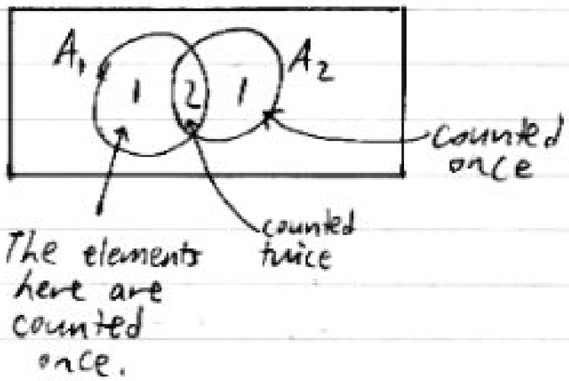
5.5: INCLUSION-EXCLUSION

n=2 sets

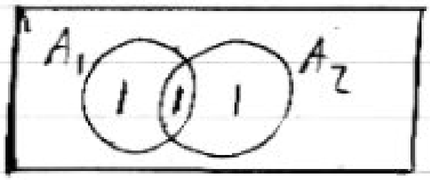
Let A_1, A_2 be finite sets.

Then, $|A_1 \cup A_2| = |A_1| + |A_2| - \underbrace{|A_1 \cap A_2|}_{=0 \text{ if } A_1, A_2 \text{ disjoint}}$

If we just count $|A_1| + |A_2|$:



When we $-|A_1 \cap A_2|$



Now, all the elements in $A_1 \cup A_2$ are counted once.

Ex To take a discrete math class, a student must be a math major or a CS major.
The class has 50 students.
30 are math majors.
40 are CS majors.
How many students are joint math/CS majors?

Let $M = \{\text{math majors}\}$
Let $C = \{\text{CS majors}\}$
$\Rightarrow M \cap C = \{\text{joint math/CS majors}\}$

Formula:

$$|M \cup C| = |M| + |C| - |M \cap C|$$

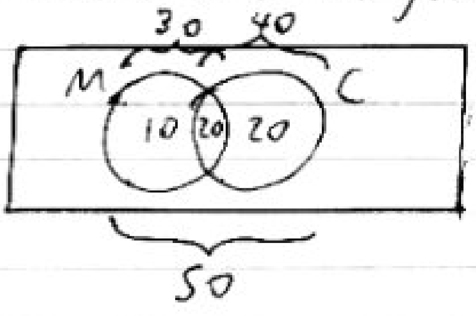
$$50 = 30 + 40 - \underbrace{|M \cap C|}_{\text{"}x\text{"} = \text{answer}}$$

$$50 = 70 - x$$

$$x = 20$$

$$|M \cap C| = 20$$

There are 20 joint math/CS majors.

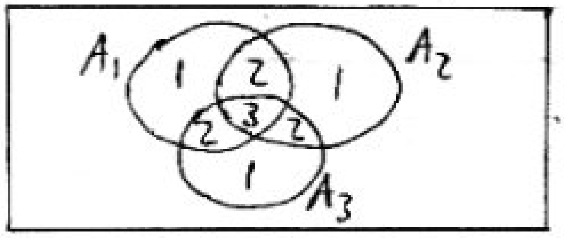


<u>$n=3$ sets</u>

Let A_1, A_2, A_3 be finite sets.

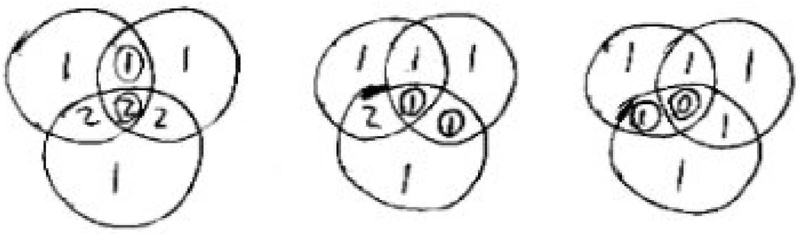
How do we find $|A_1 \cup A_2 \cup A_3|$?

Start with $|A_1| + |A_2| + |A_3|$

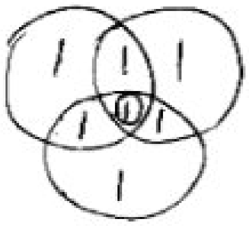


See figure 4 (p. 356)

Now, $-|A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3|$



We need $+|A_1 \cap A_2 \cap A_3|$



$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$$
$$- |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3|$$
$$+ |A_1 \cap A_2 \cap A_3|$$

<u>For general $n \in \mathbb{Z}^+$</u>

Let $A_1, A_2, \ldots, A_n$ be finite sets.

$|A_1 \cup A_2 \cup \ldots \cup A_n|$

$= |A_1| + |A_2| + \ldots + |A_n|$

$- |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots$ $\leftarrow \binom{n}{2}$ pairs of sets

$+ |A_1 \cap A_2 \cap A_3| + \ldots$ $\leftarrow \binom{n}{3}$ triples of sets

$\vdots$

$+ (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$ $\leftarrow \binom{n}{n} = 1$ n-set p361 **Ex** (Sieve of Eratosthenes)

How many prime numbers are there between 1 and 47, inclusive?

By the "$\sqrt{n}$ rule," composite numbers are divisible by 2, 3, or 5.

How many integers between 1 and 47, inclusive, are divisible by 2, 3, or 5?

Let $D_2 = \{x \mid x \in \mathbb{Z}, 1 \le x \le 47, 2 \mid x\}$
Let $D_3 = \{x \mid$ '' '' $, 3 \mid x\}$
Let $D_5 = \{x \mid$ '' '' $, 5 \mid x\}$

$|D_2 \cup D_3 \cup D_5|$

$= |D_2| + |D_3| + |D_5|$

$- |D_2 \cap D_3| - |D_2 \cap D_5| - |D_3 \cap D_5|$

$+ |D_2 \cap D_3 \cap D_5|$

Rule: $|D_i \cap D_j| = |D_{lcm(i,j)}| \underset{\text{if } i,j \text{ rel. prime}}{=\!=} |D_{ij}|$

$= |D_2| + |D_3| + |D_5|$

$- |D_6| - |D_{10}| - |D_{15}|$

$+ |D_{30}|$

$$= \left\lfloor \frac{47}{2} \right\rfloor + \left\lfloor \frac{47}{3} \right\rfloor + \left\lfloor \frac{47}{5} \right\rfloor$$

$$- \left\lfloor \frac{47}{6} \right\rfloor - \left\lfloor \frac{47}{10} \right\rfloor - \left\lfloor \frac{47}{15} \right\rfloor$$

$$+ \left\lfloor \frac{47}{30} \right\rfloor$$

$$= 23 + 15 + 9$$
$$- 7 - 4 - 3$$
$$+ 1$$

$$= 34$$

34 integers between 1, 47 are divisible by 2, 3, and 5

So, $47 - 34 = 13$ are not

<u>But</u> -1 (exclude "1")
$+3$ (include 2, 3, 5)

15 primes

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47