4.3: PERMUTATIONS AND COMBINATIONS

PERMUTATIONS

Ex How many ways are there to order n runners? (No ties). ($n \in \mathbb{Z}^+$)

$$\underset{1st}{\underline{n}}\ \underset{2nd}{\underline{n-1}}\ \underset{3rd}{\underline{n-2}}\ \cdots\ \underset{nth}{\underline{1}}$$

$$n(n-1)(n-2)\cdots(2)(1) = n!$$

"n factorial"

There are $n!$ ways to order n distinct objects.
There are $n!$ <u>permutations</u> of n distinct objects.

<u>$n!$</u>

$$0! = 1$$
$$1! = 1$$
$$2! = (2)(1) = 2$$
$$3! = (3)(2)(1) = 6 \qquad 2\times 4$$
$$4! = 24 \qquad 2\times 5$$
$$5! = 120$$
$$\vdots$$
$$n! = n\times(n-1)!$$

$n!$ grows rapidly

$$15! > 10^{12}$$

<u>Ex</u> # ways to order a 52-card deck.

$$52! \approx 8 \times 10^{67} \quad \text{(astronomical)}$$

Not just "orders"!

<u>Ex</u> # ways to put n people in n rooms
(1 person per room)

Imagine that the rooms are #ed from 1 to n

Poor sucker next to laundry room.

$$\underset{\substack{\text{Room}\\1}}{\underline{n}} \quad \underset{\substack{\text{Room}\\2}}{\underline{n-1}} \quad \underset{\substack{\text{Room}\\3}}{\underline{n-2}} \quad \cdots \quad \underset{\substack{\text{Room}\\n}}{\underline{1}}$$

$n!$

<u>Ex</u> Traveling Salesman Problem (TSP).
Find the most efficient route through n cities.
Important unsolved CS problem: find an efficient, optimal algorithm.

$\Theta((n-1)!)$

Brute-force: $\Theta(n!)$ - bad
There <u>are</u> excellent approximation algorithms.

r-Permutations of n distinct objects

$P(n,r)$ = # of ways to choose and order a subset of r elements from a set of n elements.

$$= \frac{n!}{(n-r)!} \quad \text{or} \quad n(n-1)\cdots(n-r+1)$$
after cancelling

Ex $n=7$ Olympic runners in a race
Find # possible G/S/B assignments. (No ties)

$$\underset{G}{\underline{7}} \times \underset{S}{\underline{6}} \times \underset{B}{\underline{5}} = 210$$

$$P(7,3) = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 7 \times 6 \times 5 = 210$$

Idea: $\underset{G}{\underline{\quad}}\ \underset{S}{\underline{\quad}}\ \underset{B}{\underline{\quad}} \,\Big|\, \underbrace{\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}}$

We don't care about the order among the 4 losers.

$\frac{7!}{4!}$ ← # ways to order all 7

COMBINATIONS

What if we don't care about the order among the winners, either?

Ex $n = 7$ runners
Find # of possible combos of "medalists." (No ties)

What do you think the formula is?

$$\underbrace{\overset{\text{winners}}{\underset{1st}{7}\ \underset{2nd}{6}\ \underset{3rd}{5}}}_{\div 3!}\ \Big|\ \underbrace{\overset{\text{losers}}{\underset{4th}{4}\ \underset{5th}{3}\ \underset{6th}{2}\ \underset{7th}{1}}}_{\div 4!}$$

$$\frac{7!}{3!4!} = \frac{7\times 6\times 5\times 4!}{3\times 2\times 1\times 4!} = 35$$

$C(n,r) = \binom{n}{r}$ "n choose r"

$=$ # of r-subsets of an n-set

Ex There are 35 3-subsets of a 7-set.
In a set, order doesn't matter.

$$= \frac{n!}{r!(n-r)!}$$

Proof $\underbrace{P(n,r)}_{\substack{\text{\# ways}\\ \text{to choose}\\ \text{and order}\\ r \text{ elts}\\ \text{from an } n\text{-set}}} = \underbrace{C(n,r)}_{\substack{\text{\# ways}\\ \text{to choose}\\ \text{an } r\text{-subset}}} \times \underbrace{r!}_{\substack{\text{\# ways}\\ \text{to order}\\ r \text{ elts.}}} \Leftrightarrow \frac{n!}{(n-r)!} = C(n,r)\times r!$

Ex How many possible 5-card hands are there?
(Order irrelevant in a "hand")

$$\binom{52}{5} = \frac{52!}{5!\,47!} = \frac{52\times51\times50\times49\times48\times\cancel{47!}}{5\times4\times3\times2\times1\times\cancel{47!}}$$

$$= 2{,}598{,}960$$

Symmetry Property

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!\,r!}$$

# ways to choose r winners from an n-set	=	# ways to choose n-r losers from an n-set

GOTO L-3-20 to 21 in M119 after my examples

HW Tips

Ex 6 students are in a class.
Grades: A, B, C, D, F.
Find the # of possible "grade reports" in which at least one student has an A.

