

S.1: RECURRENCE RELATIONS

#13, Snuvaldi 2014

$$\text{Ex } a_n = 3a_{n-2} - 2a_{n-3} \quad (n \geq 3)$$

$n \geq 2$ assumed

is a recurrence relation ("RR").

It has order or degree 3, because a_n is defined in terms of the previous 3 terms (except a_{n-1} , -OK). i.e., To compute a_n , we must "look back" up to three places before it.

$$\left. \begin{array}{l} a_0 = 1 \\ a_1 = 0 \\ a_2 = 0 \end{array} \right\} \begin{array}{l} \text{Initial conditions/values} \\ \text{"seeds"} \end{array}$$

These specify the terms that immediately precede the case (here, $n=3$) where the RR "begins" or "takes effect."

If the RR has order = k (here, $k=3$), we must specify k initial conditions in order to determine a unique sequence $\{a_n\}$.

: :

$$\left\{ \begin{array}{l} a_n = 3a_{n-2} - 2a_{n-3} \quad (n \geq 3) \\ a_0 = 1 \\ a_1 = 0 \\ a_2 = 0 \end{array} \right. \quad \text{determine a unique sequence}$$

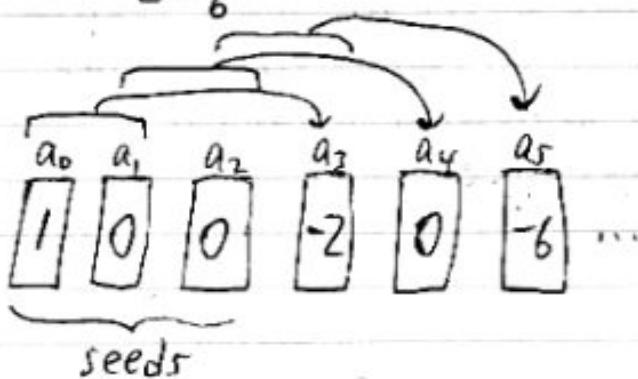
Find the first six terms of this sequence.

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 0 \\ a_2 &= 0 \end{aligned}$$

$$\begin{aligned} a_3 &= 3a_1 - 2a_0 && \leftarrow n=3 \text{ case: RR "begins"} \\ &= 3(0) - 2(1) \\ &= -2 \end{aligned}$$

$$\begin{aligned} a_4 &= 3a_2 - 2a_1 \\ &= 3(0) - 2(0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} a_5 &= 3a_3 - 2a_2 \\ &= 3(-2) - 2(0) \\ &= -6 \end{aligned}$$



Ex 4 (Rosin) breeding rabbits and the Fibonacci Numbers $\{f_n\}$

$$(f_0 = 0 \text{ - optional})$$

$$f_1 = 1$$

$$f_2 = 1$$

$$f_n = f_{n-1} + f_{n-2} \quad (n \geq 3) \leftarrow \text{2nd-order RR}$$

$$\{f_n\}: 1, 1, 2, 3, 5, 8, \dots$$

formula - S.2

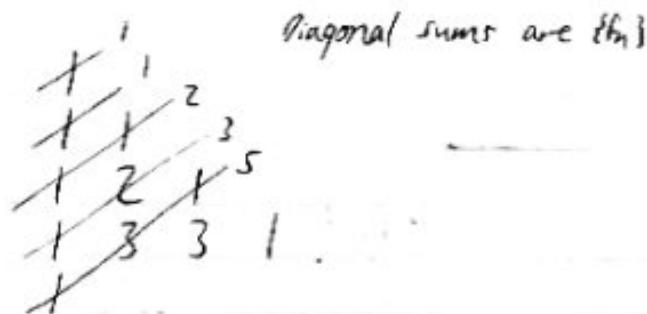
In nature:

pineapples, cacti, pinecones, sunflowers,
leaf arrangements of almost all plants
(phyllotaxis)

Conway 112

$$f_{n+2} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} \dots$$

Neat: Pascal Δ



Ex 6 (Rosen)

Let a_n = # bit strings of length n
that do not have two consecutive "0's."
No "00."

