5.1: RECURRENCE RELATIONS

#13, Brualdi 284

Ex $a_n = 3a_{n-2} - 2a_{n-3}$ $(n \geq 3)$
$n \in \mathbb{Z}$ assumed

is a recurrence relation ("RR").

It has order or degree 3, because a_n is defined in terms of the previous 3 terms (except a_{n-1} - OK).
i.e., To compute a_n, we must "look back" up to three places before it.

$$\left.\begin{array}{l} a_0 = 1 \\ a_1 = 0 \\ a_2 = 0 \end{array}\right\} \text{Initial conditions / values} \quad \text{"Seeds"}$$

These specify the terms that immediately precede the case (here, $n=3$) where the RR "begins" or "takes effect."

If the RR has order $= k$ (here, $k=3$), we must specify k initial conditions in order to determine a unique sequence $\{a_n\}$.

⋮

$$\left.\begin{cases} a_n = 3a_{n-2} - 2a_{n-3} \quad (n \geq 3) \\ a_0 = 1 \\ a_1 = 0 \\ a_2 = 0 \end{cases}\right\} \rightarrow \text{determine a unique sequence}$$

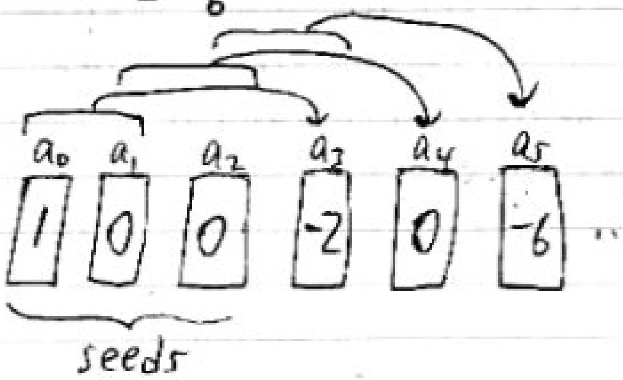
Find the first six terms of this sequence.

$a_0 = 1$
$a_1 = 0$
$a_2 = 0$

$$\begin{aligned} a_3 &= 3a_1 - 2a_0 \quad \leftarrow n=3 \text{ case: RR "begins"} \\ &= 3(0) - 2(1) \\ &= -2 \end{aligned}$$

$$\begin{aligned} a_4 &= 3a_2 - 2a_1 \\ &= 3(0) - 2(0) \\ &= 0 \end{aligned}$$

$$\begin{aligned} a_5 &= 3a_3 - 2a_2 \\ &= 3(-2) - 2(0) \\ &= -6 \end{aligned}$$



<u>Ex 4</u> (Rosen) Breeding Rabbits and the Fibonacci Numbers $\{f_n\}$

$(f_0 = 0$ - optional$)$

$f_1 = 1$

$f_2 = 1$

$f_n = f_{n-1} + f_{n-2} \quad (n \geq 3)$ ← 2nd-order RR

$\{f_n\}$: 1, 1, 2, 3, 5, 8, ...

Formula - 5.2

In nature:

pineapples, cacti, pinecones, sunflowers, leaf arrangements of almost all plants (phyllotaxis)

Conway 112
$f_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} \ldots$

Neat: Pascal Δ

Diagonal sums are $\{f_n\}$

1				1
1	1			1
1	2	1		2
1	3	3	1	3
1				5

Ex 6 (Rosen)

Let a_n = # bit strings of length n that do not have two consecutive "0"s. NO "00."

Find a recurrence relation for $\{a_n\}$

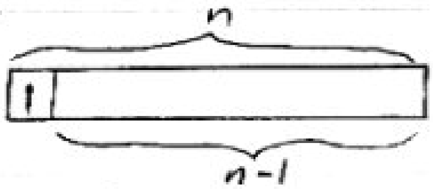
we can deal w/ the dinky cases later.

For now, assume n is large.

Count the n-bit strings w/out "00".

Rosen focuses on right end - by symmetry, it doesn't matter

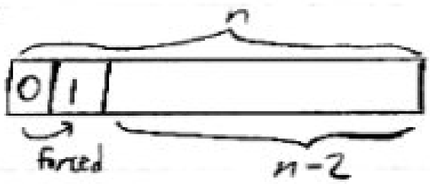
Case 1



← a_{n-1} of this type

There are a_{n-1} substrings w/no "00."

