CH. 14: VECTORS and SURFACES

14.1: VECTORS IN 2D

(A) Scalars

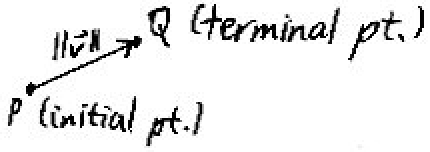
are real #s

(B) Vectors

have magnitude and direction

Ex A vector $\vec{v}$ (or boldfaced "**v**")

$\vec{v} = \overrightarrow{PQ}$



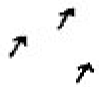
Exs Displacement, Velocity, Force

55mph

$\|\vec{v}\|$ = magnitude, length, or norm of $\vec{v}$

Equal vectors:

Harry Potter flies



(Position is irrelevant.)

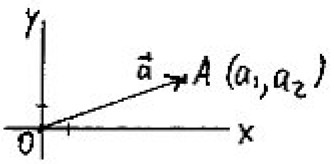
(C) $\mathbb{R}^2$

(using Cartesian / Rectangular Coordinate System) vs. Polar

$= \{ (x, y) \mid x \text{ and } y \text{ are real \#s} \}$

the set of all — ordered pairs — such that

"2-space"



The position vector for A or $\vec{a}$

$= \langle a_1, a_2 \rangle$

horiz. — vertical

components

⟨⟩ are angle brackets

⟨⟩ see Math Dictionary Borowski/Borwein

Many books simply use $\mathbb{R}^2$ instead of V_2.

$\vec{a}$ is in $V_2 = \{ \langle x,y \rangle \mid x$ and y are real #s $\}$

$$\boxed{\begin{aligned} \|\vec{a}\| &= \| \langle a_1, a_2 \rangle \| \\ &= \sqrt{a_1^2 + a_2^2} \end{aligned}}$$

① from Pyth. Thm. / Distance Formula
② $\|\vec{a}\| \geq 0$ always

Ⓓ Standard Unit Vectors in V_2

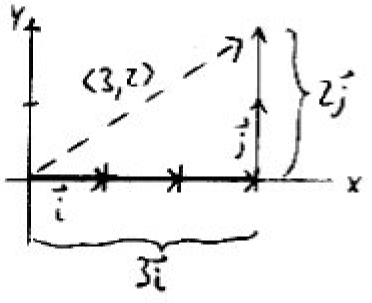
length 1

$$\boxed{\begin{aligned} &\vec{i} = \langle 1,0 \rangle \\ &\vec{j} = \langle 0,1 \rangle \\ &\langle a_1, a_2 \rangle = a_1 \vec{i} + a_2 \vec{j} \end{aligned}}$$

Physics!

<u>Ex</u> $\langle 3,2 \rangle = 3\vec{i} + 2\vec{j}$

Treasure Map



Basic Example of ...

Ⓔ Vector Addition

Ex $\langle 1,4\rangle + \langle 2,3\rangle$

$\vec{v}$ ↗   → $\vec{w}$

Add corresponding components.

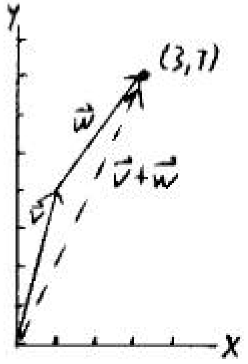
$= \langle 1+2, 4+3\rangle$

$= \boxed{\langle 3,7\rangle}$

Triangle Law

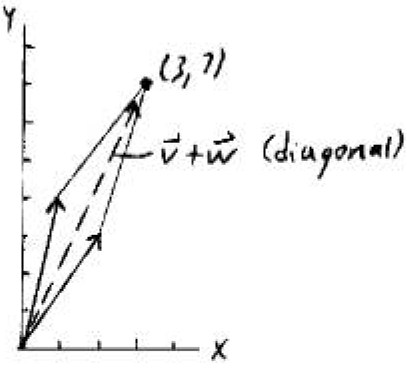
"Head-to-tail"

Treasure Map



Parallelogram Law

"Tail-to-tail"



$\vec{v}+\vec{w}$ = resultant

Exs Net displacement
Net force

(F) Scalar Multiples of $\vec{v}$

<u>Ex</u> $3\langle -1, 2\rangle$

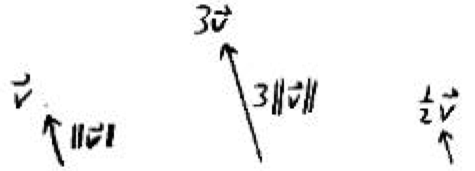
Multiply each component by 3.