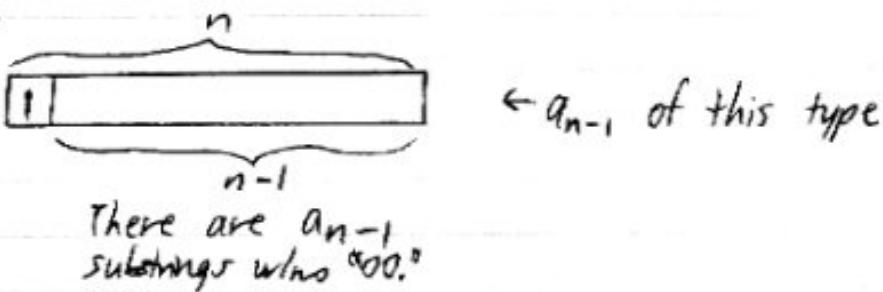
Find a recurrence relation for $\{a_n\}$

We can deal w/
the double cases
later.

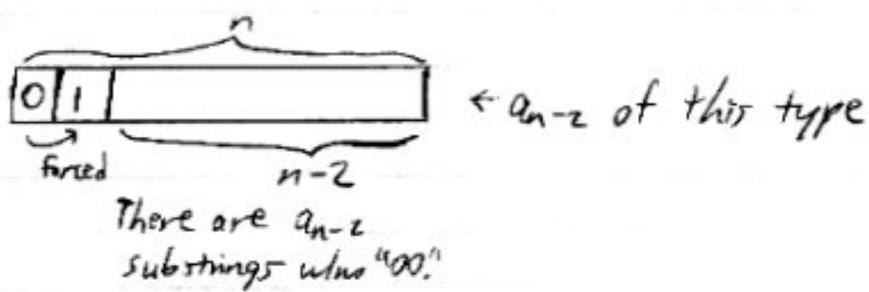
For now, assume n is large.

Count the n -bit strings w/out "00".

Case 1



Case 2



$$a_n = a_{n-1} + a_{n-2}$$

\leftarrow 2nd-order, so we
need 2 initial conditions.

Give the initial conditions

Two possible groups

Group 1

The empty string has no "00", so $a_0 = 1$.

① and ② have no "00", so $a_1 = 2$.

$$\begin{cases} a_0 = 1 \\ a_1 = 2 \end{cases} \leftarrow \text{The "n=0" case fits into our argument.}$$

Group 2

If you're unsure whether or not the "n=0" case fits in.

$$\begin{cases} a_1 = 2 \\ a_2 = 3 \end{cases} (\varnothing, 01, 10, 11)$$

How can we define $\{a_n\}$?

$$\begin{cases} a_0 = 1 \\ a_1 = 2 \\ a_n = a_{n-1} + a_{n-2} \quad (n \geq 2) \end{cases}$$

"Back-of-book"
method
(They try to incorporate n=0 case.)

$$\begin{cases} (a_0 = 1) \\ a_1 = 2 \\ a_2 = 3 \\ a_n = a_{n-1} + a_{n-2} \quad (n \geq 3) \end{cases}$$

can collapse
"Ex 6"
method

$\frac{n}{(0)}$	a_n (1)	f_n (2)
1	2	1
2	3	1
3	5	2
4	8	3
5	13	5

$\{a_n\}$ is just a shifted Fibonacci sequence

$$a_n = f_{n+2} \quad (n \geq 0)$$

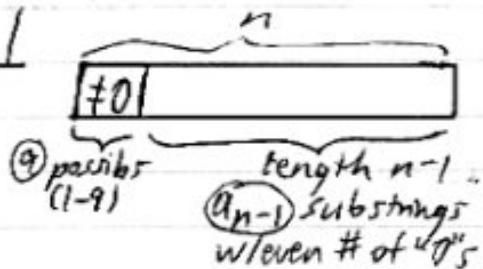
Ex 7 (Rosen)

let $a_n = \#$ of valid n -digit codewords

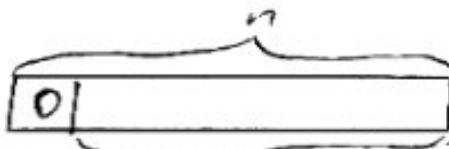
has an even
 # of "0's"
 not necessarily
 consecutive

What might
be a good
way to split?

Case 1



$\leftarrow 9a_{n-1}$ of this type

Case 2

We need this substring to have an odd # of "0's" (invalid).