Case 2



← a_{n-2} of this type

There are a_{n-2} substrings w/no "00."

$$\boxed{a_n = a_{n-1} + a_{n-2}}$$

← 2nd-order, so we need 2 initial conditions.

Give the initial conditions

Two possible groups

Group 1

The empty string has no "00", so $a_0 = 1$.

$\boxed{0}$ and $\boxed{1}$ have no "00", so $a_1 = 2$.

$\begin{cases} a_0 = 1 \\ a_1 = 2 \end{cases}$ ← The "$n=0$" case fits into our argument.

Group 2

If you're unsure whether or not the "$n=0$" case fits in.

$\begin{cases} a_1 = 2 \\ a_2 = 3 \end{cases}$ $(\cancel{00}, 01, 10, 11)$

How can we define $\{a_n\}$?

$\begin{cases} a_0 = 1 \\ a_1 = 2 \\ a_n = a_{n-1} + a_{n-2} \ (n \geq 2) \end{cases}$ or $\begin{cases} (a_0 = 1) \\ a_1 = 2 \\ a_2 = 3 \\ a_n = a_{n-1} + a_{n-2} \ (n \geq 3) \end{cases}$

"Back-of-book" method
(They try to incorporate $n=0$ case.)

can collapse

"Ex 6" method

n	a_n	f_n
(0)	(1)	(0)
1	2	1
2	3	1
3	5	2
4	8	3
5	13	5

$\{a_n\}$ is just a shifted fibonacci sequence

$$a_n = f_{n+2} \quad (n \geq 0)$$

Ex 7 (Rosen)

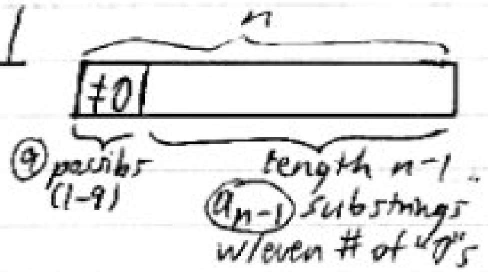
Let a_n = # of valid n-digit codewords

has an even # of "0"s not necessarily consecutive

0-9

What might be a good way to split?

Case 1



$\leftarrow 9a_{n-1}$ of this type

<u>Case 2</u>

$$\overbrace{\boxed{0}\underbrace{\boxed{\qquad\qquad\qquad\qquad\qquad}}_{\text{length } n-1}}^{n}$$

We need this substring to have an odd # of "0"'s (invalid).

How many $(n-1)$-digit strings are there?

$$\underset{1}{\underline{10}}\ \underset{2}{\underline{10}}\ \cdots\ \underset{n-1}{\underline{10}} \rightarrow 10^{n-1}$$

How many of these are <u>valid</u>? a_{n-1}

So, there are $10^{n-1} - a_{n-1}$ invalid strings of length $n-1$.

Combine the cases:

$$a_n = 9a_{n-1} + (10^{n-1} - a_{n-1})$$
$$= 8a_{n-1} + 10^{n-1} \quad \leftarrow \text{1st-order}$$

Define $\{a_n\}$:

$$\begin{cases} a_0 = 1 & \text{(empty string has zero "0"'s)} \\ a_n = 8a_{n-1} + 10^{n-1} & (n \geq 1) \end{cases} \quad \underline{\text{or}} \quad \begin{cases} (a_0 = 1) \\ a_1 = 9 & (\boxed{1}\boxed{2}\cdots\boxed{9}) \\ a_n = 8a_{n-1} + 10^{n-1} & (n \geq 2) \end{cases}$$

can collapse

SOLVING RECURRENCE RELATIONS

A solution of a RR is any sequence whose terms satisfy the RR.

