horizontal Line Test

If $e^a = e^b$, then $a = b$





f in <u>Ex 2</u> ($f(x) = x^2$, $f: \mathbb{Z} \to \mathbb{Z}$) is <u>not</u> one-to-one



fails Horizontal Line Test

$x^2 = 49$

$x = \pm 7$ ← 2 pre-images

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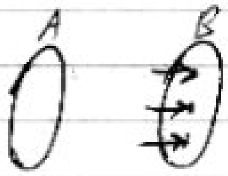
ONTO FUNCTIONS



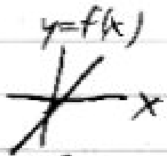
Every element in the
codomain must get "hit".
(Codomain = Range)

A function $f: A \to B$ is onto or surjective $\Leftrightarrow$
$\forall b \in B\ \exists a \in \underset{\checkmark}{\underline{A}}\ [f(a)=b]$
uod for b   uod for A

f in Ex 2 ($f(x)=x^2$, $f: \mathbb{Z} \to \mathbb{Z}$) is not onto,
because not every element in the
codomain ($\mathbb{Z}$) is in the range ($\{0,1,4,9,..\}$).

Ex $f: \mathbb{R} \to \mathbb{R}$
$f(x) = 3x$

Approach 1: Graph



Range = $\mathbb{R}$

So, Codomain = Range.
So, f is onto.

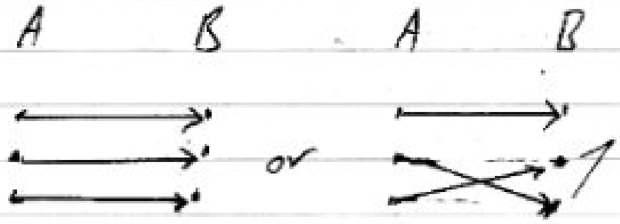
Approach 2:
Let b be any arbitrary element in the codomain ($\mathbb{R}$).
Can we always find a pre-image for b? YES
$b = 3x$
$\Rightarrow x = \frac{b}{3}$ (pre-image)
(in domain ✓)

$? \longrightarrow b$

So, f is onto.

BIJECTIONS

A bijection (or a one-to-one correspondence) is a function that is both one-to-one and onto.

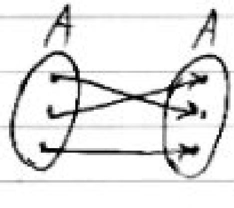


could switch anyway, etc.

(You can get this picture, maybe by moving dots.)

Special Case

$f: A \to A$, where A is a finite set

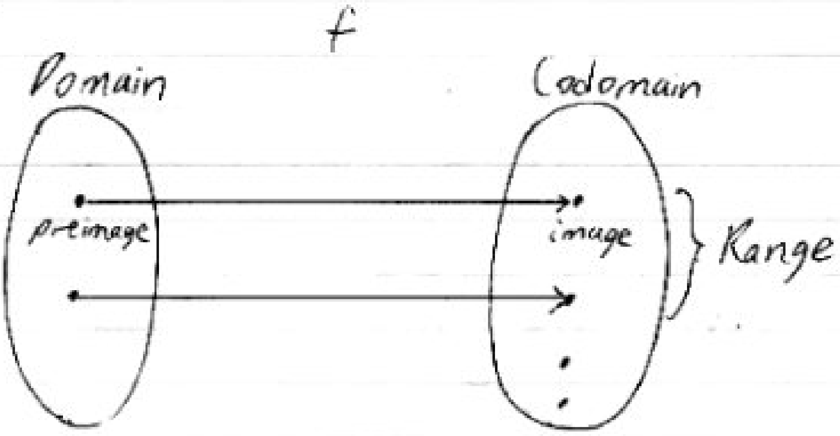


Key: same-size sets

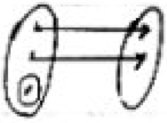
f is onto $\Leftrightarrow$ f is 1-1   ($\Rightarrow$ would be a waste!)

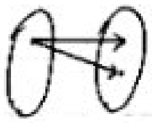
If f is 1-1 or onto $\Rightarrow$ f is a bijection

<u>Review</u>

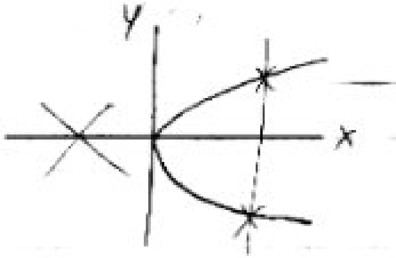


<u>Not functions</u>





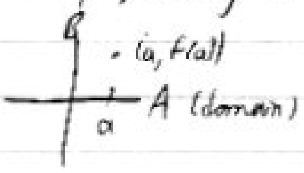
f is <u>not</u> a func from $\mathbb{R}$ to $\mathbb{R}$:

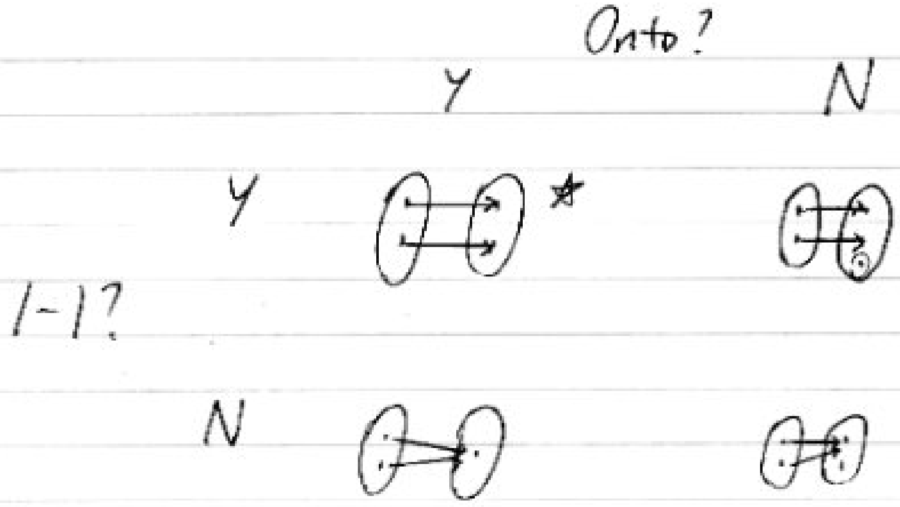


plot $(\overset{x}{\text{preimage}}, \overset{y}{\text{image}})$
y is <u>not</u> a func of x