How many $(n-1)$ -digit strings are there?

$$\frac{10}{1} \frac{10}{2} \dots \frac{10}{n-1} \rightarrow 10^{n-1}$$

How many of these are valid? a_{n-1}

So, there are $10^{n-1} - a_{n-1}$ invalid strings of length $n-1$.

Combine the cases:

$$\begin{aligned} a_n &= 9a_{n-1} + (10^{n-1} - a_{n-1}) \\ &= 8a_{n-1} + 10^{n-1} \quad \leftarrow \text{1st-order} \end{aligned}$$

Define $\{a_n\}$:

$$\begin{cases} a_0 = 1 & (\text{empty string has zero '0's}) \\ a_n = 8a_{n-1} + 10^{n-1} & (n \geq 1) \end{cases} \quad \text{or} \quad \begin{cases} (a_0 = 1) \\ a_1 = 9 \\ a_n = 8a_{n-1} + 10^{n-1} & (n \geq 2) \end{cases}$$

$\underbrace{\hspace{10em}}$
can collapse

SOLVING RECURRENCE RELATIONS

A solution of a RR is any sequence whose terms satisfy the RR.

Exercise 2

$$\text{Ex RR: } a_n = -3a_{n-1} + 4a_{n-2}$$

Show that $\{a_n\}$ is a solution if

$$\textcircled{1} \quad a_n = 1 \rightarrow \text{constant sequence } 1, 1, 1, 1, \dots$$

$$\begin{aligned} a_n &\stackrel{?}{=} -3a_{n-1} + 4a_{n-2} \\ 1 &\stackrel{?}{=} -3(1) + 4(1) \\ 1 &\stackrel{?}{=} -3 + 4 \\ 1 &\stackrel{?}{=} 1 \end{aligned} \quad \textcircled{Y}$$

$$\textcircled{2} \quad a_n = (-4)^n$$

$$\begin{aligned} a_n &\stackrel{?}{=} -3a_{n-1} + 4a_{n-2} \\ (-4)^n &\stackrel{?}{=} -3(-4)^{n-1} + 4(-4)^{n-2} \\ &= (-4)^{n-2} \underbrace{[-3(-4) + 4]}_{\substack{=12+4 \\ =16}} \\ &= 16(-4)^{n-2} \\ &= (-4)^2(-4)^{n-2} \\ &= (-4)^n \end{aligned} \quad \textcircled{Y}$$

If initial conditions are given, a solution must satisfy them.

If k (=order) consecutive ~~e.g.~~ are given, there is a unique solution.

Note $\begin{cases} a_0 = 0 \\ a_n = 0, n \geq 1 \end{cases}$ No sol'n

Ex 3 Compound Interest

Exercise #8

Similar Ex World population in 1999 = 6B

Assume it grows 1.3% a year.

Let P_n = World pop. n years after 1999.

Each year, the world pop. is

$$100\% + 1.3\% = 101.3\%$$

of the previous year's pop.

$$\begin{cases} P_n = 1.013 P_{n-1} \\ P_0 = 6B \end{cases} \quad \text{determines a unique sequence}$$

"Solving" a RR usually means giving a "nice" formula for the general term (P_n here)

Iterative approach:

$P_n = 1.013 P_{n-1}$ "unwrap" this until you reach P_0

$$= (1.013)(1.013)P_{n-2}$$

$$= \underbrace{(1.013)(1.013)(1.013)}_{3 \text{ copies}} P_{n-3}$$

$$= \underbrace{(1.013)(1.013) \cdots (1.013)}_{n \text{ copies}} P_0$$

Think P_{n-n}

$$= (1.013)^n P_0$$

$$= (1.013)^n \overset{6B}{\cancel{6B}} \leftarrow \text{"explicit formula"}$$

Can verify using induction.

$$\text{Year 2020: } n=21 \quad P_{21} = (1.013)^{21} 6B$$

Ex (HW #5d)

$$\text{Solve } \begin{cases} a_n = a_{n-1} + 2n + 3 \\ a_0 = 4 \end{cases}$$

using an iterative approach

Assume n is large.

$$a_n = \underbrace{a_{n-1} + 2n + 3}_{\substack{\uparrow \\ \text{Unwrap}}}$$

$$= [a_{n-2} + 2(n-1) + 3] + 2n + 3$$

$$= a_{n-2} + \underbrace{2(n-1)}_{\substack{\uparrow \\ \text{group}}} + \underbrace{2n + 3 + 3}_{\substack{\curvearrowleft \\ \text{group}}}$$