$= \langle 3(-1), 3(2)\rangle$

$= \boxed{\langle -3, 6\rangle}$

$c\vec{v}$

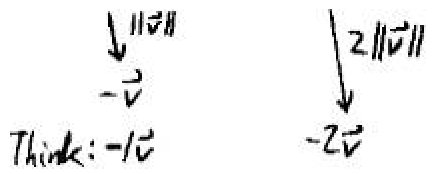
scalar can (1) rescale $\vec{v}$
(2) flip direction (if $c<0$)



$c>0$
$\Rightarrow$ same direction as $\vec{v}$

• $\vec{0} = \langle 0,0\rangle$
has <u>every</u> direction

$c=0$



$c<0$
$\Rightarrow$ opposite direction

$$\boxed{\|c\vec{v}\| = |c|\,\|\vec{v}\|}$$ never negative

$\vec{w}$ is parallel to $\vec{v}$ ($\vec{w}\parallel\vec{v}$) $\Leftrightarrow$ if and only if $\vec{w}=c\vec{v}$ for some scalar c (or $\vec{v}=\vec{0}$)

i.e., The vectors parallel to $\vec{v}$ are the scalar multiples of $\vec{v}$. ($\vec{v}\neq\vec{0}$)

$\boxed{\underline{\text{Def'n}} \quad \frac{\vec{v}}{c} = \frac{1}{c}\vec{v} \quad (c \neq 0)}$

Divide each component by c.

$\boxed{\underline{\text{Def'n}} \quad \vec{v} - \vec{w} = \vec{v} + (-\vec{w})}$

$\underline{\text{Ex}}$ $\langle 3,2\rangle - \langle 5,1\rangle = \langle 3-5, 2-1\rangle$

$= \langle -2, 1\rangle$

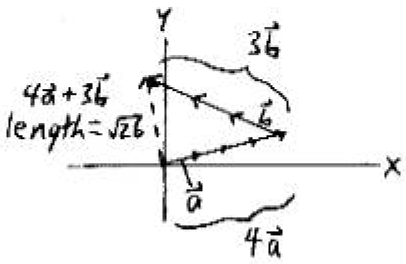
$\vec{w} + (\text{what?}) = \vec{v}$
$\vec{v}-\vec{w}$

⑥ $\underline{\text{Ex}}$

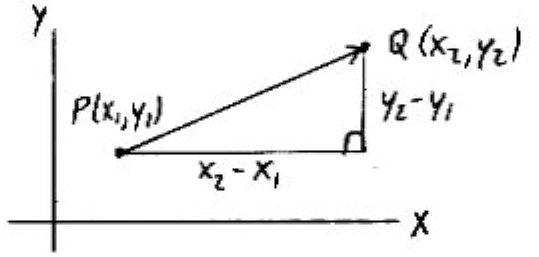
If $\vec{a} = \langle 2, \frac{1}{2}\rangle$, $\vec{b} = \langle -3, 1\rangle$, find $\|4\vec{a}+3\vec{b}\|$.

$$\begin{aligned}
\|4\vec{a}+3\vec{b}\| &= \|4\langle 2,\tfrac{1}{2}\rangle + 3\langle -3,1\rangle\| \\
&= \|\langle 8,2\rangle + \langle -9,3\rangle\| \\
&= \|\langle 8+(-9), 2+3\rangle\| \\
&= \|\langle -1,5\rangle\| \\
&= \sqrt{(-1)^2+(5)^2} \\
&= \boxed{\sqrt{26}}
\end{aligned}$$

up to 19



(H) An Initial Point and a Terminal Point Determine a Vector



$$\boxed{\begin{gathered}\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle \\ \Rightarrow \|\overrightarrow{PQ}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ \text{Distance Formula!}\end{gathered}}$$

$$\overrightarrow{QP} = -\overrightarrow{PQ}$$

(I) The Unit Vector in the Direction of $\vec{v}$ $(\vec{v} \neq \vec{0})$

$$\boxed{\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} \quad \text{or} \quad \frac{\vec{v}}{\|\vec{v}\|}}$$

Normalizing $\vec{v}$

$\vec{v}$ $\vec{u}$ length 1

(J) Ex

$P(1, -2)$; $Q(-3, -1)$. Find the unit vector in the direction of $\overrightarrow{PQ}$.

