**S.1: RECURRENCE RELATIONS**

**Example**

\[ a_n = 3a_{n-2} - 2a_{n-3} \quad (n \geq 3) \]

is a recurrence relation ("RR").

It has order or degree 3 because \( a_n \) is defined in terms of the previous 3 terms (except \( a_{n-1} \), OK).

i.e., To compute \( a_n \), we must "look back" up to three places before it.

\[ a_0 = 1 \quad \text{Initial conditions/values} \]
\[ a_1 = 0 \quad \text{"Seeds"} \]
\[ a_2 = 0 \]

These specify the terms that immediately precede the case (here, \( n = 3 \)) where the RR "begins" or "takes effect."

If the RR has order = \( k \) (here, \( k = 3 \)), we must specify \( k \) initial conditions in order to determine a unique sequence: \( \{a_n\} \).
\[
\begin{align*}
\begin{cases}
a_n &= 3a_{n-2} - 2a_{n-3} \quad (n \geq 3) \\
a_0 &= 1 \\
a_1 &= 0 \\
a_2 &= 0
\end{cases}
\end{align*}
\]

\text{determine a unique sequence}

Find the first six terms of this sequence.

\[
a_0 = 1 \\
a_1 = 0 \\
a_2 = 0
\]

\[
a_3 = 3a_1 - 2a_0 = 3(0) - 2(1) = -2
\]

\[
a_4 = 3a_2 - 2a_1 = 3(0) - 2(0) = 0
\]

\[
a_5 = 3a_3 - 2a_2 = 3(-2) - 2(0) = -6
\]

\[
\begin{array}{cccccc}
a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\
1 & 0 & 0 & -2 & 0 & -6
\end{array}
\]

\text{seeds}
Ex 4 (Rosen) Breeding Rabbits and the Fibonacci Numbers \( \{f_n\} \)

\[
\begin{align*}
(f_0 &= 0 \text{ - optional}) \\
(f_1 &= 1) \\
(f_2 &= 1) \\
fn &= fn-1 + fn-2 \quad (n \geq 3) \quad \text{2nd-order RR}
\end{align*}
\]

\( \{f_n\} : 1, 1, 2, 3, 5, 8, \ldots \)

Formula - S, Z

In nature:
- pineapples, cacti, pinecones, sunflowers,
- leaf arrangements of almost all plants (PhylloTaxis)

Neat: Pascal \( \Delta \)

\[
\begin{array}{cccc}
& & 1 & \\
& 1 & & 2 \\
1 & & 3 & & 5 \\
& 1 & 3 & 3 & 1
\end{array}
\]

Diagonal sums are \( f_n \)
Ex 6 (Rosen)

Let \( a_n \) = \# bit strings of length \( n \) that do not have two consecutive "0"s.
No "00."

Find a recurrence relation for \( \{a_n\} \)

For now, assume \( n \) is large.

Count the \( n \)-bit strings w/out "00."

**Case 1**

![Diagram for Case 1]

There are \( a_{n-1} \) subtrings w/o "00."

**Case 2**

![Diagram for Case 2]

There are \( a_{n-2} \) subtrings w/o "00."

\[ a_n = a_{n-1} + a_{n-2} \]

\( \leq \) 2nd-order, so we need 2 initial conditions.
Give the initial conditions

Two possible groups

Group 1

The empty string has no "00", so $a_0 = 1$.

□ and □ have no "00", so $a_1 = 2$.

\[
\begin{align*}
  a_0 &= 1 \\
  a_1 &= 2
\end{align*}
\]

The "n=0" case fits into our argument.

Group 2

If you're unsure whether or not the "n=0" case fits in.

\[
\begin{align*}
  a_1 &= 2 \\
  a_2 &= 3 \quad (00, 01, 10, 11)
\end{align*}
\]

How can we define $a_{n+3}$?

\[
\begin{align*}
  a_0 &= 1 \\
  a_1 &= 2 \\
  a_n &= a_{n-1} + a_{n-2} \quad (n \geq 2)
\end{align*}
\]

or

\[
\begin{align*}
  a_0 &= 1 \\
  a_1 &= 2 \\
  a_2 &= 3 \\
  a_n &= a_{n-1} + a_{n-2} \quad (n \geq 3)
\end{align*}
\]

