

4.1: BASICSSUM RULE

If  $A_1, A_2, \dots, A_m$  are disjoint (non-overlapping) finite sets, then

$$|A_1 \cup A_2 \cup \dots \cup A_m| = |A_1| + |A_2| + \dots + |A_m|$$

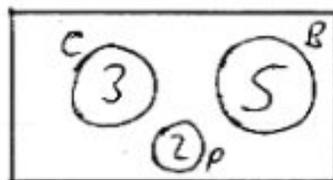
Ex A high school's science dept. has  
 3 chem teachers  $\leftarrow$  "C"  
 5 bio teachers  $\leftarrow$  "B"  
 2 physics teachers  $\leftarrow$  "P"

(a) Assume each teacher does only one subject.

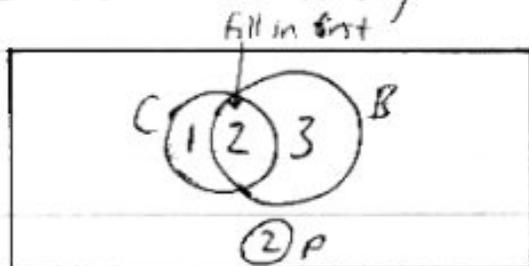
How many science teachers are there?

$$\begin{aligned} & |C \cup B \cup P| \\ &= |C| + |B| + |P| \\ &= 3 + 5 + 2 \\ &= \textcircled{10} \end{aligned}$$

There are 10 choices for a dept. chair.



⑥ Assume there are two joint chem/bio teachers.



$$\begin{aligned} |C| &= 3 \\ |B| &= 5 \\ |P| &= 2 \end{aligned}$$

How many science teachers are there?

$$1 + 2 + 3 + 2 = 8$$

Another approach

Inclusion-Exclusion Principle (Sec 5.5)

$$|C \cup B| = |C| + |B| - |C \cap B|$$



to correct  
for double-counting

$$\begin{aligned} &= 3 + 5 - 2 \\ &= 6 \end{aligned}$$

$$\begin{aligned} \text{So, } |C \cup B \cup P| &= |C \cup B| + |P| \\ &= 6 + 2 \\ &= 8 \end{aligned}$$

PRODUCT RULE

Consider a sequence of  $m$  "decisions."  
 There are  $n_1$  "choices" for the 1<sup>st</sup> decision.  
 There are  $n_2$  "choices" for the 2<sup>nd</sup> decision,  
 regardless of the 1<sup>st</sup> decision.

⋮  
 There are  $n_m$  "choices" for the  $m^{\text{th}}$  decision,  
 regardless of the 1<sup>st</sup>  $(m-1)$  decisions.

Then, there are  $n_1 n_2 \cdots n_m$  possible sequences.

Special Case

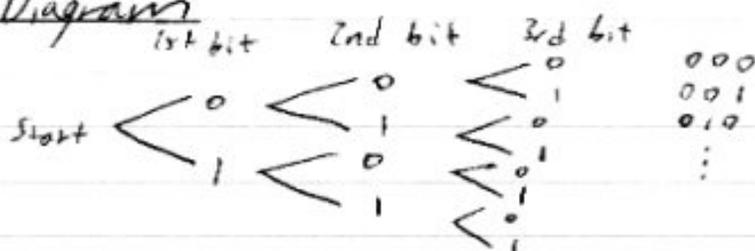
$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| |A_2| \cdots |A_m|$$

fixed, finite sets

Ex A "bit" can be a "0" or a "1".  
 How many "bit strings" of length 3 are there?

$$\text{Here, } A_1 = A_2 = A_3 = \{0, 1\}$$

$$\underbrace{2}_{0 \text{ or } 1} \times \underbrace{2}_{0 \text{ or } 1} \times \underbrace{2}_{0 \text{ or } 1} = 2^3 = 8$$

Tree Diagram

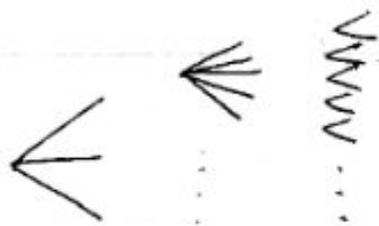
(old)

Ex 3 chem teachers  
5 bio teachers  
2 physics teachers  $\rightarrow$  no overlaps

A science committee consists of exactly one teacher from each science dept.  
How many possible committees are there?