Sol'n

$$\begin{aligned}\overrightarrow{PQ} &= \langle -3-1, -1-(-2) \rangle \\ &= \langle -4, 1 \rangle\end{aligned}$$

Think: "Q−P"

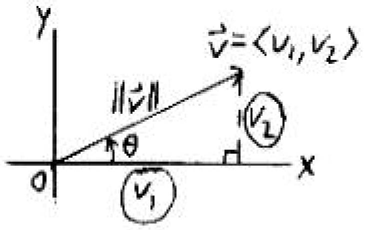
$$\begin{aligned}\|\overrightarrow{PQ}\| &= \sqrt{(-4)^2 + (1)^2} \\ &= \sqrt{17}\end{aligned}$$

$$\vec{u} = \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \frac{\langle -4, 1 \rangle}{\sqrt{17}} = \boxed{\left\langle -\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \right\rangle}$$

can rat'lize the denom.

Up to 31

(K) Finding Horiz., Vertical Components of a Vector in V_2



θ = direction angle for $\vec{v}$

Given $\|\vec{v}\|, \theta \Rightarrow$ find v_1, v_2

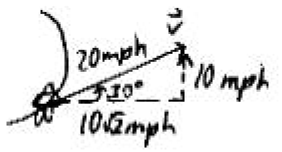
$$\cos\theta = \frac{v_1}{\|\vec{v}\|} \qquad \sin\theta = \frac{v_2}{\|\vec{v}\|}$$

$$v_1 = \|\vec{v}\|\cos\theta \qquad v_2 = \|\vec{v}\|\sin\theta$$

$$\boxed{\vec{v} = \langle \underbrace{\|\vec{v}\|\cos\theta}_{v_1}, \underbrace{\|\vec{v}\|\sin\theta}_{v_2} \rangle}$$

v_1 = horiz. comp. of $\vec{v}$ v_2 = vertical " "

Ex (Slow) car on a track
tank!



$$\begin{aligned}\vec{v} &= \langle \|\vec{v}\|\cos\theta, \|\vec{v}\|\sin\theta \rangle \\ &= \langle 20\cos 30°, 20\sin 30° \rangle \\ &= \langle 20(\tfrac{\sqrt{3}}{2}), 20(\tfrac{1}{2}) \rangle \\ &= \langle 10\sqrt{3}, 10 \rangle\end{aligned}$$

Up to 47

Given $v_1, v_2 \Rightarrow$ find $\|\vec{v}\|, \theta$

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$$

$\tan\theta = \frac{v_2}{v_1}$, Watch Quadrant! (I or III? II or IV?)

① Vector Properties (Swokowski p.687)

$\vec{a}, \vec{b}, \vec{c}, \vec{0}$ are vectors in V_n (n is a fixed natural #),
c, d are scalars.

Vector "+"	☆(i)	$\vec{a}+\vec{b}=\vec{b}+\vec{a}$	Vector "+" is commutative
	(ii)	$\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$	associative
	(iii)	$\vec{a}+\vec{0}=\vec{a}$	$\vec{0}$ is the additive identity
	(iv)	$\vec{a}+(-\vec{a})=\vec{0}$	$-\vec{a}$ is the additive inverse of $\vec{a}$
Scalar Mult.	☆(v)	$c(\vec{a}+\vec{b})=c\vec{a}+c\vec{b}$	Scalar mult. distributes over vector "+"
	(vi)	$(c+d)\vec{a}=c\vec{a}+d\vec{a}$	Vector " " scalar "+"
	(vii)	$(cd)\vec{a}=c(d\vec{a})=d(c\vec{a})$	Mult. by scalars is flexible
	(viii)	$1\vec{a}=\vec{a}$	1
	(ix)	$0\vec{a}=\vec{0}=c\vec{0}$	0

☆ Proof p.687

Like #54 but with c, not p.