Exercise 2

Ex RR: $a_n = -3a_{n-1} + 4a_{n-2}$

Show that $\{a_n\}$ is a solution if

① $a_n = 1$ → Constant sequence $1, 1, 1, 1, \ldots$

$$\begin{aligned} a_n &\stackrel{?}{=} -3a_{n-1} + 4a_{n-2} \\ 1 &\stackrel{?}{=} -3(1) + 4(1) \\ 1 &\stackrel{?}{=} -3 + 4 \\ 1 &\stackrel{?}{=} 1 \end{aligned}$$

Y

② $a_n = (-4)^n$

$$\begin{aligned} a_n &\stackrel{?}{=} -3a_{n-1} + 4a_{n-2} \\ (-4)^n &\stackrel{?}{=} -3(-4)^{n-1} + 4(-4)^{n-2} \\ &= (-4)^{n-2}\underbrace{[-3(-4) + 4]}_{\substack{=12+4 \\ =16}} \\ &= 16(-4)^{n-2} \\ &= (-4)^2(-4)^{n-2} \\ &= (-4)^n \end{aligned}$$

Y

If initial conditions are given, a solution must satisfy them.
If k (= order) consecutive ICs are given, there is a unique solution.

Note $\begin{cases} a_0 = 0 \\ a_n = \frac{1}{a_{n-1}} \ (n \geq 1) \end{cases}$ No sol'n

Ex 3 Compound Interest

Exercise #8 Similar Ex World population in 1999 = 6B
Assume it grows 1.3% a year.
Let P_n = World pop. n years after 1999.

Each year, the world pop. is
$100\% + 1.3\% = 101.3\%$
of the previous year's pop.

$$\left\{\begin{array}{l} P_n = 1.013\,P_{n-1} \\ P_0 = 6B \end{array}\right\} \text{determines a unique sequence}$$

"Solving" a RR usually means giving a "nice" formula for the general term (P_n here)

Iterative approach:

"unwrap" this until you reach P_0

$$\begin{aligned} P_n &= 1.013\,P_{n-1} \\ &= (1.013)(1.013\,P_{n-2}) \\ &= \underbrace{(1.013)(1.013)(1.013)}_{3 \text{ copies}}P_{n-3} \\ &\ \ \vdots \\ &= \underbrace{(1.013)(1.013)\cdots(1.013)}_{n \text{ copies}}P_0 \quad \text{Think } P_{n-n} \\ &= (1.013)^n \underbrace{P_0}_{6B} \\ &= (1.013)^n\,6B \leftarrow \text{"explicit formula"} \end{aligned}$$

Can verify using induction.

Year 2020: $n = 21$ $P_{21} = (1.013)^{21}\,6B$

<u>Ex</u> (HW #5d)

Solve $\begin{cases} a_n = a_{n-1} + 2n + 3 \\ a_0 = 4 \end{cases}$

using an iterative approach

Assume n is large.

$$a_n = \underbrace{a_{n-1}} + 2n + 3$$

$$= [a_{n-2} + 2(n-1) + 3] + 2n + 3$$

$$= a_{n-2} + \underbrace{2(n-1) + 2n}_{\text{group}} + \underbrace{3 + 3}_{\text{group}}$$

Unwrap

$$= [a_{n-3} + 2(n-2) + 3] + 2(n-1) + 2n + 3 + 3$$

$$= a_{n-3} + 2(n-2) + 2(n-1) + 2n + 3 + 3 + 3$$

$$\vdots$$

$$= \underbrace{a_0}_{4} + 2(1) + 2(2) + \ldots + 2n + \underbrace{3 + 3 + \ldots + 3}_{n \text{ copies}}$$

$$= 4 + 2\,[\underbrace{1 + 2 + 3 + \ldots + n}_{\frac{n(n+1)}{2}}] + 3n$$

$$= 4 + n(n+1) + 3n$$

$$= 4 + n^2 + n + 3n$$

$$= n^2 + 4n + 4 \quad (n \geq 0) \leftarrow \text{Formula for } a_n$$

HW Tip

From 3.2, Quiz

$$\underbrace{1+2+2^2+\dots+2^n}_{\text{\# bit strings of length } \le n} = \underbrace{2^{n+1}-1}_{\text{\# bit strings of length } n+1 \text{ excluding } 1'}$$

Extra Challenge: Describe a bijection.

