4.1 BASICS

**SUM RULE**

If \( A_1, A_2, \ldots, A_m \) are disjoint (non-overlapping) finite sets, then

\[
|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|
\]

**Example**

A high school's science dept. has

- 3 chem teachers \( \leftrightarrow \) "C"
- 5 bio teachers \( \leftrightarrow \) "B"
- 2 physics teachers \( \leftrightarrow \) "P"

Assume each teacher does only one subject.

How many science teachers are there?

\[
|C \cup B \cup P| = |C| + |B| + |P|
= 3 + 5 + 2 = 10
\]

There are 10 choices for a dept. chair.
(b) Assume there are two joint chem/bio teachers. 

\[ |C| = 3 \]
\[ |B| = 5 \]
\[ |P| = 2 \]

\[ |C \cup B| = |C| + |B| - |C \cap B| \]

Another approach:

**Inclusion-Exclusion Principle (Sec 5.5)**

\[ |C \cup B| = |C| + |B| - |C \cap B| \]

\[ = 3 + 5 - 2 \]

\[ = 6 \]

So, \[ |C \cup B \cup P| = |C \cup B| + |P| \]

\[ = 6 + 2 \]

\[ = 8 \]
PRODUCT RULE

Consider a sequence of \( m \) "decisions."
There are \( n_1 \) "choices," for the 1st decision.
There are \( n_2 \) "choices," for the 2nd decision,
regardless of the 1st decision.

\[ \vdots \]

There are \( n_m \) "choices" for the \( m \)th decision,
regardless of the 1st \((m-1)\) decisions.

Then, there are \( n_1 n_2 \cdots n_m \) possible sequences.

**Special Case**

\[ |A_1 \times A_2 \times \ldots \times A_m| = |A_1| |A_2| \cdots |A_m| \]

fixed, finite sets

Ex. A "bit" can be a "0" or a "1".
How many "bit strings" of length 3 are there?

Here, \( A_1 = A_2 = A_3 = \{0, 1\} \)

\[ \overbrace{2 \times 2 \times 2}^{0 \text{ or } 1 \text{ or } 0 \text{ or } 1} = 2^3 = 8 \]

Tree Diagram

\[ \begin{array}{ccc}
\text{1st bit} & \text{2nd bit} & \text{3rd bit} \\
\text{Start} & 0 & 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \\
\end{array} \]
Ex 3 chem teachers
5 bio teachers
2 physics teachers

\[ |C \times B \times P| = \frac{3!\cdot 5!\cdot 2!}{3! \cdot 5! \cdot 2!} = \frac{3 \times 5 \times 2}{3 \times 5 \times 2} = 30 \]

\[ \frac{3}{chem} \times \frac{5}{Bio} \times \frac{2}{Phys} = 30 \]

Ex You need a five character password
made up of uppercase letters and/or digits 0-9.
It must begin with a "6", and it must end in a vowel.
How many choices are there?

\[ \frac{1}{6} \times \frac{36}{A-Z} \times \frac{36}{A-Z} \times \frac{36}{A-Z} \times \frac{5}{A,E,I,O,U} \]
\[ = 233,280 \]
In general, the $A_i$'s need not be fixed!

Password Ex
Exactly 5 chars long
A-Z or 0-9
Begins with "6"
Ends in a vowel.
* No character can be repeated.

```
a_1 a_2 a_3 a_4 a_5
```

\[
\begin{array}{c}
\frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6}
\end{array}
\]

\[\frac{5}{A,E,I,O, or U}\]

\[1 \times 34 \times 33 \times 32 \times 5 = 179,520\]

\[\frac{A-Z,}{0-9,}\]
\[\text{except } 6, \text{a}_5, \text{e}, \text{a}_3, \text{e}, \text{a}_1, \text{e}, \text{a}_3\]

\(\approx 77\%\) of total # where repetitions were allowed
SPLITTING INTO CASES

We can combine Sum, Product Rules.

Password Ex
- Up to 5 chars long
- A-Z or 0-9
- Begins with "6."
- Ends in a vowel.
- No character can be repeated.

Let \( l \) = length of password

\( l \) can't be 1, because "6" isn't a vowel.