"Back-of-book" method can collapse "Ex 6" method

(They try to incorporate n=0 case.)
<table>
<thead>
<tr>
<th>( n )</th>
<th>( a_n )</th>
<th>( f_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0)</td>
<td>(1)</td>
<td>(0)</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>13</td>
<td>5</td>
</tr>
</tbody>
</table>

\([a_n]\) is just a shifted Fibonacci sequence

\[ a_n = f_{n+2} \quad (n \geq 0) \]

Ex 7 (Rosen)

let \( a_n \) = # of valid \( n \)-digit codewords

<table>
<thead>
<tr>
<th>0-9</th>
</tr>
</thead>
<tbody>
<tr>
<td>has an even # of '0's</td>
</tr>
<tr>
<td>not necessarily consecutive</td>
</tr>
</tbody>
</table>

What might be a good way to split?

Case 1

\( \begin{array}{c}
\text{length } n-1 \\
\text{(1-9)} \\
\end{array} \)

\( (a_n-1) \) substrings w/even # of '0's

\( \leftarrow 9a_{n-1} \) of this type
Case 2

We need this substring to have an odd # of "0"'s (invalid).

How many \((n-1)\)-digit strings are there?

\[
\begin{array}{cccc}
1 & 0 & 1 & 0 \\
& 2 & \cdots & 10 \\
& \phantom{2} & \cdots & \phantom{10}
\end{array}
\Rightarrow 10^{n-1}
\]

How many of these are valid? \(a_{n-1}\)

So, there are \(10^{n-1} - a_{n-1}\) invalid strings of length \(n-1\).

Combine the cases:

\[
a_n = 9a_{n-1} + (10^{n-1} - a_{n-1})
\]

\[
= 8a_{n-1} + 10^{n-1} \quad \text{\textless \text{1st-order}}
\]

Define \(a_n\) as:

\[
\begin{align*}
\{ a_0 = 1 & \quad \text{(empty string has zero "0"s)} \\
, a_n = 8a_{n-1} & \quad (n \geq 1)
\end{align*}
\]

or

\[
\begin{align*}
\{ a_0 = 1 & \quad \text{(empty string has zero "0"s)} \\
, a_1 = 9 & \quad (11, 12, \ldots, 19) \\
, a_n = 8a_{n-1} + 10^{n-1} & \quad (n \geq 2)
\end{align*}
\]

\text{can collapse}
SOLVING RECURRANCE RELATIONS

A solution of a RR is any sequence whose terms satisfy the RR.

Example RR: \(a_n = -3a_{n-1} + 4a_{n-2}\)

Show that \(\{a_n\}\) is a solution if

1. \(a_n = 1\) → Constant sequence \(1, 1, 1, 1, \ldots\)

\[a_n = 1 - 3a_{n-1} + 4a_{n-2}\]
\[1 = -3(1) + 4(1)\]
\[1 = -3 + 4\]
\[1 = 1\]

2. \(a_n = (-4)^n\)

\[a_n = (-4)^n - 3a_{n-1} + 4a_{n-2}\]
\[(-4)^n = -3(-4)^{n-1} + 4(-4)^{n-2}\]
\[= (-4)^{n-2}[12 + 1] = 16\]
\[= 16(-4)^{n-2}\]
\[= (-4)^2(-4)^{n-2}\]
\[= (-4)^n\]

If initial conditions are given, a solution must satisfy them. If \(k\) (order) are given, there is a unique solution.

Note: \(\{a_n = 0\}\) No solution.
Ex. 3  Compound Interest

Example Ex 1: World population in 1999 = 6B

Assume it grows 1.3% a year.
Let \( P_n \) = World pop. \( n \) years after 1999.

Each year, the world pop. is
\[ 100\% + 1.3\% = 101.3\% \]

of the previous year's pop.

\[ \begin{align*}
\{ P_n &= 1.013 \cdot P_{n-1} \\
P_0 &= 6B \}
\end{align*} \]

Determines a unique sequence

"Solving" a RR usually means giving a "nice" formula for the general term (\( P_n \) here)

Iterative approach:

\[ P_n = 1.013 \cdot P_{n-1} \]

\[ = (1.013) \cdot (1.013) \cdot P_{n-2} \]

\[ = (1.013)^2 \cdot P_{n-3} \]

3 copies

\[ = (1.013)^3 \cdot P_{n-4} \]

\[ = \cdots \]

\[ = (1.013)^n \cdot P_0 \]

\[ = (1.013)^{6B} \]

\[ = (1.013)^n \cdot 6B \] \( \sim \) "explicit formula"

Can verify using induction.

