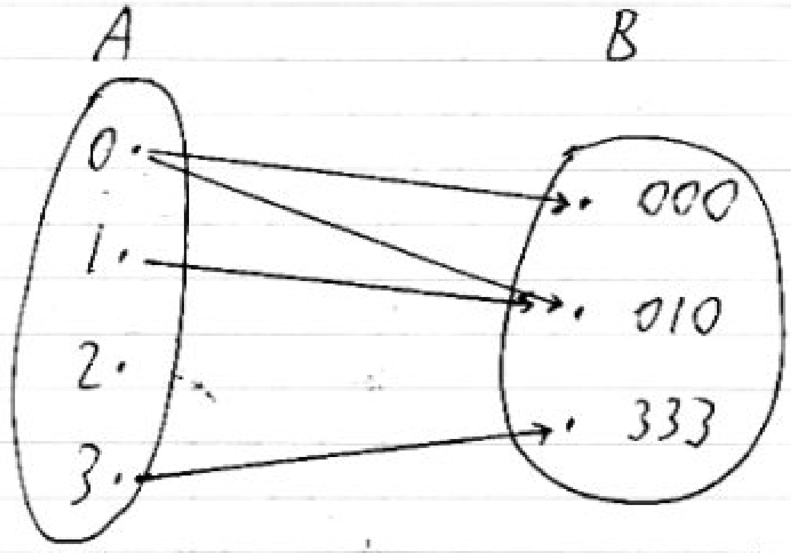
6.1

A <u>binary relation</u> (usually denoted by "R") from a set A to a set B is a subset of $A \times B$. It's a set of ordered pairs.

<u>Ex</u> $A = \{0, 1, 2, 3\}$

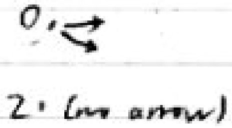
$B = \{000, 010, 333\}$
Let's say R is a relation from A to B s.t.
$(a, b) \in R \leftrightarrow$ digit a is in string b.

Picture of R:



Directed graph (digraph) - arrows on edges.

Some relations (not this one) are functions.
Why not? Our R isn't a function



Function

$R = \{ (0,000), (0,010), (1,010), (3,333) \} \subseteq A \times B$

Notation: $aRb \leftrightarrow (a,b) \in R$

$a \not R b \leftrightarrow (a,b) \notin R$

<u>Ex</u> $1 \, R \, 010$

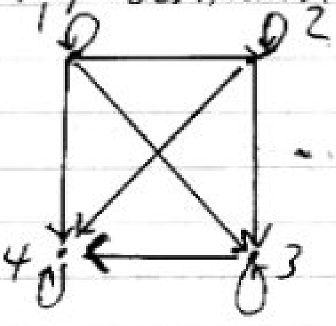
<u>Special Case</u>

R can be a relation from set A to itself.
"R is a relation on A."

<u>Ex</u> $A = \{1, 2, 3, 4\}$

$R = \{ (a_i, a_j) \mid a_i \leq a_j \}$ is a relation on A

Digraph representation of R:



$R = \{ (1,1), (1,2), (2,2),$
$(1,3), (2,3), (3,3),$
$(1,4), (2,4), (3,4), (4,4) \}$

6.2: n-ary Relations

An n-ary relation on the sets: $A_1, A_2, \ldots, A_n$ is a subset of $A_1 \times A_2 \times \ldots \times A_n$ (n = "degree").

Ex (Database of students)

Fields (A_i sets)

Last Name	First Name	ID #	Major
Jones	Paul	312	CS
Smith	John	463	CS
Smith	Paul	110	Math

Records – elements of R

The ID # field is a primary key, since there are no repeats. An ID# determines a 4-tuple.