$$|C \times B \times P| = |C| |B| |P| \\ = 3 \times 5 \times 2 \\ = \textcircled{30}$$

$$\frac{3}{\text{Chem}} \times \frac{5}{\text{Bio}} \times \frac{2}{\text{Phys}} = \textcircled{30}$$



Ex You need a five character password made up of uppercase letters and/or digits 0-9. It must begin with a "6", and it must end in a vowel. How many choices are there?

$$\frac{1}{\text{"6"} \text{ (forced)}} \times \frac{36}{\text{A-Z, 0-9}} \times \frac{36}{\text{A-Z, 0-9}} \times \frac{36}{\text{A-Z, 0-9}} \times \frac{5}{\text{A, E, I, O, U}} = 233,280$$



SPLITTING INTO CASES

We can combine Sum, Product Rules.

Password Ex

★ Up to 5 chars. long

A-Z or 0-9

Begins with "6."

Ends in a vowel.

No character can be repeated.

Let  $l$  = length of password

$l$  can't be 1, because "6" isn't a vowel.

Case 1:  $l=2$

$$\frac{1}{\text{"6"}} \times \frac{5}{\text{vowel}} = \textcircled{5}$$

Case 2:  $l=3$

$$\frac{1}{\text{"6"}} \times \frac{34}{\text{vowel}} \times \frac{5}{\text{vowel}} = \textcircled{170}$$

Case 3:  $l=4$

$$\frac{1}{\text{"6"}} \times \frac{34}{\text{vowel}} \times \frac{33}{\text{vowel}} \times \frac{5}{\text{vowel}} = \textcircled{5610}$$

Case 4:  $l=5$  From before,  $\textcircled{179,520}$

Total:  $\textcircled{185,305}$

## "AT LEAST ONE"

Ex A password consists of 5 characters  
(uppercase letters <sup>and</sup> or digits 0-9).  
At least one character must be a digit.

Total # of 5-character strings (A-Z, 0-9)

$$\underbrace{36}_{\substack{A-Z, \\ 0-9}} \times \underbrace{36} \times \underbrace{36} \times \underbrace{36} \times \underbrace{36} = 36^5$$

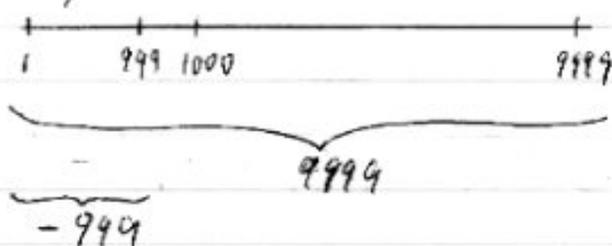
# of 5-character strings with no digits.

$$\underbrace{26}_{A-Z} \times \underbrace{26} \times \underbrace{26} \times \underbrace{26} \times \underbrace{26} = 26^5$$

$$\text{Answer: } 36^5 - 26^5 \quad 48584800$$

#20 Consider the positive integers <sup>can't start w/0</sup> [with exactly 4 decimal digits]  
(i.e., between 1000 and 9999 inclusive).

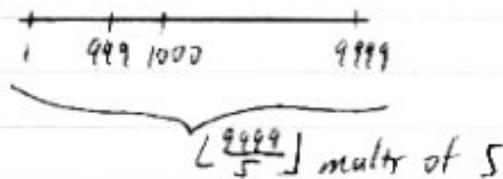
How many are there? 9000



e) How many are divisible by 5 or 7?

Let  $D_5 =$  subset of these integers that are divisible by 5  
 $D_7 =$

$$|D_5| = \left\lfloor \frac{9999}{5} \right\rfloor - \left\lfloor \frac{999}{5} \right\rfloor$$



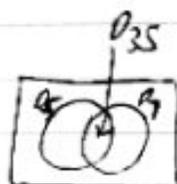
exclude the  
 $\left\lfloor \frac{999}{5} \right\rfloor$  mults of 5  
here

$$= 1999 - 199$$

$$= \underline{1800}$$

$$\begin{aligned}
 |D_7| &= \left\lfloor \frac{9999}{7} \right\rfloor - \left\lfloor \frac{999}{7} \right\rfloor \\
 &= 1428 - 142 \\
 &= \boxed{1286}
 \end{aligned}$$

Inclusion-Exclusion:



$$|D_5 \cup D_7| = |D_5| + |D_7| - \underbrace{|D_5 \cap D_7|}$$

What integers have both 5 and 7 in their prime factor?