Year 2020: \( n = 21 \)

\( P_{21} = (1.013)^{21} \cdot 6B \)
Ex (HW #5d)

Solve \( \begin{cases} a_n = a_{n-1} + 2n + 3 \\ a_0 = 4 \end{cases} \)

using an iterative approach.

Assume \( n \) is large.

\[
a_n = a_{n-1} + 2n + 3
\]

\[
= [a_{n-2} + 2(n-1) + 3] + 2n + 3
\]

\[
= a_{n-2} + 2(n-1) + 2n + 3 + 3
\]

\[
= a_{n-2} + 2(n-2) + 2(n-1) + 2n + 3 + 3
\]

\[
= a_{n-3} + 2(n-2) + 2(n-1) + 2n + 3 + 3
\]

\[
= a_0 + 2(1) + 2(2) + \ldots + 2n + 3 + 3 \sum_{i=1}^{n-1} + 3
\]

\[
= 4 + 2 \left[ 1 + 2 + 3 + \ldots + n \right] + 3n
\]

\[
= 4 + 2 \frac{n(n+1)}{2} + 3n
\]

\[
= 4 + n(n+1) + 3n
\]

\[
= 4 + n^2 + n + 3n
\]

\[
= n^2 + 4n + 4 \quad (n \geq 0)
\]

\[\text{Formula for } a_n\]
**HW Tip**

From 3.2, Quiz

\[ 1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1 \]

---

Extra Challenge: describe a bijection.

Generalization

\[ \sum_{i=0}^{n} r^i = 1 + r + r^2 + \ldots + r^n = \frac{r^{n+1} - 1}{r - 1} \quad (r \neq 1) \]

If \( r = 2 \), \( \frac{2^{n+1} - 1}{2 - 1} \)

---

**Ex 5 (Rosen) Tower of Hanoi**

\( H_n = \# \text{ moves required to solve if } n \text{ disks} \)

---

\[ H_n = 2H_{n-1} + 1, \quad H_1 = 1 \quad \Rightarrow H_n = 2^{n-1} - 1 \]
Extra Challenge

\[ 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1 \]

\[ |S| \quad \text{and} \quad |T| \]

\( S = \text{set of all bit strings of length } \leq n \)

\( T = \text{set of all bit strings of length } n+1, \text{ except } 1 \)

A bijective proof:

Set up a 1-1 correspondence between the strings in \( S \) and the strings in \( T \).

Strings in \( T \):

If it starts w/ "1"

Example: \( \overline{001011} \)

says the rest of the string is the corresponding string in \( S \).

\( 1001011 \rightarrow 001011 \)

in \( T \) \quad \text{in } S \)
If it starts w/ "0"

Ex \[ \text{0001011} \]

says we eat the string at the leftmost "1"

\[ \text{0001011} \rightarrow \text{011} \]

in \( T \) in \( S \)

Ex \( n=4 \)

\[
\begin{array}{c|c}
T \ (\text{length}) & S \ (\text{length}) \\
\hline
\text{0000} & \leftrightarrow & \text{0000} \\
\text{0001} & \leftrightarrow & \Phi \\
\text{0010} & \leftrightarrow & \text{0010} \\
\text{0100} & \leftrightarrow & \text{0} \\
\text{0110} & \leftrightarrow & 1 \\
\end{array}
\]

00000 \leftrightarrow \text{excluded string}

"but there's no"
In general,

\[ 1 + r + r^2 + \ldots + r^n = \frac{r^{n+1} - 1}{r - 1} \]

Extend to \( r = 3, 4, 5, \ldots \)

\[ r = 3; \quad 1 + 3 + 3^2 + \ldots + 3^n = \frac{3^{n+1} - 1}{3 - 1} = \frac{1}{2} \left( 3^{n+1} - 1 \right) \]

\[ \frac{1}{5} \] \[ \frac{5}{11} \]