Unwrap .

$$= [a_{n-3} + 2(n-2) + 3] + 2(n-1) + 2n + 3 + 3$$

$$= a_{n-3} + 2(n-2) + 2(n-1) + 2n + 3 + 3 + 3$$

⋮

$$= a_0 + 2(1) + 2(2) + \dots + 2n + \underbrace{3 + 3 + \dots + 3}_{\substack{\uparrow \\ n \text{ copies}}}$$

$$= 4 + 2 \underbrace{[1 + 2 + 3 + \dots + n]}_{\frac{n(n+1)}{2}} + 3n$$

$$= 4 + n(n+1) + 3n$$

$$= 4 + n^2 + n + 3n$$

$$= n^2 + 4n + 4 \quad (n \geq 0) \leftarrow \text{Formula for } a_n$$

HW Tip

From 3.2, Quiz

$$\underbrace{1+2+2^2+\dots+2^n}_{\# \text{ bit strings of length } n} = \underbrace{2^{n+1}-1}_{\# \text{ bit strings of length } n+1}$$

bit strings
of length n

bit strings
of length $n+1$
excluding 1

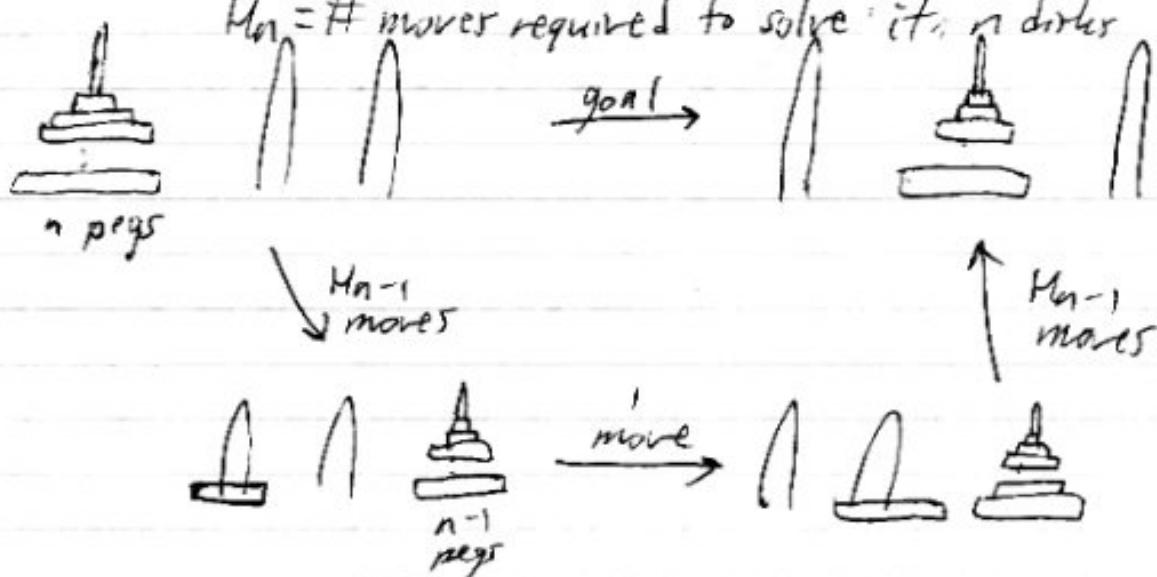
Extra Challenge: Describe a bijection.
Generalization

$$\sum_{i=0}^n r^i = 1+r+r^2+\dots+r^n = \frac{r^{n+1}-1}{r-1} \quad (r \neq 1)$$

$$\text{If } r=2, \frac{2^{n+1}-1}{2-1}$$

Ex 5 (Koren) Tower of Hanoi

$H_n = \# \text{ moves required to solve it, } n \text{ disks}$



$$H_n = 2H_{n-1} + 1, H_1 = 1 \Rightarrow H_n = 2^n - 1$$

Extra Challenge

$$\underbrace{1 + 2 + 2^2 + \dots + 2^n}_{|S|} = 2^{n+1} - 1$$

$|S|$

$S = \text{set of all bit strings of length } \leq n$

$|T|$

$T = \text{set of all bit strings of length } n+1, \text{ except } 1$

A bijective proof:

Set up a 1-1 correspondence between the strings in S and the strings in T .