Cases? NO!

$$\begin{pmatrix}\text{\# of unrestricted} \\ \text{reports}\end{pmatrix} - \begin{pmatrix}\text{\# reports} \\ \text{w/ 0 "A"s}\end{pmatrix}$$

Ex

$$\begin{array}{rcccccccccccccc} & \overset{\{A,B,C,D,F\}}{5} & 5 & 5 & 5 & 5 & 5 & - & \overset{\{B,C,D,F\}}{4} & 4 & 4 & 4 & 4 & 4 \\ \text{Students} \rightarrow & J & K & L & M & N & O & & J & K & L & M & N & O \end{array}$$

$$5^6 - 4^6$$

11,529

LSAT

Ex 6 students.
Jack and Nick get the same grade.
Ken and Larry get different grades from each other.
How many possibles?

$$\begin{array}{cccccc} 5 & 5 & 4 & 5 & 1 & 5 \\ J & K & L & M & N & O \end{array}$$

2500

Ex 6 students.
Exactly 4 pass (A, B, C).
How many possibilities?

Start case-by-case analysis

Case 1: J, K, L, M pass

$$\underset{J}{\overset{ABC}{3}}\ \underset{K}{3}\ \underset{L}{3}\ \underset{M}{3}\ \underset{N}{\overset{DF}{2}}\ \underset{O}{\overset{DF}{2}} = 324$$

Shortcut!

How many combos of 4 students are there? $\binom{6}{4} = \frac{6!}{4!2!} = \frac{6\times 5}{2} = 15$

There are 15 cases, each w/324 possibil. "Symmetry"
$15 \times 324 = 4860$
IMPORTANT: The 15 cases are disjoint!

Pascal's Triangle

Like a table of "$\binom{n}{r}$"s (combinatorial coefficients).
binomial

To construct:
Begin, end each row with a "1."
Any other entry = sum of the two entries immediately above

$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$ $\quad n, r \in \mathbb{Z}^+$, $n \geq r$

Base: $\binom{n}{0} = \binom{n}{n} = 1 \ \forall n \in \mathbb{Z}^{\geq 0}$

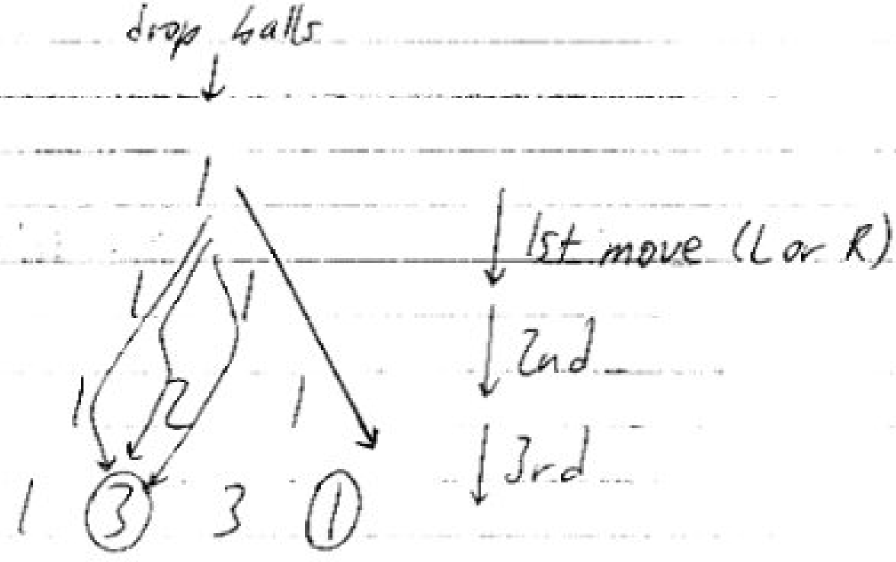
Row # (n)		
0	1	$\leftarrow \binom{0}{0} = 1$
1	1 1	$\leftarrow \binom{1}{0} = 1, \binom{1}{1} = 1$
2	1 2 1	$\leftarrow \binom{2}{0} = 1, \binom{2}{1} = 2, \binom{2}{2} = 1$
3	1 3 3 1	
4	1 4 6 4 1	

Observe:
$\binom{n}{0} = \binom{n}{n} = 1$
Symmetry

Overheads/Handouts – Chaos

Why does this give us "$\binom{n}{r}$"'s?

Imagine that the entries are pins in a pinboard.
(Plinko in $ is Right, Pachinko)



Each entry indicates the # of ways a ball can drop from the top to its pin.

To get to 3, you must take 1R and 2Ls in any order.

\# orderings/perms. of RLL
= # ways to "choose" exactly 1R from a sequence of 3 mov

$$= \frac{3!}{1!\,2!} = \binom{3}{1} = 3$$

Namely, LLR, LRL, RLL

1

1 1

1 way to hit — (1) (2) — 2 ways to hit 1

1 3 3 1

So, 3 ways to hit

So, our construction of the Δ is appropriate.
Our physical model and combo. interp. justify this.