Case 1: \( l = 2 \)

\[
\frac{1}{6} \times \frac{5}{vowel} = \underline{5}
\]

Case 2: \( l = 3 \)

\[
\frac{1}{6} \times 34 \times \frac{5}{vowel} = \underline{170}
\]

Case 3: \( l = 4 \)

\[
\frac{1}{6} \times 34 \times 33 \times \frac{5}{vowel} = \underline{5610}
\]

Case 4: \( l = 5 \) From before, \( 179,520 \)

Total: \( 185,305 \)
"AT LEAST ONE"

Ex: A password consists of 5 characters (uppercase letters and digits 0-9). At least one character must be a digit.

Total # of 5-character strings (A-2, 0-9)

\[
\frac{36 \times 36 \times 36 \times 36 \times 36}{A-Z, 0-9} = 36^5
\]

# of 5-character strings with no digits

\[
\frac{26 \times 26 \times 26 \times 26 \times 26}{A-Z} = 26^5
\]

Answer: \(36^5 - 26^5 = 48584800\)
#20 Consider the positive integers \([\text{w/exactly 4 decimal digits}]\) (i.e., between 1000 and 9999 inclusive).

How many are there? \(9000\)

\[\begin{array}{c}
\text{1} & \text{999} & \text{1000} & \text{9999} \\
\text{999} & \text{9999} \\
- & \text{999} \\
\end{array}\]

\(\mathbf{e)}\) How many are divisible by 5 or 7?

Let \(D_5 = \) subset of these integers that are divisible by 5
\[D_7 = \]

\[|D_5| = \left\lfloor \frac{9999}{5} \right\rfloor - \left\lfloor \frac{999}{5} \right\rfloor \]

\[|D_7| = \left\lfloor \frac{9999}{7} \right\rfloor \text{ mult of 5} \]

\[\text{exclude the} \left\lfloor \frac{999}{7} \right\rfloor \text{ mult of 7} \]

\[= 1999 - 199 = 1800\]
\[ |0_7| = \left[ \frac{9999}{\frac{1}{7}} \right] - \left[ \frac{999}{\frac{1}{7}} \right] = 1428 - 142 = 1286 \]

**Inclusion-Exclusion:**

\[ |0_5 \cup 0_7| = |0_5| + |0_7| - |0_5 \cap 0_7| \]

\[ |0_{35}| = \left[ \frac{9999}{\frac{1}{5}} \right] - \left[ \frac{999}{\frac{1}{5}} \right] = 285 - 28 = 257 \]

\[ \text{lcm}(5, 7) = 257 \]

\[ = 1800 + 1286 - 257 = 2829 \]
Ex Find the # of 4-letter strings (A-Z) that have no two consecutive letters the same.

\[
\begin{array}{cccc}
26 & 25 & 25 & 25 \\
A-Z & \text{can't} & \text{can't} & \text{can't}
\end{array}
\]

= \boxed{406,250}
4.7: PIGEONHOLE PRINCIPLE (and "EXTREME SCENARIOS")

(If \( N \) pigeons fly into \( N-1 \) pigeonholes \( (N \geq 2) \),
\( \geq 2 \) pigeonhole has \( \geq 2 \) pigeons in it.)

\[
\begin{align*}
1 & \rightarrow 01 \\
\vdots & \rightarrow 02 \\
N & \rightarrow 0N-1
\end{align*}
\]

(BASIC) PIGEONHOLE PRINCIPLE

\( N \in \mathbb{Z}, N \geq 2. \)

If \( N \) or more balls are dropped into \( N-1 \) boxes, then there is at least one box with more than one ball.  

\( \geq 2 \)

Balls = pigeons  
Boxes = holes

Tricky: Identifying what are the objects/boxes.

What is the minimum number of students in a group required to ensure that at least two students in the group have the same grade? 6

Balls = students  
Boxes = grades

You've a bunch of rubber balls... Putting them in lockers...
Consider the "extreme scenario."

\[
\begin{array}{cccccc}
& & & \text{must join another ball} & \\
& \circ & & & \\
& A & B & C & D & F \end{array}
\]

\( \Rightarrow 5 \) isn't quite enough.

Read Exs 1-3: identify balls, boxes (concepts/objects)

Ex A class has 90 students.
Prof's plan: 10 will get As
20 B5
30 C5
20 D5
10 F5

What is the minimum group size required to ensure that \( > 4 \) students in the group have As?

