1.7: SEQUENCES AND SUMMATIONS

A <u>sequence</u> represents an ordered list.

<u>Ex</u> The sequence $\{a_n\}$ ("a-sub-n") (<u>not</u> "set") usually denotes

$$a_1, a_2, a_3, a_4, \ldots$$

<u>terms</u>

We sometimes start with a_0, not a_1:

$$a_0, a_1, a_2, a_3, \ldots$$

(typical in calculus)

A <u>string</u> is a finite sequence.

<u>Ex</u> $\underset{a_1}{1}, \underset{a_2}{0}, \underset{a_3}{0}, \underset{a_4}{1}$ $\quad \{a_n\}_{n=1}^{4}$

The <u>general term</u> (a_n) in a sequence may be given by a formula or rule.

<u>Ex</u> $a_n = (-1)^n 3n$, Start with $n=1$ (assume unless told otherwise)

$n=1$: $a_1 = (-1)^1(3)(1) = (-1)(3) = -3$

$n=2$: $a_2 = (-1)^2(3)(2) = (1)(6) = 6$

$n=3$: $a_3 = (-1)^3(3)(3) = (-1)(9) = -9$

$n=4$: $a_4 = (-1)^4(3)(4) = (1)(12) = 12$

Sequence:

$-3, 6, -9, 12, \ldots$ ← an <u>alternating sequence</u> (signs alternate)

In I.Q. tests, you're asked to find the most "obvious" pattern:

<u>Ex</u> $\underbrace{3, 7, 11, 15, 19, \ldots}$ Web p.72

∞ possibs, but 23, 27, 31, ... seem most "obvious" (successively adding 4)

can input 1st few terms puzzle seqs.

<u>Special Sequence Types</u>

① Arithmetic progressions
- successively add some common difference "d"

<u>Ex</u> 3, 7, 11, 15, 19, ...

$a_1 = 3,\ d = 4$

$a_n = a_1 + \underbrace{(n-1)}_{\#\text{steps}}\underbrace{d}_{\text{stepsize}} \quad = 3 + (n-1)(4)$

② Geometric progressions
- successively multiply some common ratio "r"

<u>Ex</u> 5, 15, 45, 135, ...

$a_1 = 5,\ r = 3$

$a_n = a_1 \underbrace{r}_{\text{multiplier}}^{\overbrace{n-1}^{\#\text{steps}}} \quad = 5(3)^{n-1}$

SUMMATIONS

<u>Ex</u> $\sum_{i=1}^{n} a_i = a_1 + a_2 + \dots + a_n \quad (n \in \mathbb{Z}^+)$

(Sum of 1st n terms)

<u>Ex</u> $\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \dots + a_n \quad \left(\begin{smallmatrix} m \le n \\ m, n \in \mathbb{Z} \end{smallmatrix}\right)$

i is the index of summation
(could use $j, k, \dots$)
The index sweeps through all the integers from the lower limit (m) to the upper limit (n).

<u>Ex</u> $$\sum_{k=3}^{5} \underbrace{(k \cdot 2^k - 1)}_{a_k} = a_3 + a_4 + a_5$$

$$= [3 \cdot 2^3 - 1] + [4 \cdot 2^4 - 1] + [5 \cdot 2^5 - 1]$$

$$= 23 + 63 + 159$$

$$= 245$$

<u>Ex</u> $\sum_{i \in S} a_i$ where $S = \{1, 4, 6\}$

$i \in S$ ← indices sweep through S

$= a_1 + a_4 + a_6$

Ex (Double summation)

Arise from nested loops in programs.

$$\sum_{i=1}^{3} \sum_{j=1}^{i} \frac{i}{j} = (\text{Case } i=1) + (\text{Case } i=2) + (\text{Case } i=3)$$

$$= $$

$$i=1 \quad \sum_{j=1}^{1} \frac{1}{j}$$

$$i=2 \quad + \sum_{j=1}^{2} \frac{2}{j}$$

$$i=3 \quad + \sum_{j=1}^{3} \frac{3}{j}$$

$$= \frac{1}{1}$$

$$+ \frac{2}{1} + \frac{2}{2}$$

$$+ \frac{3}{1} + \frac{3}{2} + \frac{3}{3}$$

$$= 9\frac{1}{2}$$

Table 2 (p. 76) has special Σ formulas.