$$\begin{aligned}
 |D_{35}| &= \left\lfloor \frac{9999}{35} \right\rfloor - \left\lfloor \frac{999}{35} \right\rfloor \\
 &= 285 - 28 \\
 \text{rem}(5, 7) &= 257
 \end{aligned}$$

$$= 1800 + 1286 - 257$$

$$= \boxed{2829}$$

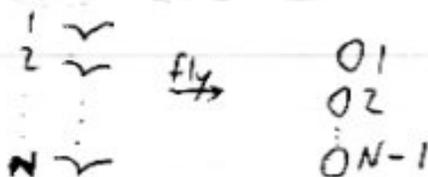
Ex Find the # of 4-letter strings (A-Z)  
that have no two consecutive letters the same.

$$\begin{array}{cccc} \underline{26} & \underline{25} & \underline{25} & \underline{25} \\ \text{A-Z} & \text{can't} & \text{can't} & \text{can't} \end{array} = (406,250)$$

↖   ↖   ↖

## 4.7: PIGEONHOLE PRINCIPLE (and "EXTREME SCENARIOS")

(If  $N$  pigeons fly into  $N-1$  pigeonholes ( $N \geq 2$ ),  
 $\geq 1$  pigeonhole has  $\geq 2$  pigeons in it.)



### (BASIC) PIGEONHOLE PRINCIPLE

$N \in \mathbb{Z}, N \geq 2$ .

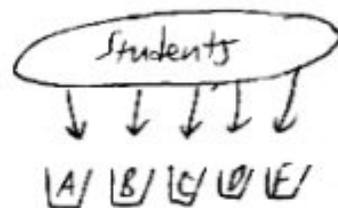
If  $N$  or more balls are dropped into  
 $N-1$  boxes, then there is at least one  
box with more than one ball.  
 (22)

Balls = pigeons  
 Boxes = holes

Ex Grades: A, B, C, D, F in Math 245.

What is the minimum number of students  
 in a group required to ensure that  
 at least two students in the group  
 have the same grade? (6)

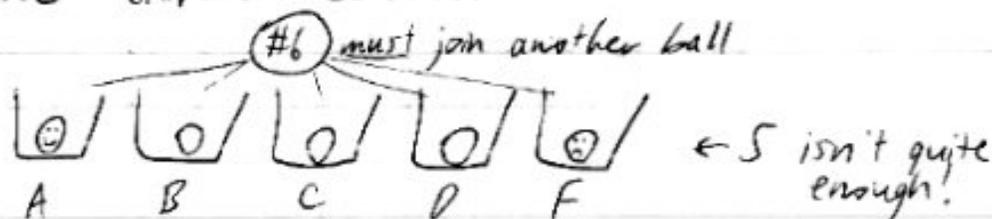
Balls = students  
 Boxes = grades



Tricky:  
 identifying  
 what are the  
 objects/boxes.

You've a bunch  
 of rubber balls...  
 - Putting them  
 in trash cans.

Consider the "extreme scenario."



Read Exs 1-3: identify balls, boxes (concepts/objects)

Ex A class has 90 students.

Prof's plan:

|    |          |     |
|----|----------|-----|
| 10 | will get | A's |
| 20 |          | B's |
| 30 |          | C's |
| 20 |          | D's |
| 10 |          | F's |

Repeat previous?

What is the minimum group size required to ensure that  $\geq 4$  students in the group have A's?

"Extreme scenario" (Worst-case)

|    |    |    |    |    |
|----|----|----|----|----|
| A  | B  | C  | D  | F  |
| 10 | 20 | 30 | 20 | 10 |

There 80 could be "picked" first.  
→ + any 4 from the A box

(84)

(GENERALIZED) PIGEONHOLE PRINCIPLE

$N, x \in \mathbb{Z}^+$

If  $N$  or more balls are dropped into  $x$  boxes, then there is at least one box with  $\geq \lceil \frac{N}{x} \rceil$  balls.

$x = \# \text{ boxes}$

Ex  $N = 11$  students (balls)  
 $x = 5$  boxes (grades)

$\lceil \frac{N}{x} \rceil = \lceil \frac{11}{5} \rceil = 3$

What does the principle say?

There is  $\geq 1$  box (grade)  
with  $\geq 3$  balls (students),  
i.e., At least 3 students must have  
the same grade.

