4.1: BASICS

SUM RULE

If $A_1, A_2, \ldots, A_m$ are disjoint (non-overlapping) finite sets, then

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|$$

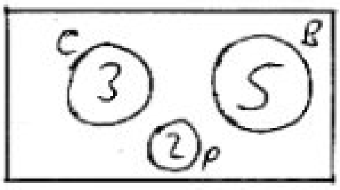
Ex A high school's science dept. has
3 chem teachers ← "C"
5 bio teachers ← "B"
2 physics teachers ← "P"

ⓐ Assume each teacher does only one subject.

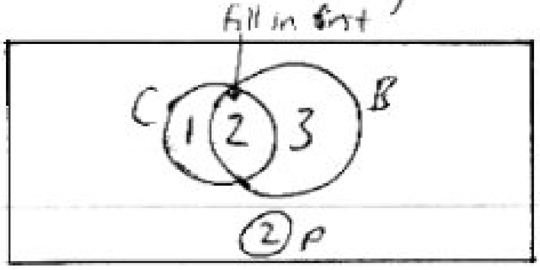
How many science teachers are there?

$$\begin{aligned} &|C \cup B \cup P| \\ &= |C| + |B| + |P| \\ &= 3 + 5 + 2 \\ &= 10 \end{aligned}$$

There are 10 choices for a dept. chair.



(b) Assume there are two joint chem/bio teachers.



$|C| = 3$
$|B| = 5$
$|P| = 2$

How many science teachers are there?

$1 + 2 + 3 + 2 = 8$

<u>Another approach</u>

<u>Inclusion-Exclusion Principle (Sec 5.5)</u>

$$|C \cup B| = |C| + |B| - |C \cap B|$$

to correct for double-counting

$$= 3 + 5 - 2$$
$$= 6$$

So, $|C \cup B \cup P| = |C \cup B| + |P|$
$= 6 + 2$
$= 8$

PRODUCT RULE

Consider a sequence of m "decisions."
There are n_1 "choices" for the 1^{st} decision.
There are n_2 "choices" for the 2^{nd} decision, regardless of the 1^{st} decision.

$\vdots$

There are n_m "choices" for the m^{th} decision, regardless of the 1^{st} $(m-1)$ decisions.

Then, there are $n_1 n_2 \cdots n_m$ possible sequences.

Special Case

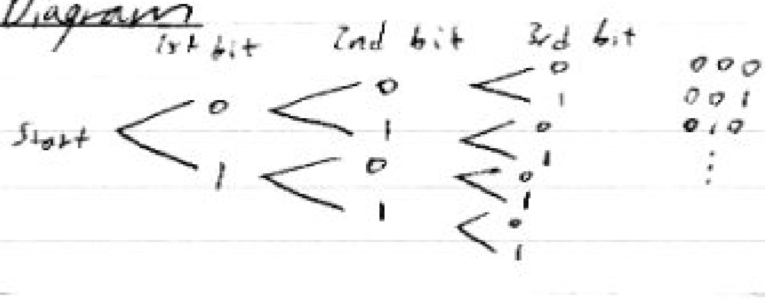
$$|\underbrace{A_1 \times A_2 \times \ldots \times A_m}_{\text{fixed, finite sets}}| = |A_1||A_2| \cdots |A_m|$$

Ex A "bit" can be a "0" or a "1".
How many "bit strings" of length 3 are there?

Here, $A_1 = A_2 = A_3 = \{0, 1\}$

$$\underset{0 \text{ or } 1}{\underline{2}} \times \underset{0 \text{ or } 1}{\underline{2}} \times \underset{0 \text{ or } 1}{\underline{2}} = 2^3 = 8$$

Tree Diagram



(old)

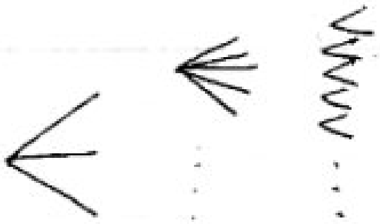
Ex 3 chem teachers
5 bio teachers
2 physics teachers

} no overlaps

A science committee consists of exactly one teacher from each science dept.
How many possible committees are there?

$$|C \times B \times P| = |C||B||P| = 3 \times 5 \times 2 = 30$$

$$\underset{\text{Chem}}{\underline{3}} \times \underset{\text{Bio}}{\underline{5}} \times \underset{\text{Phys}}{\underline{2}} = 30$$



Ex You need a five character password made up of uppercase letters and/or digits 0-9.
It must begin with a "6", and it must end in a vowel.
How many choices are there?

$$\underset{\substack{\text{"6"}\\ \text{(forced)}}}{\underline{1}} \times \underset{\substack{\text{A-Z,}\\ \text{0-9}}}{\underline{36}} \times \underset{\substack{\text{A-Z,}\\ \text{0-9}}}{\underline{36}} \times \underset{\substack{\text{A-Z,}\\ \text{0-9}}}{\underline{36}} \times \underset{\text{A,E,I,O,or U}}{\underline{5}} = 233{,}280$$

In general, the "A_i"s need not be fixed!