Exclude 0000

001012 → 012
002012 →

\( \text{leftmost} = 0, \)
\( \text{cut at} \)
\( \text{leftmost} = 1 \) or 2

101012
201012

\( \text{keep the verb} \)
\( \text{if leftmost} = 1 \) or 2

So, \( |S| = \frac{1}{2} \left( \#(n-1)\text{-ternary strings} - 1 \right) \)
5.2: SOLVING RR (MORE)

A linear homogeneous RR of order \(k\) with constant coefficients has the form

\[
\begin{align*}
    a_n &= c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \\
    \text{where each } c_i \in \mathbb{R} \text{ and } c_k \neq 0 \\
    \text{("constant coeffs")}
\end{align*}
\]

Ex. \(a_n = a_{n-2} - 4a_{n-3}\)

- order = 3
- \(c_1 = 0\) (no \(a_{n-1}\) term)
- \(c_2 = 1\)
- \(c_3 = -4\)

Ex. \(a_n = \frac{1}{a_{n-1}} + a_{n-2}\) is not linear

Ex. \(a_n = a_{n-1} a_{n-2}\) is not homogeneous

Ex. \(a_n = \frac{1}{a_{n-1} + 1}\) is not homogeneous

Ex. \(a_n = \begin{cases} a_{n-1} - 3a_{n-2} \\ \text{not a constant} \end{cases}\) does not have all coefficients as constants
We can systematically solve RR’s of form (6).

**Ex (Order = 1)**

Solve \[ \begin{cases} a_n = 3a_{n-1} & \text{for } n \geq 1 \\ a_0 = 2 \end{cases} \]

There is a unique solution – the sequence \( a_n \):

\[
\begin{align*}
2 & \quad 6 & \quad 18 & \quad 54 & \quad \ldots \\
\times 3 & \quad \times 3 & \quad \times 3
\end{align*}
\]

We want a nice formula for \( a_n \).

**Step 1 Rewrite RR**

\[ a_n = 3a_{n-1} \]

\[ a_n - 3a_{n-1} = 0 \]

**Step 2 Replace** \( \{ a_n \text{ with } r \} \)

\[ a_n - 3a_{n-1} = 0 \]

\[ r - 3 = 0 \quad \text{characteristic equation of the RR} \]

**Step 3 Find the root (solution) of the char. eq.**

\[ r - 3 = 0 \]

\[ r = 3 \]
Step 4 The solutions to the RR have $a_n = \alpha r_1^n$

where $\alpha \in \mathbb{R}$
and $r_1$ is the root of the char. eq.

Here, $a_n = \alpha \cdot 3^n$

A sequence \{a_n\} is a solution to the RR $\iff$ its $a_n$ has this form

Step 5 Use the initial condition to solve for $\alpha$

\[ a_n = \alpha \cdot 3^n \]
\[ n=0: a_0 = \alpha \cdot 3^0 \]
\[ 2 = \alpha \cdot 1 \]
\[ \alpha = 2 \]

Step 6 The unique solution \{a_n\} has

\[ a_n = 2 \cdot 3^n, \quad n \geq 0 \]
Ex (Order = 2) #46

Solve \( \begin{cases} a_n = 7a_{n-1} - 10a_{n-2} & \text{for } n \geq 2 \\ a_0 = 2 \\ a_1 = 1 \end{cases} \)

**Step 1** Rewrite RR

\[
\begin{align*}
a_n &= 7a_{n-1} - 10a_{n-2} \\
a_n - 7a_{n-1} + 10a_{n-2} &= 0
\end{align*}
\]

**Step 2** Replace \( \begin{cases} a_n \text{ with } r^2 \\ a_{n-1} \text{ with } r \\ a_{n-2} \text{ with } 1 \end{cases} \)

\[
\begin{align*}
a_n - 7a_{n-1} + 10a_{n-2} &= 0 \\
r^2 - 7r + 10 &= 0 < \text{char. eq. of RR}
\end{align*}
\]

**Step 3** Find the roots of the char. eq.

\[
\begin{align*}
r^2 - 7r + 10 &= 0 \\
(r - 5)(r - 2) &= 0 \\
r_1 &= 5, \quad r_2 = 2
\end{align*}
\]