6.3: Representing Binary Relations

1. Digraphs
2. 0-1 Matrices

Old Ex

$$
\begin{array}{cc|c}
 & & \overbrace{\begin{array}{ccc} b_1 & b_2 & b_3 \\ 000 & 010 & 333 \end{array}}^{B} \\
A & \begin{array}{c} 0 \; a_1 \\ 1 \; a_2 \\ 2 \; a_3 \\ 3 \; a_4 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

m_{ij} = entry in row i, column j
$= \begin{cases} 1 \leftrightarrow a_i R b_j \\ 0 \leftrightarrow a_i \not R b_j \end{cases}$

Old Ex

$A = \{1, 2, 3, 4\}$

$R = \{(a_i, a_j) \mid a_i \leq a_j\}$ is a relation on A

① Digraph ✓

② 0-1 Matrix Rep.

must be in same order

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6.1/6.5: Equivalence Relations

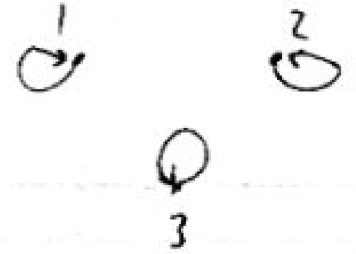
Let's say R is a relation on a set A.

R is <u>reflexive</u> $\longleftrightarrow \forall a \in A \ (a,a) \in R$
"aRa"

<u>Ex</u> $A = \{1, 2, 3\}$

R is reflexive $\leftrightarrow$ the corresponding digraph has <u>at least</u>

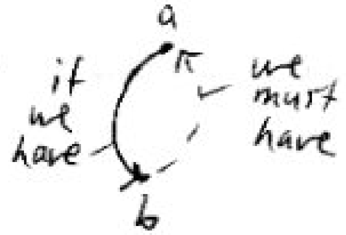
more edges are OK



Property is implied in English usage.

R is <u>symmetric</u> $\leftrightarrow [\forall a, b \in A \ (aRb \to bRa)]$

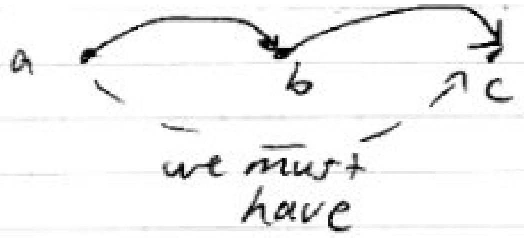
<u>Ex</u>



R is <u>transitive</u> $\leftrightarrow$

$$[\forall a,b,c \in A \ (aRb \text{ and } bRc \rightarrow aRc)]$$

<u>Ex</u> If we have



R is an <u>equivalence relation</u> $\leftrightarrow$
it is reflexive, symmetric, and transitive

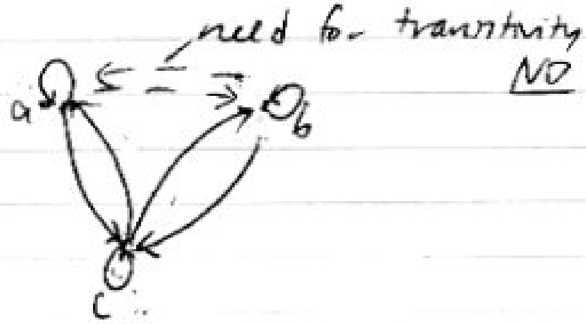
<u>Ex</u>

<u>NO</u>

not reflexive (no $\circlearrowleft_b$)

not symmetric (we have $\downarrow^a_b$, so we need $\uparrow^a_b$)

<u>Ex</u>



<u>NO</u>

In an EC, every elt is related to every other.

<u>Ex</u>

<u>YES</u>

(E.R.)
In an equivalence relation, the elements of an equivalence class are all related to one other and themselves.

<u>Ex</u> $A = \mathbb{Z}$
$R = \{(a,b) \mid a \equiv b \pmod 3\}$ is an E.R. on A.

$$\mathbb{Z}: \left| \begin{array}{c|c|c} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 0 & 1 & 2 \\ -3 & -2 & -1 \\ -6 & -5 & -4 \end{array} \right|$$

3 ECs

<u>Ex</u> A = freshmen at a college (forced into apts.)
$R = \{(a,b) \mid a$ lives in the same apt. as $b\}$ is an E.R. on A.