"Extreme scenario"

(#11) must join 2 other balls



## Reverse Ex

$x=5$  boxes (grades)

What is the minimum group size ( $N$ ) required to ensure that  $\geq 3$  students in the group have the same grade?

Think: Extreme scenario

$$2 \cdot 5 + 1 = 11$$

$\uparrow$                      $\uparrow$   
 $h-1$                  $x$   
where                = # boxes  
 $h=3$   
"height"

$$h-1 \left\lfloor \frac{8}{1} \right\rfloor \left\lfloor \frac{8}{2} \right\rfloor \dots \left\lfloor \frac{8}{x} \right\rfloor$$

## Corollary

$$x, h \in \mathbb{Z}^+$$

$x$  boxes

The minimum # of balls ( $N$ ) required to ensure that  $\geq 1$  box has  $\geq h$  balls is

$$(h-1)x + 1$$

# of complete layers in the extreme scenario

Have fun! Odds in back!

## 4.3: PERMUTATIONS AND COMBINATIONS

### PERMUTATIONS

Ex How many ways are there to order  $n$  runners? (No ties). ( $n \in \mathbb{Z}^+$ )

$$\frac{n}{1^{\text{st}}} \frac{n-1}{2^{\text{nd}}} \frac{n-2}{3^{\text{rd}}} \dots \frac{1}{n^{\text{th}}}$$

$$n(n-1)(n-2) \dots (2)(1) = n!$$

"n factorial"

There are  $n!$  ways to order  $n$  distinct objects.  
There are  $n!$  permutations of  $n$  distinct objects.

$n!$

$$\begin{aligned} 0! &= 1 \\ 1! &= 1 \\ 2! &= (2)(1) = 2 \\ 3! &= (3)(2)(1) = 6 \\ 4! &= 24 \\ 5! &= 120 \end{aligned} \quad \begin{array}{l} \\ \\ \} \times 4 \\ \} \times 5 \end{array}$$

$$\vdots$$

$$n! = n \times (n-1)!$$

$n!$  grows rapidly

$$\vdots$$

$$15! > 10^{12}$$

Ex # ways to order a 52-card deck.

$$52! \approx 8 \times 10^{67} \text{ (astronomical!)}$$

Not just "orders"!

Ex # ways to put  $n$  people in  $n$  rooms  
(1 person per room)

Imagine that the rooms are #ed from 1 to  $n$ .

For nuclear  
next to  
laundry room.

$$\frac{n}{\text{Room 1}} \quad \frac{n-1}{\text{Room 2}} \quad \frac{n-2}{\text{Room 3}} \quad \dots \quad \frac{1}{\text{Room } n}$$

$$(n!)$$

Ex Traveling Salesman Problem (TSP).

Find the most efficient route through  $n$  cities.

Important unsolved CS problem: find an  
efficient, optimal algorithm.

brute-force:  $\Theta(n!)$  - bad

There are excellent approximation algorithms.

$\Theta(n!)$

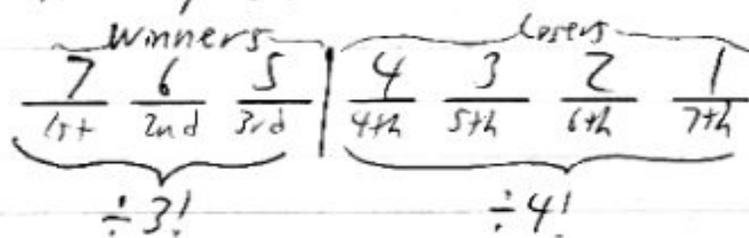


## COMBINATIONS

What if we don't care about the order among the winners, either?

Ex  $n=7$  runners

Find # of possible combos of "medalists." (No ties)



What do you think the formula is?

$$\frac{7!}{3!4!} = \frac{7 \times 6 \times 5 \times 4!}{3 \times 2 \times 1 \times 4!} = \textcircled{35}$$

$$C(n, r) = \binom{n}{r} \text{ "n choose r"}$$

= # of  $r$ -subsets of an  $n$ -set

Ex There are 35 3-subsets of a 7-set.  
In a set, order doesn't matter.

