CH 7: GRAPHS

A graph consists of dots (vertices) and edges that connect pairs of vertices.

Ex: \[ V = \{ a, b, c \} \] \[ E = \{ \{ a, b \}, \{ a, c \} \} \]

\[ |V| = 3 \text{ vertices} \quad |E| = 2 \text{ edges} \]

The graph \( G = (V,E) \) can be represented as

\[
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\]

\( a \) and \( b \) are adjacent vertices, because an edge connects them. The edge \( \{a, b\} \) is incident with \( a \) and \( b \).

A simple graph has no loops or multiple edges.

The degree of a vertex \( \nu \) is \( \deg (\nu) \)

\( = \# \) of edges incident with \( \nu \),

(a loop adds two to the degree).
Ex. \[\begin{array}{ccc}
a & b & \text{deg}(a) = 0, \ a \text{ is "isolated"} \\
d & c & \text{deg}(b) = 3 \\
\end{array}\]
\[\text{deg}(c) = 2 \]
\[\text{deg}(d) = 1 \]

\[\text{sum of degrees} = 6\]
\[|E| = 3\]

\text{Handshaking Thm.:} \quad \sum_{v \in V} \text{deg}(v) = 2 |E|

degree sum = twice the \# of edges

Why? Each edge contributes 2 to the degree sum.
\[+1 +1 0+2\]

\text{Corollary 1. The degree sum must be even.}
Ex. In a group of 5 computers, is it possible for

1 computer to be directly connected to 2 others and
2 others
Another 4 others and
3 others and
3 others and
1 other.

Vertices = computers
Edges = direct connections
Degree sum = 13 < odd, so impossible!

Corollary 2. A graph cannot have an odd number
of odd-degree vertices.
Otherwise, the degree sum would be odd.

Special Graphs

Cycle graphs

$C_n \ (n \geq 3)$

$C_3$  $C_4$  $C_5$

$|V| = n$
$|E| = n$
Each $\text{deg}(v) = 2$

Note: $C_4$ could look like

\[ \triangle \]
Complete graphs - every vertex pair is connected by exactly one edge \( K_n \) (everyone shakes everyone else's hand once)

\[ K_3 \quad K_4 \quad K_5 \]

\[ |V| = n \]

Each \( \deg(v) = n-1 \)

\[ |E| = ? \]

Each of the \( n \) vertices is adjacent to \( n-1 \) others.

\[ \frac{n(n-1)}{2} \]

Ex: \( n = 4 \)

\[ 3 \text{ edges} \quad 2 \text{ edges} \quad 1 \text{ edge} \]

\[ |E| \text{ for } K_4 = 3 + 2 + 1 \]

\[ |E| \text{ for } K_n = (n-1) + (n-2) + \ldots + 1 \]
Proofs for: \[ 1 + 2 + \ldots + n = \frac{n(n+1)}{2} \]

1) \[ S = 1 + 2 + \ldots + n \]
\[ S = n(n-1)/2 + 1 \]
\[ 2S = (n+1)(n+1) + \ldots + (n+1) \]
\[ 2S = n(n+1) \]
\[ S = \frac{n(n+1)}{2} \]

2) Induction

3) \[ |E| \leq \frac{n(n-1)}{2} \]
\[ \Rightarrow 1 + 2 + \ldots + (n-1) \]
\[ \Rightarrow \frac{(n+1)n}{2} \]

"Old friend"

In a simple graph on \( n(\geq 2) \) vertices, at least two vertices must have the same degree.

Otherwise, all \( n \) vertices must have different degrees.
**Question:** How many diagonals does a convex polygon with $n$ sides have?

Each vertex has a diagonal to all vertices except itself and both neighbors.

- A $n-3$ diagonals for each vertex.
- Total diagonals: $n(n-3)$.

**Example:**

- For a triangle ($n=3$), there are 0 diagonals.
- For a quadrilateral ($n=4$), there are 2 diagonals.

**Studying Coloring vertex/edges**

- **Graph coloring**
  - $|V|$: Number of vertices
  - $|E|$: Number of edges
  - **Coloring problems**
    - 4-color theorem
  - **Graph Theory**
  - **Combinatorics**

- **Studying graphs (TG)**
  - Weighted graphs
  - $W(G)$: Weighted graph
  - $|V(G)|$: Set of vertices
  - $E(G)$: Set of edges