**Step 4** The solutions to the RR have

\[
\begin{align*}
a_n &= x_1 r_1^n + x_2 r_2^n \\
a_n &= x_1 \cdot 5^n + x_2 \cdot 2^n
\end{align*}
\]
Step 5 Use the initial conditions to solve for $x_1, x_2$.

\[ a_n = x_1 \cdot S^n + x_2 \cdot Z^n \]

$\begin{align*}
    n=0: a_0 &= x_1 \cdot S^0 + x_2 \cdot Z^0 \\
           &= 2 = x_1 + x_2 \\
\end{align*}$

$\begin{align*}
    n=1: a_1 &= x_1 \cdot S^1 + x_2 \cdot Z^1 \\
           &= 1 = 5x_1 + 2x_2 \\
\end{align*}$

Solve this system.

$\begin{align*}
    x_1 &= -1, \\
    x_2 &= 3
\end{align*}$

Step 6 The unique solution \{an\} has

$\begin{align*}
    a_n &= (-1) \cdot S^n + 3 \cdot 2^n, \quad n \geq 0
\end{align*}$

Ex $\begin{align*}
    a_{100} &= (-1) \cdot S^{100} + 3 \cdot 2^{100} \\
           &\approx 7.9 \times 10^{69}
\end{align*}$

Beats iteration!!

Ex 4 Fibonacci numbers

$\begin{align*}
    \begin{cases}
    f_n = f_{n-1} + f_{n-2} & (n \geq 2) \\
    f_0 = 0 \\
    f_1 = 1
    \end{cases}
\end{align*}$

\[ f_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

\[ \frac{f_{n+1}}{f_n} \rightarrow \frac{1 + \sqrt{5}}{2} \quad \text{golden ratio} \]
Ex (Order = 2, repeated roots) #4f

Solve \[
\begin{cases}
  a_n = -6a_{n-1} - 9a_{n-2} \quad \text{for } n \geq 2 \\
  a_0 = 3 \\
  a_1 = -3
\end{cases}
\]

\[a_n = -6a_{n-1} - 9a_{n-2}\]

1. \[a_n + 6a_{n-1} + 9a_{n-2} = 0\]

2. \[r^2 + 6r + 9 = 0\]

3. \[(r + 3)^2 = 0\]

\[r = -3 \text{ is the sole, } \text{"repeated" root.}\]

4. If \(r\) is a repeated root, the solutions to the RR have

\[a_n = \alpha_1 r_n + \alpha_2 r^{2n} \quad (\alpha_1, \alpha_2 \in \mathbb{R})\]

Here, \(a_n = \alpha_1 (-3)^n + \alpha_2 n (-3)^n\)
(5) 
\[ a_n = \alpha_1 (-3)^n + \alpha_2 n (-3)^n \]

\[ n=0: \quad a_0 = \alpha_1 (-3)^0 + \alpha_2 (0)(-3)^0 \]

\[ 3 = \alpha_1 (1) \]

\[ \alpha_1 = 3 \]

\[ n=1: \quad a_1 = \alpha_1 (-3)^1 + \alpha_2 (1)(-3)^1 \]

\[ -3 = -3\alpha_1 - 3\alpha_2 \]

\[ 1 = \alpha_1 + \alpha_2 \]

\[ 1 = 3 + \alpha_2 \]

\[ \alpha_2 = -2 \]

(6) Solution:
\[ a_n = 3(-3)^n - 2n(-3)^n, \quad n \geq 0 \]
Order $k$, $k$ distinct roots (not tested)

2. Replace
   \[
   \begin{align*}
   a_n & \text{ with } r^k \\
   a_{n-1} & \text{ with } r^{k-1} \\
   \vdots & \\
   a_{n-k} & \text{ with } 1
   \end{align*}
   \]

4. If the $k$ distinct roots of the char. eq. are $r_1, r_2, \ldots, r_k$, the solutions to the RR have

   \[
   a_n = \sum_{i=1}^{k} x_i r_i^n
   \]

5. We get a system of $k$ linear eqs. in $k$ unknowns ($x_i$'s).

Optional: Read Ex 6

Order $k$, repeated roots (not tested)

p. 325 - Thm 4

Ex char. eq. $\Rightarrow (r-3)^3 (r-4)^2$

\[
\begin{align*}
a_n &= \left[ a_1 \cdot 3^n + a_2 n \cdot 3^n + a_3 n^2 \cdot 3^n \right] + \left[ b_4 \cdot 4^n + b_5 n \cdot 4^n \right] \\
&= c_{2,1} + c_{2,2} n + c_{3,1} n^2 + \cdots + c_{4,1} n + c_{5,1} n^2
\end{align*}
\]
**5.5: INCLUSION-EXCLUSION**

**n=2 sets**

Let $A_1, A_2$ be finite sets.

