CH 7: GRAPHS

A graph consists of dots (vertices) and edges that connect pairs of vertices.

Ex $V = \{a, b, c\}$ ← $|V| = 3$ vertices
$E = \{\{a,b\}, \{a,c\}\}$ ← $|E| = 2$ edges

The graph $G = (V, E)$ can be represented as

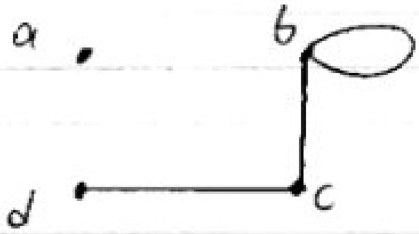
a and b are adjacent vertices, because an edge connects them.
The edge $\{a,b\}$ is incident with a and b.

A simple graph has no loops or multiple edges.

The degree of a vertex "v" = "deg(v)"

= # of edges incident with v
(a loop adds two to the degree).

Ex



$\deg(a) = 0$, a is "isolated"
$\deg(b) = 3$
$\deg(c) = 2$
$\deg(d) = 1$

sum of degrees = 6

$|E| = 3$

Handshaking Thm

$$\sum_{v \in V} \deg(v) = 2|E|$$

degree sum = twice the # of edges

Why? Each edge contributes 2 to the degree sum.

Corollary 1 The degree sum must be even.

<u>Ex</u> In a group of 5 computers, is it possible for

1 computer to be directly connected to 2 others <u>and</u>
Another " " " " " 4 others <u>and</u>
" " " " " " 3 others <u>and</u>
" " " " " " 3 others and
" " " " " " 1 other.

Vertices = computers
Edges = direct connections

Degree sum = 13 ← odd, so impossible!

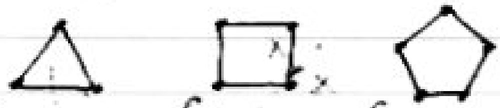
<u>Corollary 2</u> A graph cannot have an odd number of odd-degree vertices.

Otherwise, the degree sum would be odd.

<u>Special Graphs</u>

Cycle graphs

C_n $(n \geq 3)$



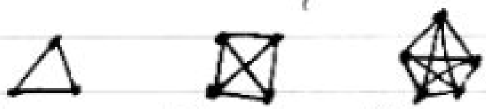
C_3 C_4 C_5

Note: C_4 could look like

$|V| = n$
$|E| = n$
Each $\deg(v) = 2$

Complete graphs - every vertex pair is connected by exactly one edge

K_n (everyone shakes everyone else's hand once)



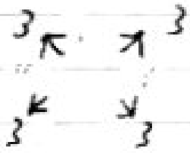
K_3 K_4 K_5

$|V| = n$

Each $\deg(v) = n-1$

$|E| = ?$

Each of the n vertices is adjacent to $n-1$ others.



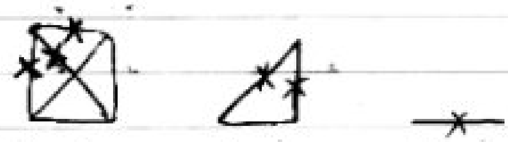
Think: Incl-Excl Sort-of → familiar?

$$\frac{n(n-1)}{2}$$

Ex $\{a,c\}$ and $\{c,a\}$ are the same edge

How can we see that $\frac{n(n-1)}{2} = 1 + 2 + \ldots + (n-1)$?

<u>Ex</u> $n=4$



3 edges 2 edges 1 edge

$|E \text{ for } K_4| = 3 + 2 + 1$

$|E \text{ for } K_n| = (n-1) + (n-2) + \ldots + 1$

Proofs for: $1+2+\dots+n = \frac{n(n+1)}{2}$

1) $S = 1+2+\dots+n$
$S = n+(n-1)+\dots+1$
$2S = (n+1)+(n+1)+\dots+(n+1)$
$2S = n(n+1)$
$S = \frac{n(n+1)}{2}$

2) Induction

3) $|E \text{ for } K_n| = \frac{n(n-1)}{2} = 1+2+\dots+(n-1) \xrightarrow[\text{n w/ n+1}]{\text{Replace}} \frac{(n+1)n}{2} = 1+2+\dots+n$

"Old friend"

Smells like pigeonhole principle

In a simple graph on $n (\geq 2)$ vertices, at least two vertices must have the same degree.

Otherwise, all n vertices must have different degrees

disconnected graph — connected graph

0 1 2 . $n-1$

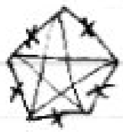
isolated vertex — vertex adjacent to all other vertices

can't have both

Convex: Every line seg connecting two pts. in the interior or boundary of the polygon lies entirely w/in this set.

Old HW: 4.1.55

How many diagonals does a convex polygon with n sides have?

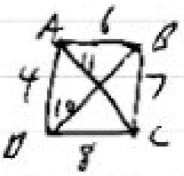
Ex  sides aren't diagonals

no

Each vertex has a diagonal to all vertices except itself and both neighbors.

↑ n-3 diagonals

$$\frac{n(n-3)}{2}$$ ← to adjust for double-counting

Weighted graphs (TSP) 

what subject - models you can buy

Many applics
- Chemistry
- Scheduling (coloring vertices/edges)
- Circuit design
- Telecommunications/Internet
- Traffic Networks
- Garbage collection
- Social sciences

(5-7 in Math 21

4-Color Thm: how many colors are needed to color a map s.t. no 2 adjacent countries get the same color. (No Michigans)

Political setups from Star Trek

4 is always enough! Countries - vertices Adjacencies - edges Described 1852 - Computer pf. 1976