Why is it R?S?T?

ECs correspond to the apts.

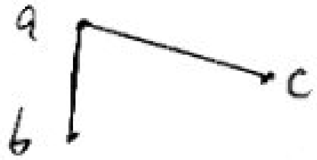
evil twin

CH 7: GRAPHS

A *graph* consists of dots (vertices) and edges that connect pairs of vertices.

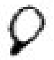
Ex $V = \{a, b, c\}$ ← $|V| = 3$ vertices
$E = \{\{a,b\}, \{a,c\}\}$ ← $|E| = 2$ edges

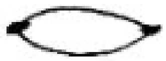
The graph $G = (V, E)$ can be represented as



a and b are adjacent vertices, because an edge connects them.
The edge $\{a,b\}$ is incident with a and b.

A simple graph has no loops or multiple edges.

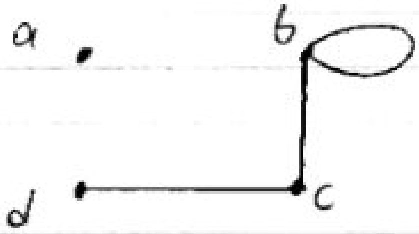




The degree of a vertex "v" = "deg(v)"

= # of edges incident with v
(a loop adds two to the degree).

Ex



$\deg(a) = 0$, a is "isolated"
$\deg(b) = 3$
$\deg(c) = 2$
$\deg(d) = 1$

sum of degrees $= 6$

$|E| = 3$

Handshaking Thm

$$\sum_{v \in V} \deg(v) = 2|E|$$

degree sum = twice the # of edges

Why? Each edge contributes 2 to the degree sum.

Corollary 1 The degree sum must be even.

Ex In a group of 5 computers, is it possible for

1 computer to be directly connected to 2 others and
Another " " " " " " 4 others and
" " " " " " " 3 others and
" " " " " " " 3 others and
" " " " " " " 1 other.

Vertices = computers
Edges = direct connections

Degree sum = 13 ← odd, so impossible!

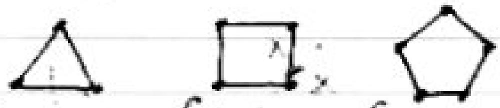
Corollary 2 A graph cannot have an odd number of odd-degree vertices.

Otherwise, the degree sum would be odd.

Special Graphs

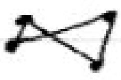
Cycle graphs

C_n $(n \geq 3)$



C_3 C_4 C_5

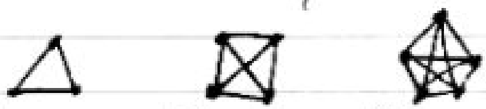
Note: C_4 could look like



$|V| = n$
$|E| = n$
Each $\deg(v) = 2$

Complete graphs - every vertex pair is connected by exactly one edge

K_n (everyone shakes everyone else's hand once)



K_3 K_4 K_5

$|V| = n$

Each $\deg(v) = n-1$

$|E| = ?$

Each of the n vertices is adjacent to $n-1$ others.

Think: Incl-Excl

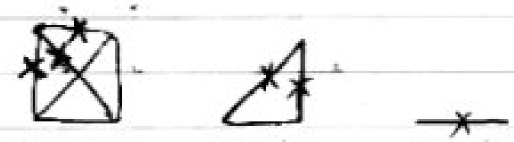
Sort-of → familiar?

$$\frac{n(n-1)}{2}$$

Ex
$\{a,c\}$ and $\{c,a\}$ are the same edge

How can we see that $\frac{n(n-1)}{2} = 1+2+\ldots+(n-1)$?