Password Ex

Exactly 5 chars. long
A-Z or 0-9
Begins with "6".
Ends in a vowel.
☆ No character can be repeated.

arbitrary password string

$\underline{a_1}\ \underline{a_2}\ \underline{a_3}\ \underline{a_4}\ \underline{a_5}$

this much is the same as before

$\underset{\text{"6"}}{\underline{1}}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underset{\text{A, E, I, O, or U}}{\underline{5}}$

$$\underline{1} \times \underset{\substack{\text{A-Z,}\\ \text{0-9,}\\ \text{except}\\ \text{"6", } a_5}}{\underline{34}} \times \underset{\substack{\text{except}\\ \text{"6", } a_5,\\ a_2}}{\underline{33}} \times \underset{\substack{\text{except}\\ \text{"6", } a_5,\\ a_2,\ a_3}}{\underline{32}} \times \underline{5} = 179{,}520$$

($\approx 77\%$ of total # where repetitions were allowed)

SPLITTING INTO CASES

We can combine Sum, Product Rules.

Password Ex

☆ Up to 5 chars. long
A-Z or 0-9
Begins with "6."
Ends in a vowel.
No character can be repeated.

Let ℓ = length of password

ℓ can't be 1, because "6" isn't a vowel.

Case 1: $\ell = 2$

$$\underset{\text{"6"}}{\underline{1}} \times \underset{\text{vowel}}{\underline{5}} = 5$$

Case 2: $\ell = 3$

$$\underset{\text{"6"}}{\underline{1}} \times \underline{34} \times \underset{\text{vowel}}{\underline{5}} = 170$$

Case 3: $\ell = 4$

$$\underset{\text{"6"}}{\underline{1}} \times \underline{34} \times \underline{33} \times \underset{\text{vowel}}{\underline{5}} = 5610$$

Case 4: $\ell = 5$ from before, 179,520

Total: 185,305

"AT LEAST ONE"

Ex A password consists of 5 characters (uppercase letters and/or digits 0–9).
At least one character must be a digit.

Total # of 5-character strings (A-Z, 0-9)

$$\underset{A-Z,\ 0-9}{\underline{36}} \times \underline{36} \times \underline{36} \times \underline{36} \times \underline{36} = 36^5$$

# of 5-character strings with no digits.

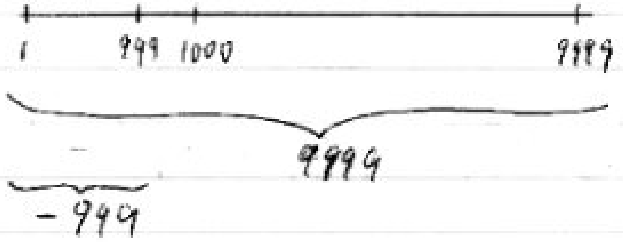
$$\underset{A-Z}{\underline{26}} \times \underline{26} \times \underline{26} \times \underline{26} \times \underline{26} = 26^5$$

Answer: $36^5 - 26^5$ 48,584,800

can't start w/ 0

#20 Consider the positive integers [w/ exactly 4 decimal digits] (i.e., between 1000 and 9999 inclusive).

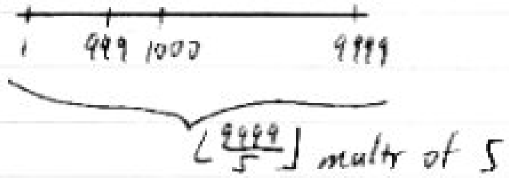
How many are there? 9000



e) How many are divisible by 5 or 7?

Let D_5 = subset of these integers that are divisible by 5
D_7 = ⟃ 7

$$|D_5| = \left\lfloor \frac{9999}{5} \right\rfloor - \left\lfloor \frac{999}{5} \right\rfloor$$



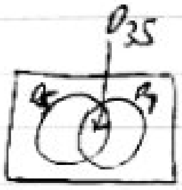
exclude the $\lfloor \frac{999}{5} \rfloor$ mults of 5 here

$= 1999 - 199$

$= 1800$

$$|D_7| = \left\lfloor \frac{9999}{7} \right\rfloor - \left\lfloor \frac{999}{7} \right\rfloor$$
$$= 1428 - 142$$
$$= 1286$$

Inclusion-Exclusion:



What integers have both 5 and 7 in their prime factors?

$$|D_5 \cup D_7| = |D_5| + |D_7| - \underbrace{|D_5 \cap D_7|}$$

$$|D_{35}| = \left\lfloor \frac{9999}{35} \right\rfloor - \left\lfloor \frac{999}{35} \right\rfloor$$
$$= 285 - 28$$
$$\mathrm{lcm}(5,7) = 257$$

$$= 1800 + 1286 - 257$$

$$= 2829$$

<u>Ex</u> Find the # of 4-letter strings (A-Z) that have no two consecutive letters the same.

$$\underset{\text{A-Z}}{\underline{26}}\ \underset{\text{can't}}{\underline{25}}\ \underset{\text{can't}}{\underline{25}}\ \underset{\text{can't}}{\underline{25}} = 406{,}250$$