"Extreme scenario" (Worst-case)

\[
\begin{array}{cccccc}
A & B & C & D & F \\
10 & 20 & 30 & 20 & 10
\end{array}
\]

There 30 could be "picked" first.
*any* 4 from the A box

[84]
(Generalized) Pigeonhole Principle

\[ N, x \in \mathbb{Z}^+ \]

If \( N \) or more balls are dropped into \( x \) boxes, then there is at least one box with \( \geq \left\lceil \frac{N}{x} \right\rceil \) balls.

Example:

\( N = 11 \) students (balls)
\( x = 5 \) boxes (grades)

\[ \left\lceil \frac{11}{5} \right\rceil = \left\lceil 2.2 \right\rceil = 3 \]

What does the principle say?

There is \( \geq 1 \) box (grade) with \( \geq 3 \) balls (students), i.e., At least 3 students must have the same grade.

"Extreme scenario"

\( 11 \) must join 2 other balls

\[ \begin{array}{ccccccc}
  A & B & C & D & E \\
  8 & 8 & 0 & 0 & 8 & 8
\end{array} \]
Reverse Ex.  
x = 5 boxes (grades)  
What is the minimum group size \(N\) required to ensure that \(\geq 3\) students in the group have the same grade?

Think: Extreme scenario  
\[
2 \cdot 5 + 1 = 11
\]

- \(h-1\)  \(x\)  
  \(h\) boxes  
  \(h = 3\)  
  "height"  
  \(h-1\{8/8/8\} 12x\)

Corollary  
x, h \in \mathbb{Z}^+  
x boxes  
The minimum \# of balls \(N\) required to ensure that \(\geq 1\) box has \(2h\) balls is  
\[
(h-1) x + 1
\]

# of complete layers in the extreme scenario

Have fun! Odds in back!
4.3: Permutations and Combinations

Permutations

Ex: How many ways are there to order n runners? (No ties). (n \in \mathbb{Z}^+)

\[
\begin{array}{c}
\frac{n}{1^{st}} \quad \frac{n-1}{2^{nd}} \quad \frac{n-2}{3^{rd}} \quad \cdots \quad \frac{1}{n^{th}} \\
\end{array}
\]

\[
n(n-1)(n-2)\cdots(2)(1) = n!
\]

"n factorial"

There are n! ways to order n distinct objects.

There are n! permutations of n distinct objects.

n!

0! = 1
1! = 1
2! = (2)(1) = 2
3! = (3)(2)(1) = 6 \quad 2 \times 3
4! = 24 \quad 2 \times 3 \times 4
5! = 120 \quad 2 \times 3 \times 4 \times 5

n! = n \times (n-1)!

n! grows rapidly

15! > 10^{12}
Ex # ways to order a 52-card deck.

$52! \approx 8 \times 10^{67}$ (astronomical!)

Not just "orders"!

Ex # ways to put n people in n rooms
(1 person per room)

Imagine that the rooms are numbered from 1 to n

\[
\begin{array}{cccc}
\text{Room 1} & \text{Room 2} & \text{Room 3} & \text{Room n} \\
\frac{n}{1} & \frac{n-1}{2} & \frac{n-2}{3} & \frac{1}{n}
\end{array}
\]

Ex Traveling Salesman Problem (TSP).
Find the most efficient route through n cities.

Important unsolved CS problem: find an efficient, optimal algorithm.

Brute-force: $\Theta(n!)$ - bad
There are excellent approximation algorithms.
\( r \)-Permutations of \( n \) distinct objects

\[ P(n,r) = \# \text{ of ways to choose and order a subset of } r \text{ elements from a set of } n \text{ elements.} \]

\[ = \frac{n!}{(n-r)!}, \quad \text{after cancelling} \]

\[ \text{Ex } n=7 \text{ Olympic runners in a race find # possible } 6/5/3 \text{ assignments. (No ties)} \]

\[ \frac{7 \times 6 \times 5}{6 \times 5} = 210 \]

\[ P(7,3) = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 7 \times 6 \times 5 = 210 \]

Idea:

\[ \begin{array}{ccc}
\hline
G & S & B \\
\hline
\end{array} \]

We don't care about the order among the 9 losers.

\[ \frac{7!}{4!} \leftarrow \# \text{ ways to order all 7} \]
**COMBINATIONS**

What if we don't care about the order among the winners, either?

