## 5.2: SOLVING RRs (MORE)

A linear homogeneous RR of order (or degree) k with constant coefficients has the form

$a_n$ is a linear combo of the previous terms.

$$\bigstar \begin{cases} a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \\ \text{where each } \underbrace{c_i \in \mathbb{R}}_{\text{("constant coeffs")}} \text{ and } c_k \neq 0 \text{ (the RR has order k)} \end{cases}$$

Ex $a_n = a_{n-2} - 4a_{n-3}$

order = 3
$c_1 = 0$ (no $a_{n-1}$ term)
$c_2 = 1$
$c_3 = -4$

Ex $a_n = \frac{1}{a_{n-1}} + a_{n-2}^2$ is not linear
Ex $a_n = a_{n-1} a_{n-2}$
Ex $a_n = a_{n-1} + \underbrace{1}_{\text{extra term}}$ is not homogeneous

Ex $a_n = n^2 a_{n-1} - 3a_{n-2}$ does not have all coefficients as constants

not a constant

We can systematically solve RRs of form (☆).

Ex (Order = 1)

Solve $\begin{cases} a_n = 3a_{n-1} & \text{for } n \geq 1 \\ a_0 = 2 \end{cases}$

There is a unique solution – the sequence $\{a_n\}$:

$$\begin{array}{cccccccc} a_0 & & a_1 & & a_2 & & a_3 & \\ 2 & \xrightarrow{\times 3} & 6 & \xrightarrow{\times 3} & 18 & \xrightarrow{\times 3} & 54 & \cdots \end{array}$$

We want a nice formula for $a_n$.

Step 1 Rewrite RR

$$a_n = 3a_{n-1}$$
$$a_n - 3a_{n-1} = 0$$

Step 2 Replace $\begin{cases} a_n \text{ with } r \\ a_{n-1} \text{ with } 1 \end{cases}$

$$a_n - 3a_{n-1} = 0$$
$$r - 3(1) = 0$$
$$r - 3 = 0$$ ← characteristic equation of the RR

Step 3 Find the root (solution) of the char. eq.

$$r - 3 = 0$$
$$r = 3$$

<u>Step 4</u> The solutions to the RR have $a_n = \alpha r_1^n$

where $\alpha \in \mathbb{R}$
and $r_1$ is the root of the char. eq.

Here, $a_n = \alpha \cdot 3^n$

A sequence $\{a_n\}$ is a solution to the RR $\longleftrightarrow$ its $a_n$ has this form

<u>Step 5</u> Use the initial condition to solve for $\alpha$

$$a_n = \alpha \cdot 3^n$$
$$n=0: \; a_0 = \alpha \cdot 3^0$$
$$2 = \alpha \cdot 1$$
$$\alpha = 2$$

<u>Step 6</u> The unique solution $\{a_n\}$ has

$$a_n = 2 \cdot 3^n, \; n \geq 0$$

Ex (Order=2) #46

Solve $\begin{cases} a_n = 7a_{n-1} - 10a_{n-2} \text{ for } n \geq 2 \\ a_0 = 2 \\ a_1 = 1 \end{cases}$

Step 1 Rewrite RR

$$a_n = 7a_{n-1} - 10a_{n-2}$$
$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$

Step 2 Replace $\begin{cases} a_n \text{ with } r^2 \\ a_{n-1} \text{ with } r \\ a_{n-2} \text{ with } 1 \end{cases}$

$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$
$$r^2 - 7r + 10 = 0$$ ← char. eq. of RR

Step 3 Find the roots of the char. eq.

$$r^2 - 7r + 10 = 0$$
$$(r-5)(r-2) = 0$$
$$r_1 = 5, \; r_2 = 2$$

} last resort: quadratic formula

Step 4 The solutions to the RR have

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$
$$a_n = \alpha_1 \cdot 5^n + \alpha_2 \cdot 2^n$$

<u>Step 5</u> Use the initial conditions to solve for $\alpha_1, \alpha_2$.

$$a_n = \alpha_1 \cdot 5^n + \alpha_2 \cdot 2^n$$

$n=0$: $a_0 = \alpha_1 \cdot 5^0 + \alpha_2 \cdot 2^0$
$2 = \alpha_1 + \alpha_2$

$n=1$: $a_1 = \alpha_1 \cdot 5^1 + \alpha_2 \cdot 2^1$
$1 = 5\alpha_1 + 2\alpha_2$

Solve this system.