<u>Ex</u> $n=4$



3 edges 2 edges 1 edge

$|E \text{ for } K_4| = 3+2+1$

$|E \text{ for } K_n| = (n-1)+(n-2)+\ldots+1$

Proofs for: $1+2+\ldots+n = \frac{n(n+1)}{2}$

1)
$$\begin{aligned} S &= 1+2+\ldots+n \\ S &= n+(n-1)+\ldots+1 \\ \hline 2S &= (n+1)+(n+1)+\ldots+(n+1) \\ 2S &= n(n+1) \\ S &= \frac{n(n+1)}{2} \end{aligned}$$

2) Induction

3) $|E \text{ for } K_n| = \underbrace{\frac{n(n-1)}{2}}_{=1+2+\ldots+(n-1)} \xrightarrow[\text{n w/ n+1}]{\text{Replace}} \underbrace{\frac{(n+1)n}{2}}_{=1+2+\ldots+n}$

"Old friend"

Smells like pigeonhole principle

In a simple graph on $n (\geq 2)$ vertices, at least two vertices must have the same degree.

Otherwise, all n vertices must have different degrees

disconnected graph

connected graph

0 1 2 $\cdot$ $n-1$

isolated vertex

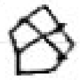
vertex adjacent to all other vertices

can't have both

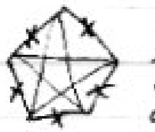
Old HW: 4.1.55

Convex: Every line seg connecting two pts. in the interior or boundary of the polygon lies entirely w/in this set.

How many diagonals does a convex polygon with n sides have?



Ex



sides aren't diagonals

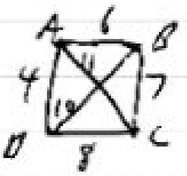
no

Each vertex has a diagonal to all vertices except itself and both neighbors.

↑ n-3 diagonals

$$\frac{n(n-3)}{2}$$ ← to adjust for double-counting

Weighted graphs (TSP)



what subject - models you can buy

Many applics
- Chemistry
- Scheduling (coloring vertices/edges)
- Circuit design
- Telecommunications/Internet
- Traffic Networks
- Garbage collection
- Social sciences

(5-7 in Math 21

4-Color Thm: how many colors are needed to color a map s.t. no 2 adjacent countries get the same color. (so Michigan)

Political setups from Star Trek

4 is always enough! Countries - vertices, Adjacencies - edges. Described 1852 - Computer pf. 1976

Critical # Th'y Theorem:
Fermat's Last Theorem (Conjecture ~350 yrs. old)
Proven by Andrew Wiles at Princeton.

$$x^2 + y^2 = z^2$$

Find $x, y, z \in \mathbb{Z}^+$ that solve this.
There are many triples:

$3^2 + 4^2 = 5^2$ 

$6^2 + 8^2 = 10^2$

$5^2 + 12^2 = 13^2$

$8^2 + 15^2 = 17^2$

$\vdots$

∞ many triples

Thm: for $n \in \mathbb{Z}$, $n \geq 3$

1983 - Gerd Faltings proved: finitely many soln.

$$x^n + y^n = z^n$$

has no solution (x, y, z)
where $x, y, z \in \mathbb{Z}^+$

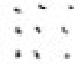
$$x^3 + y^3 = z^3$$
$$x^4 + y^4 = z^4$$
$$\vdots$$

Fermat wrote in a margin

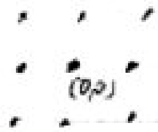
A Personal Favorite

Consider the $\mathbb{Z}\times\mathbb{Z}$ lattice of pts.

$\quad\quad\quad\quad$ || set of all pts. (x,y)
$\quad\quad\quad\quad\quad\quad\quad\quad$ both integer



Is there a straight line that passes through $(0,0)$ and no other lattice point?



Let's assume we have a straight line that hits $(0,0)$ and some other lattice point (a,b).

The slope must be $\frac{b}{a}$ $\ni \in \mathbb{Z}$ or undefined $\Updownarrow$
The slope must be rational. ($\in \mathbb{Q}$).
or undefined.

We've shown
If a straight line hits $(0,0)$ and another lattice point (a,b), then its slope is rational or undefined.

The contrapositive must be true:

If the slope of a line is irrational, then the straight line can't hit both $(0,0)$ and another lattice pt.

Ex $y = \pi x$

Ex $y = \sqrt{2} x$

$\vdots$

Infinitely many lines, each corresp. to an irrat'l #.