Picture / Histograms

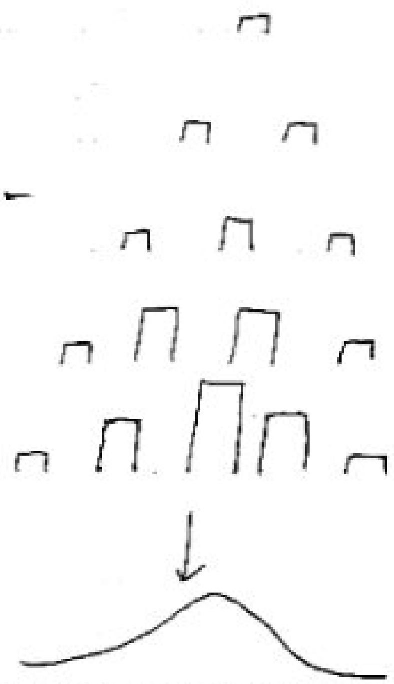
1

1 1

1 2 1

1 3 3 1

1 4 6 4 1



Revisit in 4-3, 5-6

In a science museum
Drop a lot of balls →
Assume $P(L)=P(R)$ at each move.

Normal dist'n

M245 $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

<u>Another proof</u>

$\binom{n+1}{r}$ = # bit strings of length $n+1$ with exactly r "1"s

<u>Case 1</u> $\overbrace{\boxed{1}\underbrace{\boxed{\qquad\qquad\qquad}}_{\binom{n}{r-1} \text{ ways to place } r-1 \text{ "1"s}}}^{n+1}$ $\leftarrow \binom{n}{r-1}$ of this type

<u>Case 2</u> $\boxed{0}\underbrace{\boxed{\qquad\qquad\qquad}}_{\binom{n}{r} \text{ ways to place } r \text{ "1"s}}$ $\leftarrow \binom{n}{r}$ of this type

$\therefore \binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

$\binom{n}{r}$ = # r-subsets of an n-set

Row Sum of Pascal's $\triangle$ (Row n) $= 2^n$

$\binom{n}{0}$ = # 0-subsets of an n-set $= 1$ $(\emptyset)$

$\binom{n}{1}$ = ⋮ $= n$

$\binom{n}{n}$ = ⋮ $= 1$

$\sum_{r=0}^{n} \binom{n}{r}$ = total # subsets of an n-set $= 2^n \leftarrow |P(\underset{\text{some } n\text{-set}}{S})|$

Binomial Theorem

Pascal's Δ:

$x+y \neq 0$

n										$(x+y)^n$
0					1					$(x+y)^0 = 1$
1				1		1				$(x+y)^1 = 1x + 1y$
2			1		2		1			$(x+y)^2 = 1x^2 + 2xy + 1y^2$
3		1		3		3		1		$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + y^3$
4	1		4		6		4		1	

Why does this work?

$$(x+y)^2 = (x+y)(x+y) = \underbrace{xx + xy + yx + yy}_{\text{sum of all possible "choice-products"}}$$

Each term "chooses" x or y from each factor.

$$= 1x^2 + 2xy + 1y^2$$
$$= \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2$$

ways to choose 0 "y"s   # ways to choose 1 "y"   2 "y"s

$$(x+y)^4 = \overset{1}{\binom{4}{0}}x^4 + \overset{4}{\binom{4}{1}}x^3y + \overset{6}{\binom{4}{2}}x^2y^2 + \overset{4}{\binom{4}{3}}xy^3 + \overset{1}{\binom{4}{4}}y^4$$

Can evaluate $\binom{n}{r}$ from Pascal's Δ or $\frac{n!}{r!(n-r)!}$.

$$(x+y)(x+y)(x+y)(x+y)$$

There are $\binom{4}{2} = 6$ ways of choosing 2 "x"s and 2 "y"s.

Recall Pascal's Identity:

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r} \qquad n, r \in \mathbb{Z}^+, \; n \geq r$$

Also: $\binom{n}{0} = \binom{n}{n} = 1 \quad \forall n \in \mathbb{Z}^{\geq 0}$ (Base cases).

a (two variable) __recurrence relation__ – terms depend on previous terms

Add

New term

__A proof__

$\binom{n+1}{r}$ = # bit strings of length $n+1$ with exactly r "1"'s.

__Case 1__ $\quad \overbrace{\boxed{1}\underbrace{\boxed{\qquad\qquad\qquad}}_{n}}^{n+1}$ $\leftarrow \binom{n}{r-1}$ of this type

There are $\binom{n}{r-1}$ ways to place $r-1$ "1"'s.

__Case 2__ $\quad \overbrace{\boxed{0}\underbrace{\boxed{\qquad\qquad\qquad}}_{n}}^{n+1}$ $\leftarrow \binom{n}{r}$ of this type

$\binom{n}{r}$ ways to place r "1"'s.

$$\therefore \binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$