<u>Ex</u> Prove $c(\vec{a}-\vec{b})=c\vec{a}-c\vec{b}$. ($\vec{a},\vec{b}$ in V_2, c scalar)

Let $\vec{a}=\langle a_1, a_2\rangle$
$\vec{b}=\langle b_1, b_2\rangle$

Trick: Boil things down to properties of real #s; rebuild up.

$$\begin{aligned}
c(\vec{a}-\vec{b}) &= c(\langle a_1,a_2\rangle-\langle b_1,b_2\rangle)\\
&= c(\langle a_1-b_1,\ a_2-b_2\rangle)\\
&= \langle c(a_1-b_1),\ c(a_2-b_2)\rangle\\
&= \langle ca_1-cb_1,\ ca_2-cb_2\rangle\\
&= \langle ca_1, ca_2\rangle-\langle cb_1, cb_2\rangle\\
&= c\langle a_1,a_2\rangle-c\langle b_1,b_2\rangle\\
&= c\vec{a}-c\vec{b}
\end{aligned}$$

QED (end of proof)
Quod Erat Demonstrandum

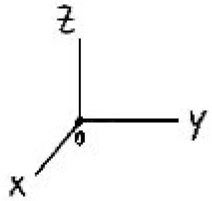
14.2: VECTORS IN 3D

(A) $\mathbb{R}^3$ (using Cartesian coords.)

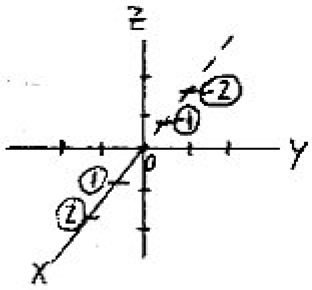
$= \{ (x, y, z) \mid x, y, \text{ and } z \text{ are real \#s} \}$

ordered triples

"3-space"



(sticks out at you)



rude

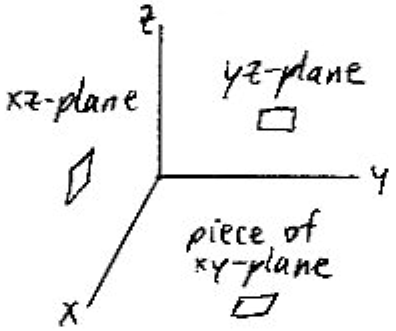
These coordinate axes are mutually perpendicular (or orthogonal, or $\perp$).

How to see ortho: upright
orthodontist

(1,1,1) pt. isn't 0!

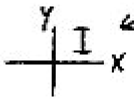
Beware of distortion! 3D → 2D paper

Coordinate planes



Mutually $\perp$

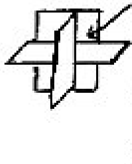
$\mathbb{R}^2$



$\leftarrow \begin{pmatrix} x>0 \\ y>0 \end{pmatrix}$

4 quadrants

$\mathbb{R}^3$



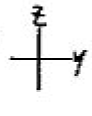
$\mathrm{I} \begin{pmatrix} x>0 \\ y>0 \\ z>0 \end{pmatrix}$

others not numbered

8 octants

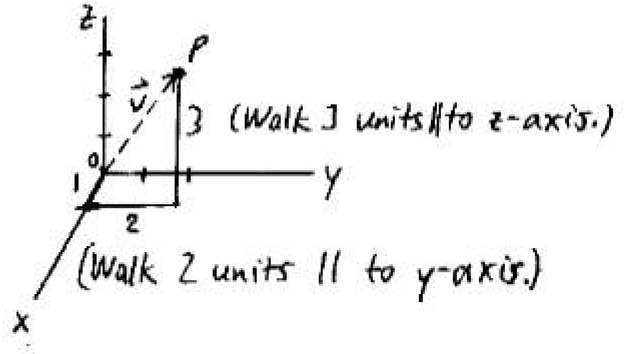
4 in front of yz-plane, $(x>0)$

4 behind $(x<0)$



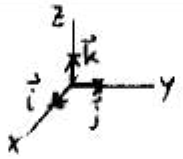
xy, yz-planes stick out at you

<u>Ex</u> Plot the point $P(1,2,3)$, and draw $\vec{v} = \langle 1,2,3 \rangle$, a vector in V_3.



Ⓑ Standard Unit Vectors in V_3

$$\vec{i} = \langle 1,0,0 \rangle$$
$$\vec{j} = \langle 0,1,0 \rangle$$
$$\vec{k} = \langle 0,0,1 \rangle$$



$$\langle a_1, a_2, a_3 \rangle = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

Physics!