Strings in T :

If it starts w/ "1"

Ex 1001011

says the rest of the string is the corresponding string in S .

$$\begin{array}{ccc} 1001011 & \longrightarrow & 001011 \\ \text{in } T & & \text{in } S \end{array}$$

If it starts w/ "0"

Ex ~~0~~001011

↑
says
we cut
the string
at the
leftmost
"0"

~~0~~001011 → 011
in T in S

Ex ($n=4$)

T (length)	↔	S (length)
<u>10000</u>	↔	0000
<u>00001</u>	↔	∅
<u>10010</u>	↔	0010
<u>00010</u>	↔	0
<u>00011</u>	↔	1
	:	
00000	← excluded string	
	but there's	
	no 1	

In general,

$$1+r+r^2+\dots+r^n = \frac{r^{n+1}-1}{r-1}$$

Extend to $r=3, 4, 5, \dots$

$$r=3: \underbrace{1+3+3^2+\dots+3^n}_{|S|} = \frac{3^{n+1}-1}{3-1} = \underbrace{\frac{1}{2}(3^{n+1}-1)}_{|T|}$$

\sum
Exclude 00...0

$$\cancel{001012} \rightarrow 012$$

$$\cancel{002012} \rightarrow$$

↑
leftmost = 0,
cut at
leftmost = 1 or 2

$$\begin{array}{r} \boxed{101012} \\ \boxed{201012} \end{array} \rightarrow 01012$$

↑
keep the rest
if leftmost =
1 or 2

$$\text{So, } |S| = \frac{1}{2} (\#(n-1)\text{-ternary strings} - 1)$$

S.2: SOLVING RR's (MORE)

A linear homogeneous RR of order (or degree) k with constant coefficients has the form

a_n is a linear
combo of the
previous terms.

$$\star \left\{ \begin{array}{l} a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \\ \text{where each } \underbrace{c_i \in \mathbb{R}}_{\text{"constant coeffs."}} \text{ and } \overbrace{c_k \neq 0}^{\text{(the RR has order } k\text{)}} \end{array} \right.$$

Ex $a_n = a_{n-2} - 4a_{n-3}$

order = 3

$c_1 = 0$ (no a_{n-1} term)

$c_2 = 1$

$c_3 = -4$

Ex $a_n = \frac{1}{a_{n-1}} + a_{n-2}$ is not linear

Ex $a_n = a_{n-1} a_{n-2}$

Ex $a_n = a_{n-1} + \underbrace{1}_{\text{extra term}}$ is not homogeneous

Ex $a_n = n^2 a_{n-1} - 3a_{n-2}$ does not have all
coefficients as constants
not a constant

We can systematically solve RRs of form (A).

Ex (Order=1)

Solve $\begin{cases} a_n = 3a_{n-1} & \text{for } n \geq 1 \\ a_0 = 2 \end{cases}$

There is a unique solution - the sequence $\{a_n\}$:

$$\begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 & & \\ 2 & 6 & 18 & 54 & \dots & \\ \xrightarrow{x3} & \xrightarrow{x3} & \xrightarrow{x3} & & & \end{array}$$

We want a nice formula for a_n .

Step 1 Rewrite RR

$$\begin{aligned} a_n &= 3a_{n-1} \\ a_n - 3a_{n-1} &= 0 \end{aligned}$$

Step 2 Replace $\begin{cases} a_n \text{ with } r \\ a_{n-1} \text{ with } 1 \end{cases}$

$$\begin{aligned} a_n - 3a_{n-1} &= 0 \\ r - 3(1) &= 0 \\ r - 3 &= 0 \end{aligned} \leftarrow \text{characteristic equation of the RR}$$

Step 3 Find the root (solution) of the char.eq.

$$\begin{aligned} r - 3 &= 0 \\ r &= 3 \end{aligned}$$

Step 4 The solutions to the RR have $a_n = \alpha r_i^n$

where $\alpha \in \mathbb{R}$

and r_i is the root of the char.eq.