$$= \frac{n!}{r!(n-r)!}$$

Proof  $P(n, r) = C(n, r) \times r! \Leftrightarrow \frac{n!}{(n-r)!} = \binom{n}{r} r!$

|                                                                                |                                              |                                        |
|--------------------------------------------------------------------------------|----------------------------------------------|----------------------------------------|
| # ways<br>to <u>choose</u><br>and <u>order</u><br>$r$ elts<br>from an $n$ -set | # ways<br>to <u>choose</u><br>an $r$ -subset | # ways<br>to <u>order</u><br>$r$ elts. |
|--------------------------------------------------------------------------------|----------------------------------------------|----------------------------------------|

Ex How many possible 5-card hands are there?  
(Order irrelevant in a "hand")

$$\binom{52}{5} = \frac{52!}{5!47!} = \frac{52 \times 51 \times 50 \times 49 \times 48 \times 47!}{5 \times 4 \times 3 \times 2 \times 1 \times 47!}$$

$$= \boxed{2,598,960}$$

Symmetry Property

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!}$$

# ways to choose  $r$  winners from an  $n$ -set = # ways to choose  $n-r$  losers from an  $n$ -set

GOTO L-3-20 to 21 in M119 after my examples



Ex 6 students.  
Exactly 4 pass (A, B, C).  
How many possibilities?

Start case-by-case analysis

Case 1: J, K, L, M pass

$$\begin{array}{cccccc} \text{A, B, C} & & & & \text{D, E} & \text{D, F} \\ \frac{3}{J} & \frac{3}{K} & \frac{3}{L} & \frac{3}{M} & \frac{2}{N} & \frac{2}{O} \end{array} = 324$$

Shortcut!

How many combos of 4 students are there?  $\binom{6}{4} = \frac{6!}{4!2!} = \frac{6 \times 5}{2} = 15$

There are 15 cases, each w/ 324 possibi. "Symmetry"

$15 \times 324 = 4860$

IMPORTANT: The 15 cases are disjoint!

Pascal's Triangle

Like a table of " $\binom{n}{r}$ "s (binomial combinatorial coefficients).

To construct:

Begin, end each row with a "1."

Any other entry = sum of the two entries

immediately above.

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r} \quad \begin{matrix} n, r \in \mathbb{Z}^+ \\ n \geq r \end{matrix}$$

Base:  $\binom{n}{0} = \binom{n}{n} = 1 \quad \forall n \in \mathbb{Z}^+$

Row # (n)

|   |   |   |   |   |                                                                   |
|---|---|---|---|---|-------------------------------------------------------------------|
| 0 | 1 |   |   |   | $\leftarrow \binom{0}{0} = 1$                                     |
| 1 | 1 | 1 |   |   | $\leftarrow \binom{1}{0} = 1, \binom{1}{1} = 1$                   |
| 2 | 1 | 2 | 1 |   | $\leftarrow \binom{2}{0} = 1, \binom{2}{1} = 2, \binom{2}{2} = 1$ |
| 3 | 1 | 3 | 3 | 1 |                                                                   |
| 4 | 1 | 4 | 6 | 4 | 1                                                                 |

Observe:

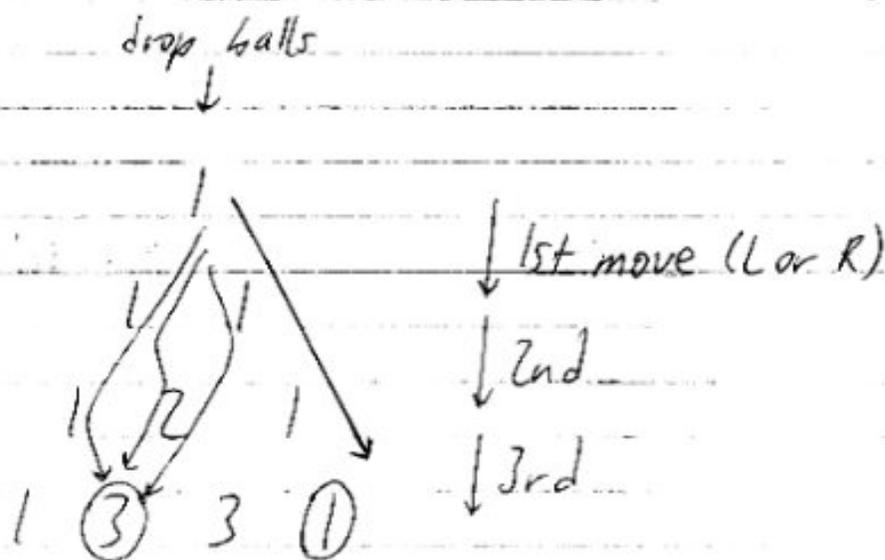
$$\binom{n}{0} = \binom{n}{n} = 1$$

Symmetry  $\frac{1}{2}$

Overheads/Handouts - Chaos

Why does this give us " $\binom{n}{r}$ "s?