$\alpha_1 = -1, \alpha_2 = 3$

<u>Step 6</u> The unique solution $\{a_n\}$ has

$$a_n = (-1) \cdot 5^n + 3 \cdot 2^n, \; n \geq 0$$

<u>Ex</u> $a_{100} = (-1) \cdot 5^{100} + 3 \cdot 2^{100}$
$\approx 7.9 \times 10^{69}$

Beats iteration!!

<u>Ex 4</u> (Rosen) Fibonacci numbers

$$\begin{cases} f_n = f_{n-1} + f_{n-2} \quad (n \geq 2) \\ f_0 = 0 \\ f_1 = 1 \end{cases}$$

$$\Rightarrow f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \overbrace{\frac{1}{\sqrt{5}}\left(\underbrace{\frac{1-\sqrt{5}}{2}}_{\approx -0.62}\right)^n}^{\to 0 \text{ as } n \to \infty}$$

$\frac{f_{n+1}}{f_n} \to \frac{1+\sqrt{5}}{2}$ "golden ratio"

Ex (Order = 2, repeated roots) #4f

Solve $\begin{cases} a_n = -6a_{n-1} - 9a_{n-2} & \text{for } n \geq 2 \\ a_0 = 3 \\ a_1 = -3 \end{cases}$

$$a_n = -6a_{n-1} - 9a_{n-2}$$

① $a_n + 6a_{n-1} + 9a_{n-2} = 0$

② $r^2 + 6r + 9 = 0$

③ $(r+3)^2 = 0$

$r = -3$ is the sole, "repeated" root.

④ If $r_1$ is a repeated root, the solutions to the RR have

$$a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n \quad (\alpha_1, \alpha_2 \in \mathbb{R})$$

Here, $\boxed{a_n = \alpha_1 (-3)^n + \alpha_2 n (-3)^n}$

(5) $a_n = \alpha_1(-3)^n + \alpha_2 n(-3)^n$

$n=0: a_0 = \alpha_1(-3)^0 + \cancel{\alpha_2(0)(-3)^{n}}\, 0$

$3 = \alpha_1(1)$

$\alpha_1 = 3$

$n=1: a_1 = \alpha_1(-3)^1 + \alpha_2(1)(-3)^1$

$-3 = -3\alpha_1 - 3\alpha_2 \quad )\div(-3)$

$1 = \alpha_1 + \alpha_2$

$1 = 3 + \alpha_2$

$\alpha_2 = -2$

(6) Solution:

$a_n = 3(-3)^n - 2n(-3)^n, n \geq 0$

Order k, k distinct roots (not tested)

② Replace $\begin{cases} a_n \text{ with } r^k \\ a_{n-1} \text{ with } r^{k-1} \\ \vdots \\ a_{n-k} \text{ with } 1 \end{cases}$

④ If the k distinct roots of the char. eq. are $r_1, r_2, \ldots, r_k$, the solutions to the RR have

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n \text{, each } \alpha_i \in \mathbb{R}$$

$$= \sum_{i=1}^{k} \alpha_i r_i^n$$ ← linear combo of the $n^{th}$ powers of the roots

⑤ We get a system of k linear eqs. in k unknowns ($\alpha_i$'s).

Optional: Read Ex 6

Order k, repeated roots (not tested)

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Ex char. eq. → $(r-3)^3 (r-4)^2$

$$a_n = [\alpha_1 \cdot 3^n + \alpha_2 n \cdot 3^n + \alpha_3 n^2 \cdot 3^n] + [\alpha_4 \cdot 4^n + \alpha_5 \cdot n 4^n]$$
$$= [\alpha_1 + \alpha_2 n + \alpha_3 n^2] 3^n + [\alpha_4 + \alpha_5 n] 4^n$$