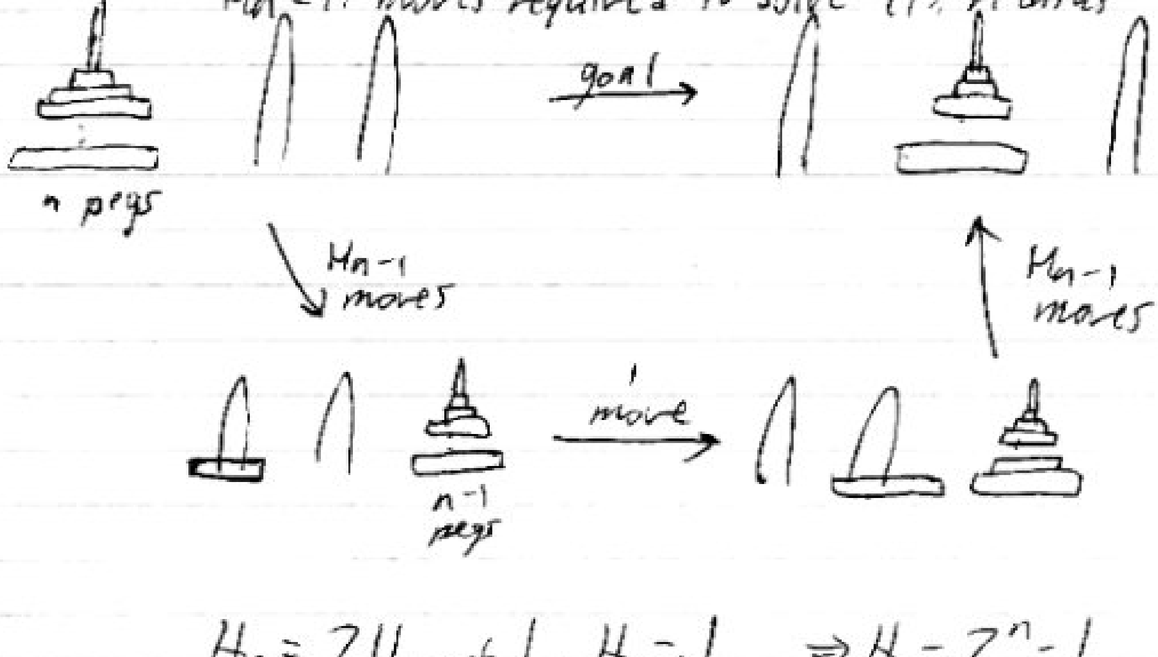
Generalization

$$\sum_{i=0}^{n} r^i = 1+r+r^2+\dots+r^n = \frac{r^{n+1}-1}{r-1} \quad (r\neq 1)$$

$$\text{If } r=2, \quad \frac{2^{n+1}-1}{2-1}$$

Ex 5 (Rosen) Tower of Hanoi

H_n = # moves required to solve it, n disks



$$H_n = 2H_{n-1}+1,\ H_1 = 1 \Rightarrow H_n = 2^n - 1$$

Extra Challenge

$$\underbrace{1+2+2^2+\ldots+2^n}_{|S|} = \underbrace{2^{n+1}-1}_{|T|}$$

S = set of all bit strings of length $\leq n$

T = set of all bit strings of length $n+1$, except 1

A bijective proof:

Set up a 1-1 correspondence between the strings in S and the strings in T.

Strings in T:

If it starts w/ "1"

Ex $\boxed{1}\underbrace{001011}$

says the rest of the string is the corresponding string in S.

$$\underset{\text{in } T}{1001011} \longrightarrow \underset{\text{in } S}{001011}$$

If it starts w/ "0"

Ex $\boxed{0}$001011

says we cut the string at the leftmost "1"

~~0001~~011 → 011

in T   in S

Ex (n=4)

T (length 5)		S (length ≤ 4)
1$\boxed{0000}$	⟷	0000
~~0000~~1	⟷	ϕ
1$\boxed{0010}$	⟷	0010
~~0001~~0	⟷	0
~~0001~~1	⟷	1
	⋮	

00000 ← excluded string

^but there's no 1

In general,

$$1+r+r^2+\dots+r^n = \frac{r^{n+1}-1}{r-1}$$

Extend to $r=3,4,5,\dots$

$$r=3: \underbrace{1+3+3^2+\dots+3^n}_{|S|} = \frac{3^{n+1}-1}{3-1} = \underbrace{\tfrac{1}{2}(3^{n+1}-1)}_{|T|}$$

I   S

Exclude $00\cdots0$

$\cancel{001}012 \rightarrow 012$

$\cancel{002}012 \nearrow$

leftmost=0,
cut at
leftmost=1 or 2

$1\boxed{01012} \rightarrow 01012$

$2\boxed{01012} \nearrow$

keep the rest
if leftmost=
1 or 2

So, $|S| = \frac{1}{2}(\#(n-1)\text{-ternary strings} - 1)$