Imagine that the entries are pins in a pinboard.  
(Plinko in \$ is right, Pachinko)

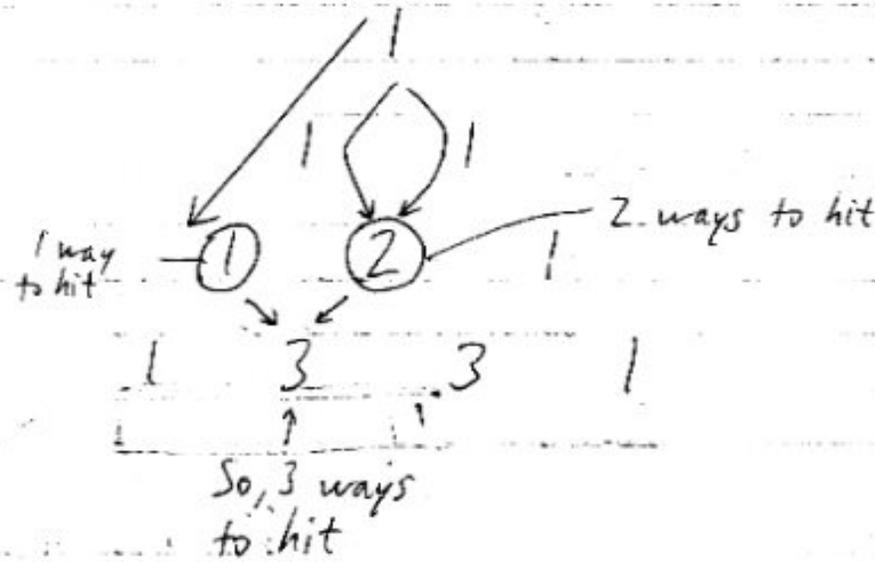


Each entry indicates the # of ways a ball can drop from the top to its pin.

To get to ③, you must take 1R and 2Ls in any order.

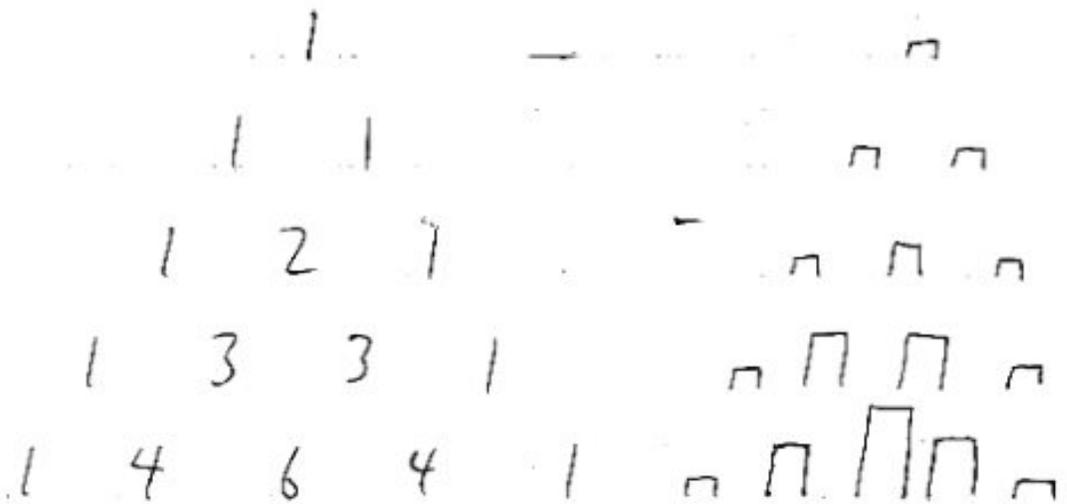
$$\begin{aligned} & \# \text{ orderings/perms. of } RLL \\ &= \# \text{ ways to "choose" exactly 1 R from a sequence of 3 mov} \\ &= \frac{3!}{1!2!} = \binom{3}{1} = 3 \end{aligned}$$

Namely, LLR, LRL, RLL

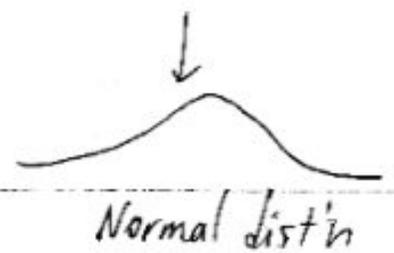


So, our construction of the  $\Delta$  is appropriate.  
Our physical model and combo. interp. justify this.