Then, $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

If we just count $|A_1| + |A_2|$: $	o 0$

The elements here are counted twice.

When we $- |A_1 \cap A_2|$

Now, all the elements in $A_1 \cup A_2$ are counted once.
Ex: To take a discrete math class, a student must be a math major or a CS major. The class has 50 students.

30 are math majors.
40 are CS majors.

How many students are joint math/CS majors?

Let $M = \{\text{math majors}\}$

Let $C = \{\text{CS majors}\}$

$\Rightarrow M \cap C = \{\text{joint math/CS majors}\}$

Formula:

$|M \cup C| = |M| + |C| - |M \cap C|$

$50 = 30 + 40 - |M \cap C|$

$50 = 70 - x$

$x = 20$

$|M \cap C| = 20$

There are 20 joint math/CS majors.

[Diagram showing a Venn diagram with 30 and 40 in separate circles and 20 in the intersection, and 50 in the outer circle.]
$n = 3$ sets

Let $A_1, A_2, A_3$ be finite sets.

How do we find $|A_1 \cup A_2 \cup A_3|$?

Start with $|A_1| + |A_2| + |A_3|$

See Figure 4 (p. 356)

Now, $- |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3|$

We need $+ |A_1 \cap A_2 \cap A_3|$
\[ |A_1 \cup A_2 \cup A_3 | = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3| \]

For general \( n \in \mathbb{Z}^+ \)

Let \( A_1, A_2, \ldots, A_n \) be finite sets.

\[ |A_1 \cup A_2 \cup \cdots \cup A_n| \]

\[ = |A_1| + |A_2| + \cdots + |A_n| \]

\[ - |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots \leftarrow (2) \text{ pairs of sets} \]

\[ + |A_1 \cap A_2 \cap A_3| + \cdots \leftarrow (3) \text{ triples of sets} \]

\[ + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n| \leftarrow (n^n) = 1 \text{ n-set} \]
Ex. (Sieve of Eratosthenes)

How many prime numbers are there between 1 and 47, inclusive?

By the \( \sqrt{n} \) rule, composite numbers are divisible by 2, 3, or 5.

How many integers between 1 and 47, inclusive, are divisible by 2, 3, or 5?

Let \( D_2 = \{ x | x \in \mathbb{Z}, 1 \leq x \leq 47, 2 | x \} \)

Let \( D_3 = \{ x | 3 | x \} \)

Let \( D_5 = \{ x | 5 | x \} \)

\[ |D_2 \cup D_3 \cup D_5| = |D_2| + |D_3| + |D_5| \]

\[ - |D_2 \cap D_3| - |D_2 \cap D_5| - |D_3 \cap D_5| + |D_2 \cap D_3 \cap D_5| \]

Rule: \( |D_i \cap D_j| = |D_{\text{rel, prime}}| \) if \( i \neq j \)

\[ = |D_2| + |D_3| + |D_5| \]

\[ - |D_i| - |D_{10}| - |D_{15}| + |D_{30}| \]
\[
\begin{align*}
&= \left(\frac{47}{2}\right) + \left(\frac{47}{3}\right) + \left(\frac{47}{6}\right) \\
&\quad - \left(\frac{47}{7}\right) - \left(\frac{47}{9}\right) - \left(\frac{47}{21}\right) \\
&\quad + \left(\frac{47}{5}\right) \\
&= 23 + 15 + 9 \\
&\quad - 7 - 4 - 3 \\
&\quad + 1 \\
&= 34
\end{align*}
\]

34 integers between 1,477 are dividible by 23 and 5.

So, 47 - 34 = 13 are not.

But \(-1\) (exclude 1) + 3 (include 2, 3, 5) = 15 primes

\[2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\]