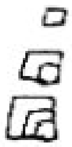
3.2: MATHEMATICAL INDUCTION

Not used to derive formulas or make discoveries.
Is used to prove guesses.

Ex 2 (pp. 189-190)
What is the sum of the first n odd positive integers?

n		Sum
1	1	1
2	$1+3$	4
3	$1+3+5$	9



Guess: n^2
Ex 2 proves this.

I'll remind later: $1+2+\dots+n = \frac{n(n+1)}{2} \approx \frac{n^2}{2}$ (+ correction) What you'd expect.

$$\forall n \; \underbrace{P(n)}$$

uod= $\mathbb{Z}^+$

sum of first n odd pos. ints is n^2

WEAK INDUCTION

$P(0)$ or $P(1)$
etc.
$P(n) \to P(n+1)$
$\therefore \forall x\, P(x)$

To prove propositions of the form $\forall n\, P(n)$, u.o.d $= \mathbb{Z}^+$:

① Basis step: Show $P(1)$ is true.

② Inductive step:
Show that, for any arbitrary $n \in \mathbb{Z}^+$,
$\underbrace{P(n)}_{\text{inductive hypothesis}} \longrightarrow P(n+1)$ is true.

i.e., If we assume $P(n)$ is true, show that $P(n+1)$ must also be true. (NOT circular reasoning!)

Falling dominoes

Inductive step guarantees that each falling domino makes the next domino fall.
Modus ponens up the ying yang!

Read Exs 2-5, 8-11

#8, p200 $\quad$ Ex Prove: $\forall n \in \mathbb{Z}^+ \quad \overbrace{1^3+2^3+\ldots+n^3 = \left[\frac{n(n+1)}{2}\right]^2}^{P(n)}$

Note: $= (1+2+\ldots+n)^2$

① Basis Step

$P(1)$: $1^3 = \left[\frac{1(1+1)}{2}\right]^2$ (plug in $n=1$)

$1 = 1$

$\therefore$ P(1) is true.

② Inductive Step

Some replace w/k.

Let n be any arbitrary positive integer.
Assume $P(n)$ is true.

(Inductive Hypothesis:)

$$1^3+2^3+\ldots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Show $P(n+1)$ is true.

Show: $1^3+2^3+\ldots+(n+1)^3 = \underbrace{\left[\frac{(n+1)(n+2)}{2}\right]^2}_{\text{Replace } n \text{ with } n+1.}$

$$\underbrace{1^3+2^3+\dots+n^3}_{\substack{\text{by the I.H.,}\\ \text{this equals}\\ \left[\frac{n(n+1)}{2}\right]^2}}+(n+1)^3$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3$$

$$= \frac{n^2(n+1)^2}{4} + (n+1)^3$$

$$= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4}$$

$$= \frac{n^2(n+1)^2 + 4(n+1)^3}{4}$$

We want: $\left[\frac{(n+1)(n+2)}{2}\right]^2$

Factor out $(n+1)^2$

$$= \frac{(n+1)^2\left[n^2+4(n+1)\right]}{4}$$

$$= \frac{(n+1)^2(n^2+4n+4)}{4}$$

$$= \frac{(n+1)^2(n+2)^2}{4}$$

$$= \left[\frac{(n+1)(n+2)}{2}\right]^2$$

$\therefore \forall n \in \mathbb{Z}^+ \; P(n)$
QED

<u>Note</u> To prove $P(n)$ for all nonnegative integers:
Basis step: Show $P(0)$ is true.

OK because such a set is well-ordered

To prove $P(n)$ is true for all integers $\geq a$ $(a \in \mathbb{Z})$
Basis step: Show $P(a)$ is true.

<u>Ex 14 (Rosen pp. 198-9)</u>

Prove that every amt. of postage of $\geq 12¢$ can be formed using just $4¢$ and $5¢$ stamps.