**Ex** $n=7$ runners

Find # of possible combos of "medalists." (Note:)

\[
\begin{array}{c|c|c|c|c}
\text{Winners} & \text{Losers} \\
\hline
\frac{7}{1st} & \frac{6}{2nd} & \frac{5}{3rd} & \frac{4}{4th} & \frac{3}{5th} & \frac{2}{6th} & \frac{1}{7th} \\
\end{array}
\]

\[
= \frac{7!}{3!4!} = \frac{7 \times 6 \times 5 \times 4!}{3 \times 2 \times 1 \times 4!} = 35
\]

\[
C(n, r) = \binom{n}{r} \quad \text{"n choose r"}
\]

= # of r-subsets of an n-set

**Ex** There are 35 3-subsets of a 7-set.

In a set, order doesn't matter.

\[
= \frac{n!}{r!(n-r)!}
\]

**Proof** $P(n, r) = C(n, r) \times r! \quad \Leftrightarrow \quad n! = C(n, r) \times r!$

\[
\begin{align*}
\text{# ways to choose r}
& \quad \text{# ways to choose r subset}
\end{align*}
\]

and order r elements
from an n-set
Ex: How many possible 5-card hands are there?
(Order irrelevant in a "hand")

\[
\binom{52}{5} = \frac{52!}{5!47!} = \frac{52 \times 51 \times 50 \times 49 \times 48 \times 47}{5 \times 4 \times 3 \times 2 \times 1 \times 47!} = 2,598,960
\]

Symmetry Property

\[
\binom{n}{r} = \binom{n}{n-r}
\]

\[
\frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!}
\]

# ways = # ways
to choose to choose
r winners nr losers
from an n-set from an n-set

GOTO L-3-20 to 21 in M119 after my examples
HW Types

Ex 6 students are in a class.
Grades: A, B, C, D, F.
Find the # of possible grade reports in which at least one student has an A.

\[
\left( \text{# of unrestricted reports} \right) - \left( \text{# reports w/ 6 "A"s} \right)
\]

Students - J K L M N O

\[
\frac{5 \times 5 \times 5 \times 5 \times 5 \times 5}{J K L M N O} = \frac{4 \times 4 \times 4 \times 4 \times 4 \times 4}{J K L M N O}
\]

\[
5^6 - 4^6
\]

\[
11,529
\]

SAT

Ex 6 students.
Jack and Nick get the same grade.
Ken and Larry get different grades from each other.
How many possibles?

\[
\frac{5 \times 5 \times 4 \times 5 \times 1 \times 5}{J K L M N O} = 500
\]
Ex 6 students. Exactly 4 pass (A,B,C).
How many possibilities?

Start case-by-case analysis

Case 1: J,K,L,M pass

\[
\begin{array}{cccccc}
A & B & C & \text{or} & \text{or} & \text{or} \\
3 & 3 & 3 & 3 & 2 & 2 \\
J & K & L & M & N & O
\end{array}
\]

= 324

Shortcut:

How many combos of 4 students are there? \( \binom{6}{4} = \frac{6!}{4!2!} = \frac{6 \times 5}{2} = 15 \)

There are 15 cases, each w/324 possibs. "Symmetry"
15 x 324 = 4860

IMPORTANT: The 15 cases are disjoint!
Pascal's Triangle

Like a table of \( \binom{n}{k} \)'s (combinatorial coefficients).

To construct:
- Begin, end each row with a "1."
- Any other entry = sum of the two entries immediately above.

\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}, \quad n \geq 0, \quad k \geq 1
\]

Base: \( \binom{0}{0} = \binom{n}{n} = 1 \quad \forall n \in \mathbb{Z}^+\)

Row \#(n):

\[
\begin{array}{ccccccc}
0 & 1 & & & & & \\
1 & 1 & 1 & & & & \\
2 & 1 & 2 & 1 & & & \\
3 & 1 & 3 & 3 & 1 & & \\
4 & 1 & 4 & 6 & 4 & 1 & \\
\end{array}
\]

Observe:
\[
\binom{n}{0} = \binom{n}{n} = 1
\]

Symmetry

Overheads/Handouts - Chaos
Why does this give us \( \binom{3}{2} \)'s?

