3.7: MATHEMATICAL INDUCTION

Not used to derive formulas or make discoveries.
Is used to prove guesses.

Ex 2 (pp. 189-190)

What is the sum of the first $n$ odd positive integers?

<table>
<thead>
<tr>
<th>$n$</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1 + 3</td>
</tr>
<tr>
<td>3</td>
<td>1 + 3 + 5</td>
</tr>
</tbody>
</table>

Guess: $n^2$
Ex 2 proves this.

$\forall n \in \mathbb{Z}^+$
$P(n) = \text{sum of first } n \text{ odd positive integers is } n^2$
WEAK INDUCTION

To prove propositions of the form \( \forall n \, P(n) \), \( n \in \mathbb{Z}^+ \):

1. **Basis step**: Show \( P(1) \) is true.

2. **Inductive step**: Show that, for any arbitrary \( n \in \mathbb{Z}^+ \), 
   \[
   P(n) \implies P(n+1)
   \]
   is true.
   
   i.e., If we assume \( P(n) \) is true, show that \( P(n+1) \) must also be true. (NOT circular reasoning!)

**Falling dominoes**

\[
\begin{align*}
P(1) & \implies I.S. \implies I.S. \implies I.S. \implies \ldots
\end{align*}
\]

Inductive step guarantees that each falling domino makes the next domino fall.

Read Exs 2-5, 8-11
Ex: Prove: \( \forall n \in \mathbb{Z}^+ \quad 1^3 + 2^3 + \ldots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2 \)

**Note:** \( = \left(1 + 2 + \ldots + n\right)^2 \)

1. **Basis Step**
   \[ P(1): \quad 1^3 = \left[ \frac{1(1+1)}{2} \right]^2 \quad \text{plug } n = 1 \]
   \[ 1 = 1 \]
   \[ \therefore P(1) \text{ is true.} \]

2. **Inductive Step**

   Let \( n \) be any arbitrary positive integer.
   Assume \( P(n) \) is true.

   (Inductive Hypothesis)
   \[ 1^3 + 2^3 + \ldots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2 \]

   Show \( P(n+1) \) is true.

   Show:
   \[ 1^3 + 2^3 + \ldots + (n+1)^3 = \left[ \frac{(n+1)(n+2)}{2} \right]^2 \]
   Replace \( n \) with \( n-1 \).
\[
\begin{align*}
&\frac{1^3 + 2^3 + \ldots + n^3 + (n+1)^3}{\text{by the I.H.,}} \\
&\text{this equals} \\
&\left(\frac{n(n+1)}{2}\right)^2 \\
&= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\
&\text{We want:} \quad \left[\frac{(n+1)(n+2)}{2}\right]^2 \\
&\text{Factor out } (n+1)^2 \\
&= \frac{(n+1)^2 \left[n^2 + 4(n+1)\right]}{4} \\
&= \frac{(n+1)^2 (n^2 + 4n + 4)}{4}
\end{align*}
\]
\[ \frac{(n+1)^2(n+2)^2}{4} = \left( \frac{(n+1)(n+2)}{2} \right)^2 \]

\[ \forall n \in \mathbb{Z}^+ \, P(n) \]

**QED**

**Note** To prove \( P(n) \) for all nonnegative integers:
Basis step: Show \( P(0) \) is true.

To prove \( P(n) \) is true for all integers \( \geq a \) (as \( a \) is well-defined)

**Ex 14 (Rosen pp. 198-9)**

Prove that every amount of postage of \( \geq 12 \) can be formed using just 4¢ and 5¢ stamps.

Let \( P(n) = \text{n cents can be formed from 4¢ and 5¢ stamps} \)

1. **Basis Step**
   
   \( P(12) \): 3 4¢ stamps \( \rightarrow 12¢ \)
② Inductive Step

Let $n$ be any integer $\geq 12$.

Assume $P(n)$ is true.

[maybe] some 4¢  [maybe] some 5¢ $\implies n$ 4¢

Show $P(n+1)$ is true

Case 1: 21 4¢ stamps can be used to form $n$ 4¢.

Then, replace 14¢ stamp with 1 5¢ stamp.

$n$ 4¢: \hspace{1cm} 4¢

\hspace{1cm} \text{same}

$(n+1)$ 4¢: \hspace{1cm} 5¢

In fact only $n=15$ is in $P(n)$ case. $n=20, 25, 30, 35, 40$ and any higher multiple of 5 can build on that.

Case 2: No 4¢ stamps can be used to form $n$ 4¢.

Then, all stamps used to form $n$ 4¢ are 5¢ stamps.

$n$ 4¢: \hspace{1cm} 5¢ 5¢ 5¢ \hspace{1cm} \leftarrow n \geq 15$ in this case

$(n+1)$ 4¢: \hspace{1cm} 4¢ 4¢ 4¢ 4¢ \hspace{1cm} \checkmark

QED
**Strong Induction**

To prove $\forall n \in \mathbb{Z}^+, \text{ } n \mod d = 2^+$

1. **Basis Step:** Show $P(1)$ is true. You may have to show $P(2), P(3), \ldots$, some $P(k)$ are true.

2. **Inductive Step:**
   - Show that, for any arbitrary $n \in \mathbb{Z}^+$, $\left( P(1) \land P(2) \land \ldots \land P(n) \right) \implies P(n+1)$

   **IH:** Assume that all previous cases are true.

Sometimes you need more than just $P(n)$ to prove $P(n+1)$.

**Ex 13 (Kosen p.198)**

Prove that every integer greater than 1 can be written as the product of primes.

1. $P(2)$ is true: $2 = 2$, prime

2. Assume $P(2), P(3), \ldots, P(n)$ are true. (IH)
   - i.e., $\forall k (2 \leq k \leq n) \implies P(k)$ is true.
Show $P(n+1)$ is true

Case 1: $n+1$ is prime
$\quad n+1 = n+1$

Case 2: $n+1$ is composite
$\Rightarrow \exists s \in \mathbb{Z} (2 \leq r \leq s \leq n) ; = rs$

Say $s$ is larger while.

By IH, $r$ and $s$ can be written as the product of primes.

So, $n+1$ can be written as a product of primes

QED

This proves the existence part of FTA.
The uniqueness proof is on p. 139 (205)
Exercise 14 (again)

(maybe) some $\forall k$ stamps $\Rightarrow$ anything

Proof by strong induction

1) Base step
   \[ P(12) \Rightarrow T : 4 + 4 + 4 = 12 \]
   \[ P(13) \Rightarrow T : 4 + 4 + 5 = 13 \]
   \[ P(14) \Rightarrow T : 4 + 5 + 5 = 14 \]
   \[ P(15) \Rightarrow T : 5 + 5 + 5 = 15 \]

2) Inductive step
   Let \( n \geq 12, n \geq 15 \)
   Assume \( \forall k \geq 12 \), \( P(k) \) is true,
   \( P(12), P(13), \ldots, P(n) \)
   Show \( P(n+1) \) is true.

   By IH and \( n \geq 15 \), we know
   \( P(n-3) \) must be true.

   \( n-3 \notin \{4,7,10\} \)
   \( n+1 \notin \{4,7,10\} \)

   QED

12 13 14 15 16 17 18

Base step