2 reasons: negative #s don't get mapped
graph fails VLT

If f is a function from A to B,
graph f by plotting the points $\{(a, f(a)) \mid a \in A\}$ $(\subseteq A \times B)$





Range=
Codomain

$f: \mathbb{R} \to \mathbb{R}$

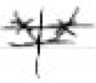
passes HLT

Ex $f(x) = e^x$

$e^a = e^b$

$\to a = b$

fails HLT



Ex $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

$a^2 = b^2$

$\not\to a = b$

★ A function that is <u>both</u> 1-1 and onto is called a <u>bijection</u> (or a <u>one-to-one correspondence</u>)

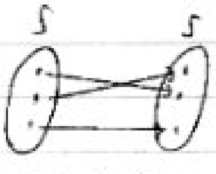
<u>Special Case</u>

If S is a finite set, then for $f: S \to S$,

$$\left\{\begin{array}{l} \text{1-1} \\ \text{onto} \\ \text{bijective} \end{array}\right.$$

If any one holds, then all 3 hold.

<u>Ex</u>



A 1-1 $f: S \to S$ must be onto, and vice versa

INVERSE FUNCTIONS

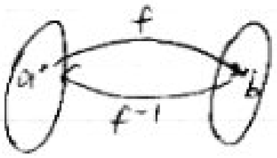
Let $f: A \to B$ be a bijection.

Then, f has an inverse function

$f^{-1}: B \to A$, which is a bijection that reverses f.

How is f^{-1} defined?

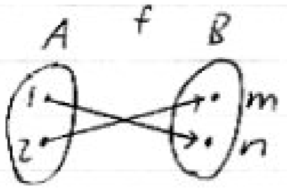
$$f(a) = b \iff f^{-1}(b) = a$$



Note: $(f^{-1})^{-1} = f$

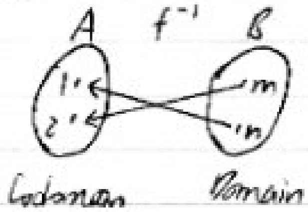
f is a bijection $\iff$ f is invertible

Ex



f is a bijection, so f is invertible.

$f(1) = n$, so $f^{-1}(n) = 1$
$f(2) = m$, so $f^{-1}(m) = 2$



Compositions combine functions (pp. 62-4; not tested): $f(g(x))$

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$\lfloor x \rfloor, \lceil x \rceil$

Two key functions that map $\mathbb{R} \longrightarrow \mathbb{Z}$

Let $x \in \mathbb{R}$.

The <u>floor function</u> (or <u>greatest integer function</u>) rounds x down.

$$f(x) = \lfloor x \rfloor \text{ or } [x]$$
$$= (\text{closest integer that's } \leq x)$$

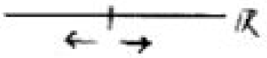
The <u>ceiling function</u> rounds x up.

$$f(x) = \lceil x \rceil \quad \leftarrow \text{"hooks" on ceiling}$$
$$= (\text{closest integer that's } \geq x)$$

<u>Exs</u>

x	$\lfloor x \rfloor$	$\lceil x \rceil$
5	5	5
9.1	9	10
-10.7	-11	-10

$\lfloor x \rfloor \leq x \leq \lceil x \rceil$ always



<u>Ex</u> We have 50 balls.
A bag carries 12 balls.
How many bags do we need?

To be safe,

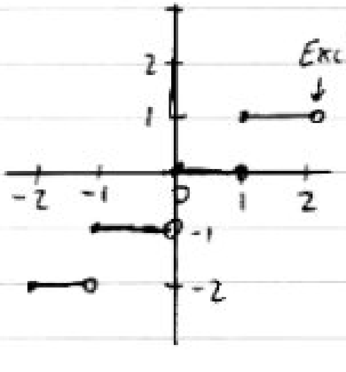
$$\left\lceil \frac{50}{12} \right\rceil = \left\lceil 4.1\overline{6} \right\rceil = 5 \text{ bags}$$

How many bags can we fill up?

$$\left\lfloor \frac{50}{12} \right\rfloor = \left\lfloor 4.1\overline{6} \right\rfloor = 4 \text{ bags}$$

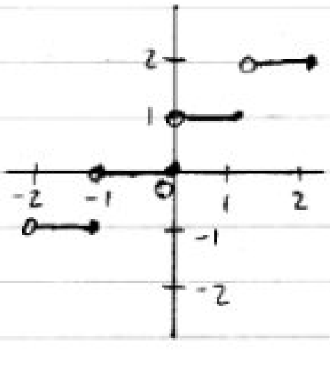
$\lfloor x \rfloor$, $\lceil x \rceil$ are <u>step functions</u>

$y = \lfloor x \rfloor$



Start at 0

$y = \lceil x \rceil$



<u>HW Tip</u>
In Rosen $\mathbb{N} = \{\underset{\wedge}{0},1,2,3,\ldots\}$
some books exclude