HW Tip

#17) $\underbrace{\sum_{i=\#}^{\#} \sum_{j=\#}^{\#}}_{\text{may be faster to work this out first}} f(i,j)$

(Not tested) <u>SPECIAL SUMS</u>

<u>Ex1</u> $\sum_{i=1}^{100} i = 1 + 2 + 3 + \ldots + 100$

Gauss - when he was 10, his teacher
wanted to occupy the class.
Slapped him - really stared.

Trick:

$$S = 1 + 2 + 3 + \ldots + 100$$
$$S = 100 + 99 + 98 + \ldots + 1 \quad \text{2 reverse}$$

$$2S = \underbrace{101 + 101 + 101 + \ldots + 101}_{100 \text{ copies}}$$

$$2S = 101(100)$$

$$S = \frac{101(100)}{2} = 5050$$

$\underline{\text{Ex 2}}$ In general, $\sum_{i=1}^{n} i = 1+2+3+\ldots+n \quad (n \in \mathbb{Z}^+)$

etc.

Same trick!

$$\begin{array}{l} S = 1 + 2 + \ldots + n \\ S = n + (n-1) + \ldots + 1 \\ \hline 2S = \underbrace{(n+1)+(n+1)+\ldots+(n+1)}_{n \text{ copies}} \end{array}$$

$$S = \frac{n(n+1)}{2}$$

More generally, let's consider the sum of any n consecutive terms in an arithmetic sequence.

Let's say $a_1 + a_2 + \ldots + a_n$

$$\text{Sum} = \underbrace{\left(\frac{a_1+a_n}{2}\right)}_{\text{Average of 1st, last terms}} \underset{\#\text{ terms}}{(n)}$$

Why does this make sense? Interpret this formula

$$S = a_1 + \overbrace{a_2}^{a_1+d} + \ldots + a_n$$

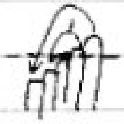
$$S = a_n + \underbrace{a_{n-1}}_{a_n-d} + \ldots + a_1$$

$$\underbrace{(a_1+a_n)+(\text{same})+\ldots+(a_1+a_n)}_{n \text{ copies}}$$

$$2S = (a_1+a_n)(n)$$

$$S = \left(\frac{a_1+a_n}{2}\right)(n)$$

2



Communism

$\underline{\text{Ex}}$ $7+8+9+10+11 = 9+9+9+9+9 = (9)(5) = 45$

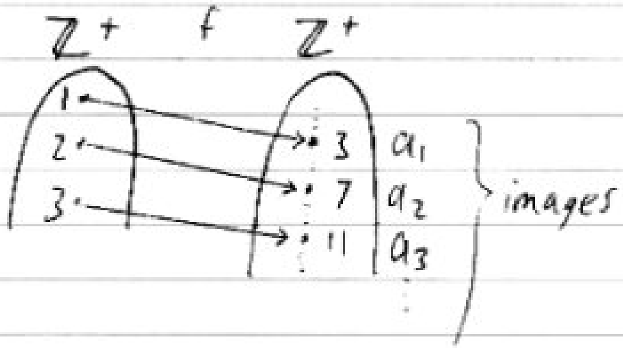
$\underline{\text{Ex}}$ $\underbrace{3+7+11+\ldots+83}_{21 \text{ terms}} = \left(\frac{3+83}{2}\right)(21) = 903$

CARDINALITY

A sequence $a_1, a_2, a_3, \ldots$ (each $a_i \in S$)

is a function $f: \mathbb{Z}^+ \to S$

Ex $3, 7, 11, 15, \ldots$ $\leftarrow$ each $\in \mathbb{Z}^+$



A set is <u>countable</u> $\Leftrightarrow$

① The set is finite.
or ② All the elements can be listed as $a_1, a_2, a_3, \ldots$

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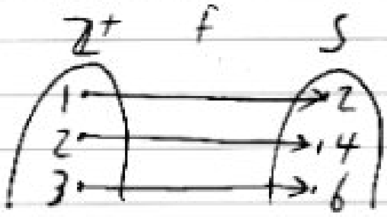
(i.e., There is a bijection (1-1 corresp.) between the set and $\mathbb{Z}^+$)

Ex Let S = set of all positive even integers
$= \{2, 4, 6, 8, \ldots\}$

S is countable.



f defined by $f(n) = 2n$ is a bijection $\mathbb{Z}^+ \to S$

1	2	3	4	…
2	4	6	8	…

Nice list!

