4.2 : PIGEONHOLE PRINCIPLE (and "EXTREME SCENARIOS")

(If N pigeons fly into $N-1$ pigeonholes ($N \geq 2$), ≥ 1 pigeonhole has ≥ 2 pigeons in it.)

(BASIC) PIGEONHOLE PRINCIPLE

$N \in \mathbb{Z}$, $N \geq 2$.
If N or more balls are dropped into $N-1$ boxes, then there is at least one box with more than one (≥ 2) ball.

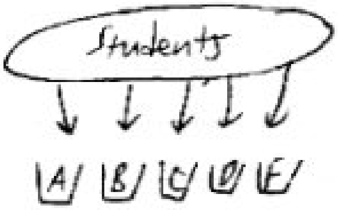
Tricky: identifying what are the objects/boxes

Balls = pigeons
Boxes = holes

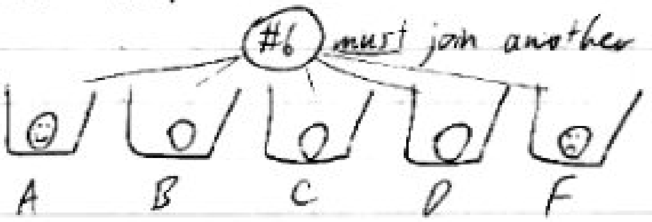
Ex Grades: A, B, C, D, F in Math 245.
What is the minimum number of students in a group required to ensure that at least two students in the group have the same grade? 6

You're a bunch of rubber balls...
– Putting you in trash cans.

Balls = students
Boxes = grades



Consider the "extreme scenario."



← 5 isn't quite enough!

<u>Read Exs 1-3</u>: identify balls, boxes (concepts/objects)

<u>Ex</u> A class has 90 students.
Prof's plan: 10 will get As
20 Bs
30 Cs
20 Ds
10 Fs

Repeat pigeons?

What is the minimum group size required to ensure that ≥ 4 students in the group have As?

"Extreme scenario" (Worst-case)

$$\begin{array}{ccccc} A & B & C & D & F \\ 10 & \underbrace{20 \quad 30 \quad 20 \quad 10} \end{array}$$

These 80 could be "picked" first.

+ any 4 from the A box

(GENERALIZED) PIGEONHOLE PRINCIPLE

$N, x \in \mathbb{Z}^+$

If N or more balls are dropped into x boxes, then there is at least one box with $\geq \lceil \frac{N}{x} \rceil$ balls.

x = # boxes

<u>Ex</u> $N = 11$ students (balls)
$x = 5$ boxes (grades)

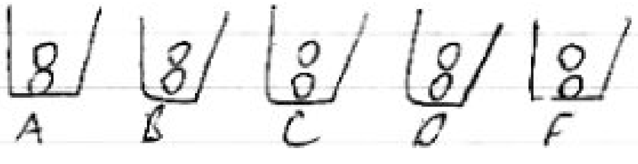
$\lceil \frac{N}{x} \rceil = \lceil \frac{11}{5} \rceil = 3$

What does the principle say?

There is ≥ 1 box (grade) with ≥ 3 balls (students),
i.e., At least 3 students must have the same grade.

"Extreme scenario"

#11 must join 2 other balls



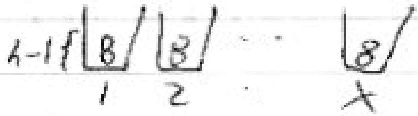
Reverse Ex

$x = 5$ boxes (grades)

What is the minimum group size (N) required to ensure that ≥ 3 students in the group have the same grade?

Think: Extreme scenario

$$\underbrace{2}_{\substack{h-1 \\ \text{where} \\ h=3 \\ \text{"height"}}} \cdot \underbrace{5}_{\substack{x \\ = \text{\# boxes}}} + 1 = 11$$



Corollary

$x, h \in \mathbb{Z}^+$

x boxes

The minimum #-of-balls (N) required to ensure that ≥ 1 box has $\geq h$ balls is

$$\underbrace{(h-1)}_{\substack{\text{\# of} \\ \text{complete layers} \\ \text{in the} \\ \text{extreme scenario}}} x + 1$$

Have fun! Odds in back!