Here, $(a_n = \alpha \cdot 3^n)$

A sequence $\{a_n\}$ is a solution to the RR
 \iff its a_n has this form

Step 5 Use the initial condition to solve for α

$$a_n = \alpha \cdot 3^n$$

$$n=0: a_0 = \alpha \cdot 3^0$$

$$2 = \alpha \cdot 1$$

$$\alpha = 2$$

Step 6 The unique solution $\{a_n\}$ has

$$(a_n = 2 \cdot 3^n, n \geq 0)$$

Ex (Order=2) #46

Solve $\begin{cases} a_n = 7a_{n-1} - 10a_{n-2} \text{ for } n \geq 2 \\ a_0 = 2 \\ a_1 = 1 \end{cases}$

Step 1 Rewrite RR

$$\begin{aligned} a_n &= 7a_{n-1} - 10a_{n-2} \\ a_n - 7a_{n-1} + 10a_{n-2} &= 0 \end{aligned}$$

Step 2 Replace $\begin{cases} a_n \text{ with } r^2 \\ a_{n-1} \text{ with } r \\ a_{n-2} \text{ with } 1 \end{cases}$

$$\begin{aligned} a_n - 7a_{n-1} + 10a_{n-2} &= 0 \\ r^2 - 7r + 10 &= 0 \leftarrow \text{char. eq. of RR} \end{aligned}$$

Step 3 Find the roots of the char. eq.

$$\begin{aligned} r^2 - 7r + 10 &= 0 \\ (r - 5)(r - 2) &= 0 && \begin{matrix} \text{last resort:} \\ \text{quadratic formula} \end{matrix} \\ r_1 &= 5, r_2 = 2 \end{aligned}$$

Step 4 The solutions to the RR have

$$\begin{aligned} a_n &= \alpha_1 r_1^n + \alpha_2 r_2^n \\ a_n &= \alpha_1 \cdot 5^n + \alpha_2 \cdot 2^n \end{aligned}$$

Step 5 Use the initial conditions to solve for α_1, α_2 .

$$a_n = \alpha_1 \cdot 5^n + \alpha_2 \cdot 2^n$$

$$\begin{aligned} n=0: a_0 &= \alpha_1 \cdot 5^0 + \alpha_2 \cdot 2^0 \\ &\quad \underbrace{2 = \alpha_1 + \alpha_2} \end{aligned}$$

$$\begin{aligned} n=1: a_1 &= \alpha_1 \cdot 5^1 + \alpha_2 \cdot 2^1 \\ &\quad \underbrace{1 = 5\alpha_1 + 2\alpha_2} \end{aligned}$$

Solve this system.

$$\alpha_1 = -1, \alpha_2 = 3$$

Step 6 The unique solution $\{a_n\}$ has

$$a_n = (-1) \cdot 5^n + 3 \cdot 2^n, n \geq 0$$

$$\begin{aligned} \text{Ex } a_{100} &= (-1) \cdot 5^{100} + 3 \cdot 2^{100} \\ &\approx 7.9 \times 10^{64} \end{aligned}$$

Beats iteration!!

Ex 4 ^(Kaggen) Fibonacci numbers

$$\begin{cases} f_n = f_{n-1} + f_{n-2} & (n \geq 2) \\ f_0 = 0 \\ f_1 = 1 \end{cases}$$

$$\Rightarrow f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \underbrace{\frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n}_{\approx -0.62} \quad \begin{array}{l} \nearrow 0 \text{ as } n \rightarrow \infty \\ \left| \frac{f_{n+1}}{f_n} \rightarrow \frac{1+\sqrt{5}}{2} \right. \end{array}$$

golden ratio"

Ex (Order = 2, repeated roots) #4f

Solve $\begin{cases} a_n = -6a_{n-1} - 9a_{n-2} & \text{for } n \geq 2 \\ a_0 = 3 \\ a_1 = -3 \end{cases}$

$$a_n = -6a_{n-1} - 9a_{n-2}$$

$$\textcircled{1} \quad a_n + 6a_{n-1} + 9a_{n-2} = 0$$

$$\textcircled{2} \quad r^2 + 6r + 9 = 0$$

$$\textcircled{3} \quad (r + 3)^2 = 0$$

$r = -3$ is the sole,
"repeated" root.