Imagine that the entries are pins in a pinboard. (Plinko in da Right, Pachinko)

drop balls

1st move (L or R)

2nd

3rd

Each entry indicates the # of ways a ball can drop from the top to its pin.

To get to 3, you must take 1R and 2Ls in any order.

\[
\text{# orderings/perms. of RLL} = \text{# ways to "choose" exactly 1R from a sequence of 3 moves} \\
= \frac{3!}{1!2!} = \binom{3}{1} = 3
\]

Namely, LLR, LRL, RLL

\[ \leq \leq \leq \]
So, our construction of the $\Delta$ is appropriate. Our physical model and combo interplay justify this.

Picture/Histograms

Revisit in 4-3, 5-6

In a science museum, drop a lot of balls → Assume $P(L) = P(R)$ at each move. Normal distr.
Another proof

\[
\binom{n+1}{r} = \text{# bi+ strings of length } n+1 \text{ with exactly } r \text{ "1"s}
\]

Case 1: \( \binom{n}{r-1} \) of this type

\( (r-1) \) ways to place \( r-1 \) "1"s.

Case 2: \( \binom{n}{r} \) of this type

\( (r) \) ways to place \( r \) "1"s.

\[
\therefore \binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}
\]

\( \binom{n}{r} = \text{# r-subsets of an n-set} \)

Row sum of Pascal's Δ (-row n) = 2^n

\( \binom{0}{n} = \text{# 0-subsets of an n-set} = 0 \) (empty)

\( \binom{n}{n} = 1 \)

\[
\sum_{r=0}^{n} \binom{n}{r} = \text{total #subsets of an n-set} = 2^n - \left| P(\emptyset) \right|
\]
Binomial Theorem

Pascal's $\Delta$:

\[
\begin{array}{cccccc}
\text{n} & 0 & 1 & 2 & 3 & 4 \\
\hline
\text{x+y=0} & 1 & 1 & 1 & 1 & 1 \\
\text{1} & 1 & 2 & 3 & 4 & 5 \\
\text{2} & 1 & 2 & 3 & 4 & 5 \\
\text{3} & 1 & 3 & 6 & 10 & 15 \\
\text{4} & 1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

\[(x+y)^0 = 1 \quad (x+y)^1 = x + y \quad (x+y)^2 = x^2 + 2xy + y^2 \]
\[(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \]

Why does this work?

\[(x+y)^2 = (x+y)(x+y) = xx + xy + yx + yy \]

Each term "chooses" sum of all possible x or y from each "choice-products" factor.

\[
(x+y)^2 = \begin{array}{c}
\frac{\binom{2}{0}}{2} x^2 + \frac{\binom{2}{1}}{2} xy + \frac{\binom{2}{2}}{2} y^2 \\
\frac{\binom{2}{0}}{2} x^2 + \frac{\binom{2}{1}}{2} xy + \frac{\binom{2}{2}}{2} y^2 \\
\end{array}
\]

\[(x+y)^4 = \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4 \]

Can evaluate \(\binom{n}{k}\) from Pascal's $\Delta$ or \[\binom{n}{k} = \frac{n!}{k!(n-k)!}\].

\[(x+y)(x+y)(x+y)(x+y)\]

There are \(\binom{4}{2} = 6\) ways of choosing 2 "x"'s and 2 "y"'s.
Recall Pascal's Identity:

\[
\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}, \quad n, r \in \mathbb{Z}^+ \\
\]

Also:

\[
\binom{n}{0} = \binom{n}{n} = 1, \quad \forall n \in \mathbb{Z}^+ \\
\text{(base cases).}
\]

A recurrence relation - terms depend on previous terms

\[
\begin{align*}
\text{Add} & \quad \downarrow \\
\text{New term} & \quad \uparrow \\
\intertext{A proof}
\binom{n+1}{r} &= \# \text{ bit strings of length } n+1 \\
&\quad \text{with exactly } r \text{ '1's.}
\end{align*}
\]

\text{Case 1:} \quad 1 \\
\quad \begin{array}{c}
\text{There are } \binom{n}{r-1} \text{ ways to place } r-1 \text{ '1's.}
\end{array}

\text{Case 2:} \quad 0 \\
\quad \begin{array}{c}
\binom{n}{r} \text{ ways to place } r \text{ '1's.}
\end{array}

\therefore \binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}