Ex Let S = set of all even integers
$= \{\ldots, -4, -2, 0, 2, 4, \ldots\}$

S is countable.

How can we make a nice list?

	5	3	1	2	4	
…	-4	-2	0	2	4	…

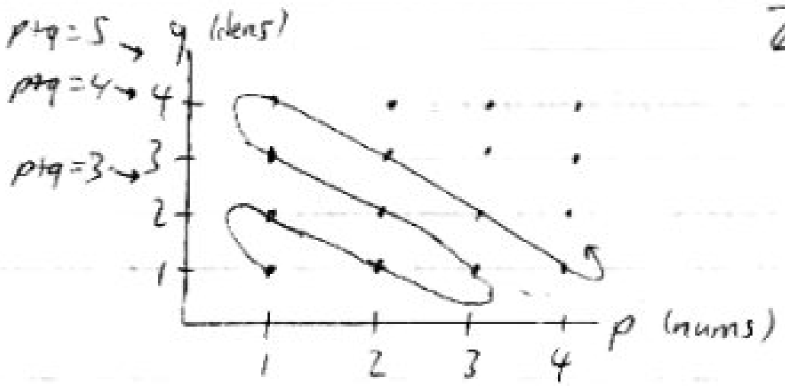
Sequence: $0, 2, -2, 4, -4, \ldots$

<u>Ex</u> $\mathbb{Q}^+$ = set of all positive rational #s

$= \left\{ x \mid x \text{ can be written as } \frac{p}{q}, \text{ where } p \in \mathbb{Z}^+, q \in \mathbb{Z}^+ \right\}$

is countable.

How can we make a nice list?



$\mathbb{Z}^+ \times \mathbb{Z}^+$ lattice

① $\frac{1}{1} = 1$ $p+q=2$
② $\frac{1}{2}$, ③ $\frac{2}{1} = 2$ $p+q=3$
④ $\frac{3}{1} = 3$, ~~⑤ $\frac{2}{2} = 1$~~ (repeat!), ⑤ $\frac{1}{3}$ $p+q=4$
⋮

List: $1, \frac{1}{2}, 2, 3, \frac{1}{3}, \ldots$

<u>Ex</u> $\mathbb{Q}$ = all rational #s is countable.

List: $0, \underbrace{1, -1}, \underbrace{\frac{1}{2}, -\frac{1}{2}}, \underbrace{2, -2} \ldots$

need negative partners

<u>Ex</u> Show $\mathbb{R}$ is uncountable.

Let $S = \{x \mid x \in \mathbb{R},\ 0 < x < 1\}$ 

It is sufficient to show S is uncountable.

<u>Proof by Contradiction</u>

Assume S is countable.
Then, all the elements of S can be listed:

$x_1 : 0.d_{11} d_{12} d_{13} \ldots$ (all d_{ij} are digits)

$x_2 : 0.d_{21} d_{22} d_{23} \ldots$

$x_3 : 0.d_{31} d_{32} d_{33} \ldots$

$\vdots$

We can construct a new # in S
that is not on the list.

$$\text{new \#} = 0.\underbrace{d_1}_{\substack{5 \text{ if } d_{11}=4 \\ 4 \text{ if } d_{11}\neq 4}}\ \underbrace{d_2}_{\substack{5 \text{ if } d_{22}=4 \\ 4 \text{ if } d_{22}\neq 4}}\ d_3 \ldots$$

Ex

$x_1 = 0.712\ldots$ Diagonalization argument
$x_2 = 0.346\ldots$
$x_3 = 0.559\ldots$
↓↓↓ etc.
New # $= 0.454\ldots$

Idea: The new # will differ from each listed #
by at least 1 digit.
In particular, the new # and x_i
will differ in the i^{th} decimal place.

BUT we assumed that all the elements in S
were listed!

Our list will never be "good enough!"

So, our assumption that S was countable was wrong.

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$\therefore S$ is uncountable.
$\therefore \mathbb{R}$ is uncountable.

See Ex 17 on pp. 77-8.