\textcircled{4} If r_1 is a repeated root,
the solutions to the RR have

$$a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n \quad (\alpha_1, \alpha_2 \in \mathbb{R})$$

Here, $(a_n = \alpha_1 (-3)^n + \alpha_2 n (-3)^n)$

$$\textcircled{5} \quad a_n = \alpha_1 (-3)^n + \alpha_2 n (-3)^n$$

$$n=0: a_0 = \alpha_1 (-3)^0 + \alpha_2 (0) (-3)^{0^+}$$

$$3 = \alpha_1 (1)$$

$$\alpha_1 = 3$$

$$n=1: a_1 = \alpha_1 (-3)^1 + \alpha_2 (1) (-3)^1$$

$$-3 = -3\alpha_1 - 3\alpha_2 \quad | :(-3)$$

$$1 = \alpha_1 + \alpha_2$$

$$1 = 3 + \alpha_2$$

$$\alpha_2 = -2$$

\textcircled{6} Solution:

$$a_n = 3(-3)^n - 2n(-3)^n, n \geq 0$$

Order k , k distinct roots (not tested)

② Replace $\begin{cases} a_n \text{ with } r^k \\ a_{n-1} \text{ with } r^{k-1} \\ \vdots \\ a_{n-k} \text{ with } 1 \end{cases}$

④ If the k distinct roots of the char. eq. are r_1, r_2, \dots, r_k , the solutions to the RR have

$$\begin{aligned} a_n &= \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_k r_k^n, \text{ each } \alpha_i \in \mathbb{R} \\ &= \sum_{i=1}^k \alpha_i r_i^n \quad \leftarrow \text{linear combos of the } n^{\text{th}} \text{ power of the roots} \end{aligned}$$

⑤ We get a system of k linear eqs. in k unknowns (α_i 's).

Optional: Read Ex 6

Order k , repeated roots (not tested)

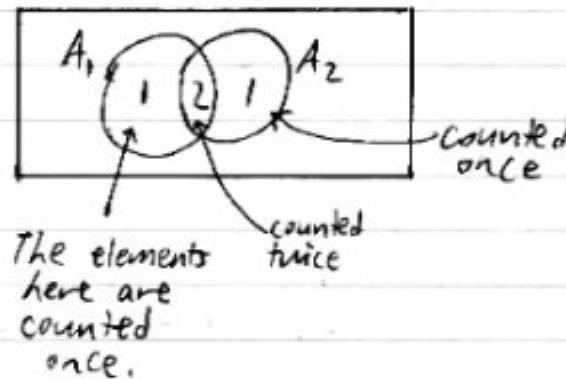
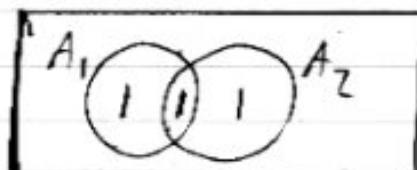
p. 325 - Thm 4

Ex char. eq. $\rightarrow (r-3)^3 (r-4)^2$

$$\begin{aligned} a_n &= [\alpha_1 \cdot 3^n + \alpha_2 n \cdot 3^n + \alpha_3 n^2 \cdot 3^n] + [\alpha_4 \cdot 4^n + \alpha_5 n \cdot 4^n] \\ &= [\alpha_1 + \alpha_2 n + \alpha_3 n^2] 3^n + [\alpha_4 + \alpha_5 n] 4^n \end{aligned}$$

S.S: INCLUSION-EXCLUSION $n=2$ setsLet A_1, A_2 be finite sets.

Then, $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
 $= 0$ if A_1, A_2 disjoint

If we just count $|A_1| + |A_2|$:When we $- |A_1 \cap A_2|$ 

Now, all the elements in $A_1 \cup A_2$ are counted once:

Ex To take a discrete math class, a student must be a math major or a CS major. The class has 50 students. 30 are math majors. 40 are CS majors. How many students are joint math/CS majors?

$$\text{Let } M = \{\text{math majors}\}$$

$$\text{Let } C = \{\text{CS majors}\}$$

$$\Rightarrow M \cap C = \{\text{joint math/CS majors}\}$$

formula:

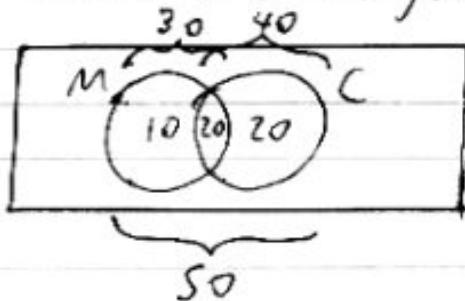
$$|M \cup C| = |M| + |C| - |M \cap C|$$

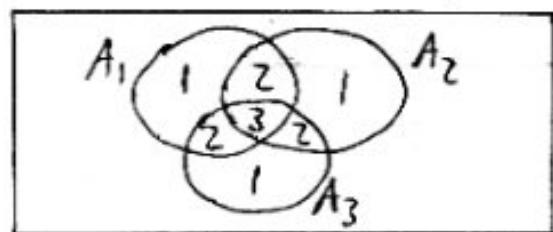
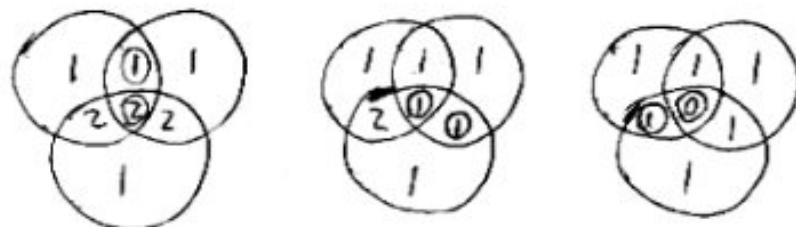
$$50 = 30 + 40 - |M \cap C|$$

$$50 = 70 - x$$
$$x = 20$$

$$|M \cap C| = 20$$

There are 20 joint math/CS majors.



$n=3$ setsLet A_1, A_2, A_3 be finite sets.How do we find $|A_1 \cup A_2 \cup A_3|$?Start with $|A_1| + |A_2| + |A_3|$ See figure 4
(p. 356)Now, $- |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3|$ We need $+ |A_1 \cap A_2 \cap A_3|$ 

$$\begin{aligned}|A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| \\&\quad - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| \\&\quad + |A_1 \cap A_2 \cap A_3|\end{aligned}$$

For general, $n \in \mathbb{Z}^+$

Let A_1, A_2, \dots, A_n be finite sets.

$$\begin{aligned}|A_1 \cup A_2 \cup \dots \cup A_n| &= |A_1| + |A_2| + \dots + |A_n| \\&\quad - |A_1 \cap A_2| - |A_1 \cap A_3| - \dots && \leftarrow \binom{n}{2} \text{ pairs of sets} \\&\quad + |A_1 \cap A_2 \cap A_3| + \dots && \leftarrow \binom{n}{3} \text{ triples of sets} \\&\quad \vdots \\&\quad + (-1)^{n+1} |A_1 \cap A_2 \cap \dots \cap A_n| && \leftarrow \binom{n}{n} = 1 \text{ } n\text{-set}\end{aligned}$$

p. 361

Ex (Sieve of Eratosthenes)

How many prime numbers are there between 1 and 47, inclusive?

By the "5n rule," composite numbers are
divisible by 2, 3, or 5.

How many integers between 1 and 47, inclusive, are divisible by 2, 3, or 5?

Let $D_2 = \{x | x \in \mathbb{Z}, 1 \leq x \leq 47, 2|x\}$
 Let $D_3 = \{x | 1 \leq x \leq 47, 3|x\}$
 Let $D_5 = \{x | 1 \leq x \leq 47, 5|x\}$

$$|\mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_5|$$

$$= |D_2| + |D_3| + |D_5|$$

$$= |D_2 \cap B| - |D_3 \cap D_5| - |D_3 \cap D_5|$$

$$+ |D_2 \cap D_3 \cap D_5|$$

Rule: $|D_i \cap D_j| = |\text{lcm}(i, j)|$ if i, j are relatively prime

$$= |D_2| + |D_3| + |D_5|$$

$$= |D_6| - |D_{10}| - |D_{15}|$$

$$+10_{30}|$$

$$= \left\lfloor \frac{47}{2} \right\rfloor + \left\lfloor \frac{47}{3} \right\rfloor + \left\lfloor \frac{47}{5} \right\rfloor$$

$$- \left\lfloor \frac{47}{6} \right\rfloor - \left\lfloor \frac{47}{10} \right\rfloor - \left\lfloor \frac{47}{15} \right\rfloor$$

$$+ \left\lfloor \frac{47}{30} \right\rfloor$$

$$= 23 + 15 + 9
- 7 - 4 - 3
+ 1$$

$$= 34$$

34 integers between 1, 47 are divisible by 2, 3, and 5.

So, $47 - 34 = 13$ are not.

but

- 1 (exclude "1")

+ 3 (include 2, 3, 5)

15 primes

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47