Picture/Histograms



In a science museum  
Drop a lot of balls  $\rightarrow$   
Assume  $P(L)=P(R)$  at each move.



M245  $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

Another proof

$\binom{n+1}{r} = \#$  bit strings of length  $n+1$  with exactly  $r$  "1"s

Case 1  $\boxed{1} \overbrace{\hspace{10em}}^{n+1} \leftarrow \binom{n}{r-1}$  of this type  
 $\underbrace{\hspace{10em}}_{(r-1)}$  ways to place  $r-1$  "1"s

Case 2  $\boxed{0} \overbrace{\hspace{10em}} \leftarrow \binom{n}{r}$  of this type  
 $\underbrace{\hspace{10em}}_{(r)}$  ways to place  $r$  "1"s

$\therefore \binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

$\binom{n}{r} = \#$   $r$ -subsets of an  $n$ -set

Row Sum of Pascal's  $\Delta$  (Row  $n$ )  $= 2^n$

$\binom{n}{0} = \#$  0-subsets of an  $n$ -set  $= 1$  ( $\emptyset$ )  
 $\binom{n}{1} = 1 = n$

$\sum_{r=0}^n \binom{n}{r} = \text{total \#subsets of an } n\text{-set} = 2^n \leftarrow |P(S)|$   
 $\uparrow$   
some  $n$ -set

# Binomial Theorem

Pascal's  $\Delta$ :

|         |     |   |   |   |   |                             |                                       |
|---------|-----|---|---|---|---|-----------------------------|---------------------------------------|
| $x+y=0$ | $n$ |   |   |   |   | $(x+y)^0 = 1$               |                                       |
|         | 0   |   | 1 |   |   |                             |                                       |
|         | 1   |   | 1 | 1 |   | $(x+y)^1 = x + y$           |                                       |
|         | 2   |   | 1 | 2 | 1 | $(x+y)^2 = x^2 + 2xy + y^2$ |                                       |
|         | 3   |   | 1 | 3 | 3 | 1                           | $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ |
|         | 4   | 1 | 4 | 6 | 4 | 1                           |                                       |

Why does this work?

$$(x+y)^2 = (x+y)(x+y) = \underbrace{xx + xy + yx + yy}$$

Each term "chooses" x or y from each factor. sum of all possible "choice-products"

$$= 1x^2 + 2xy + 1y^2$$

$$= \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2$$

# ways to choose 0 "y"s      # ways to choose 1 "y"      # ways to choose 2 "y"s

$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

Can evaluate  $\binom{n}{r}$  from Pascal's  $\Delta$  or  $\frac{n!}{r!(n-r)!}$ .

$$(x+y)(x+y)(x+y)(x+y)$$

There are  $\binom{4}{2} = 6$  ways of choosing 2 "x"s and 2 "y"s.

Recall Pascal's Identity:

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r} \quad \begin{array}{l} n, r \in \mathbb{Z}^+ \\ n \geq r \end{array}$$

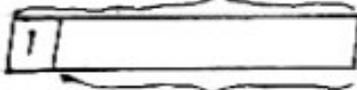
Also:  $\binom{n}{0} = \binom{n}{n} = 1 \quad \forall n \in \mathbb{Z}^{\geq 0}$   
(Base cases).

(two variable)  
a recurrence relation ~ terms depend  
on previous terms

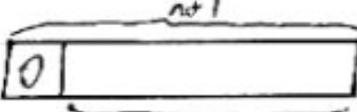
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New  
Term

A proof

$\binom{n+1}{r} = \#$  bit strings of length  $n+1$   
with exactly  $r$  "1"s.

Case 1   $\leftarrow \binom{n}{r-1}$  of this type

There are  $\binom{n}{r-1}$  ways  
to place  $r-1$  "1"s.

Case 2   $\leftarrow \binom{n}{r}$  of this type

$\binom{n}{r}$  ways to place  
 $r$  "1"s.

$$\therefore \binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$