$\exists m, n \in \mathbb{Z}^{\geq 0}$
$4m + 5n$
$\rightarrow$ get ≥ 12

Let $P(n)$ = postage of n cents can be formed from $4¢$ and $5¢$ stamps

1. Basis Step
$P(12)$: 3 $4¢$ stamps $\rightarrow 12¢$

$\underbrace{\boxed{4}\;\boxed{4}\;\boxed{4}}_{12}$

② Inductive Step

Let n be any integer ≥ 12.

Assume $P(n)$ is true.

[maybe] some 4¢ → n¢
[maybe] some 5¢ → n¢

Show $P(n+1)$ is true

<u>Case 1</u> ≥ 1 4¢ stamp can be used to form n¢.

Then, replace 1 4¢ stamp with 1 5¢ stamp.

n¢: . . . . . $\boxed{4¢}$

$(n+1)$¢: . . . . . $\boxed{5¢}$ ✓
same

In fact only $n=15$ is in this case.
$n=20$: 4+4+4+4+4
and any higher multiple of 5 can build on this.

<u>Case 2</u> No 4¢ stamps can be used to form n¢.
Then, all stamps used to form n¢ are 5¢ stamps.

n¢: $\boxed{5}$ $\boxed{5}$ $\boxed{5}$ . . . ← $n \geq 15$ in this case

What can I do?

$(n+1)$¢: $\boxed{4}$ $\boxed{4}$ $\boxed{4}$ $\boxed{4}$ . . . ✓

QED

STRONG INDUCTION

To prove $\forall n\, P(n)$, uod $= \mathbb{Z}^+$

① Basis Step: Show $P(1)$ is true.
You may have to show $P(2), P(3), \ldots$, some $P(k)$ are true.

② Inductive Step:
Show that, for any arbitrary $n \in \mathbb{Z}^+$,

$$\underbrace{[P(1) \wedge P(2) \wedge \ldots \wedge P(n)]}_{\text{IH: Assume that all previous cases are true.}} \rightarrow P(n+1)$$

Sometimes you need more than just $P(n)$ to prove $P(n+1)$.

Ex 13 (Rosen p.198)

Prove that every integer greater than 1 can be written as the product of primes.

① $P(2)$ is true : $2 = 2$, prime

② Assume $P(2), P(3), \ldots, P(n)$ are true, i.e., $\forall k\, (2 \le k \le n)\ P(k)$ is true.) IH

Show $P(n+1)$ is true

<u>Case 1</u> $n+1$ is prime ✓
$n+1 = n+1$

<u>Case 2</u> $n+1$ is composite

$\Rightarrow \exists r,s \in \mathbb{Z}\ (2 \le r \le s \le n):\ = rs$

Say s is larger WLOG.

By IH, r and s can be written as the product of primes.

So, $n+1$ can be written as a product of primes

$$\underbrace{\overbrace{p_1 p_2 \cdots p_r}^{r}\ \overbrace{p_{r+1} p_{r+2} \cdots p_{r+m}}^{s}}_{n}$$ ← can have repetitions

QED

This proves the existence part of FTA.
The uniqueness proof is on p. 139 (2.5)

Ex 14 (again)

(maybe) some $\begin{matrix}4¢\\5¢\end{matrix}$ stamps $\rightarrow$ anything $\geq 12¢$

Proof by strong induction

① Basis step

$P(12)$ is T : $4+4+4=12$

$P(13)$ is T : $4+4+5=13$

$P(14)$ is T : $4+5+5=14$

$P(15)$ is T : $5+5+5=15$

② Inductive step

Let $n \in \mathbb{Z}$, $n \geq 15$

Assume $\forall k\ (12 \leq k \leq n)\ P(k)$ is true.
$P(12), P(13), \ldots, P(n)$

Show $P(n+1)$ is true.

By IH and $n \geq 15$, we know $P(n-3)$ must be true.

Verifies only 15 needed all 5¢.

$n-3$ ¢ : ~

$n+1$ ¢ : ~ $\boxed{4¢}$ ✓

QED

$$\underbrace{12 \quad 13 \quad 14 \quad 15}_{\text{Basis step}} \quad 16 \quad 17 \quad 18 \cdots$$