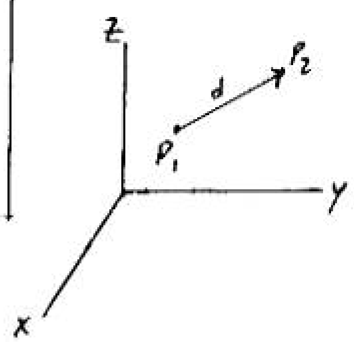
<u>Ex</u> $\langle 2,-3,-1 \rangle = 2\vec{i} - 3\vec{j} - \vec{k}$

Ⓒ $\overrightarrow{P_1P_2}$

$P_1(x_1, y_1, z_1)$; $P_2(x_2, y_2, z_2)$

$$\overrightarrow{P_1P_2} = \langle \underbrace{x_2 - x_1}_{\Delta x}, \underbrace{y_2 - y_1}_{\Delta y}, \underbrace{z_2 - z_1}_{\Delta z} \rangle$$

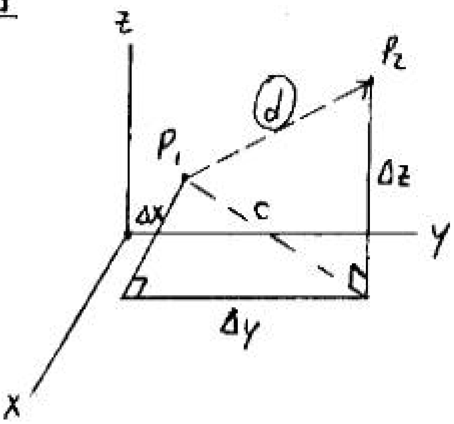
↑ uppercase delta (change in)



$d = \|\overrightarrow{P_1P_2}\|$

$=$ distance between P_1, P_2

Find d



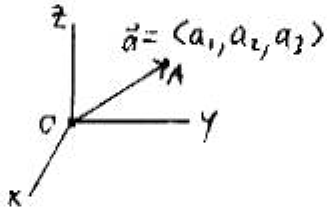
Apply Pyth. Thm. twice!

$$d^2 = \underbrace{c^2}_{} + (\Delta z)^2$$

$$= \underbrace{(\Delta x)^2 + (\Delta y)^2}_{} + (\Delta z)^2 \qquad \text{(OK if } \Delta x < 0\text{, etc.)}$$

$$\boxed{\begin{aligned} d &= \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \end{aligned}}$$

Special Case



$$\boxed{\begin{aligned} \|\vec{a}\| &= \|\langle a_1, a_2, a_3 \rangle\| \\ &= \sqrt{a_1^2 + a_2^2 + a_3^2} \end{aligned}}$$

The midpoint of $\overline{P_1P_2}$ is $\left(\underbrace{\frac{x_1 + x_2}{2}}_{\text{average of x-coords.}}, \underbrace{\frac{y_1 + y_2}{2}}_{y}, \underbrace{\frac{z_1 + z_2}{2}}_{z}\right)$

<u>Ex</u> Let ℓ be the line passing through $P_1(4,0,-2)$ and $P_2(2,5,7)$.
Find two unit vectors $\underbrace{\text{parallel to}}_{\text{“}\parallel\text{”}}$ this line.

<u>Sol'n</u>

Find $\overrightarrow{P_1P_2}$ (or $\overrightarrow{P_2P_1}$), a vector $\parallel \ell$.

$$\overrightarrow{P_1P_2} = \langle 2-4,\ 5-0,\ 7-(-2)\rangle = \langle -2,\ 5,\ 9\rangle$$

Find the unit vector in the direction of $\overrightarrow{P_1P_2}$.
<u>Normalize</u> $\overrightarrow{P_1P_2}$.

$$\|\overrightarrow{P_1P_2}\| = \sqrt{(-2)^2+(5)^2+(9)^2} = \sqrt{110}$$

$$\vec{u}_1 = \frac{\overrightarrow{P_1P_2}}{\|\overrightarrow{P_1P_2}\|}$$

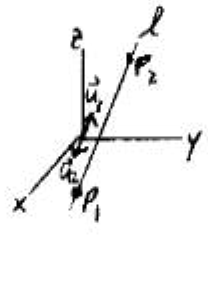
$$= \frac{\langle -2,5,9\rangle}{\sqrt{110}} \quad \text{really, } \frac{1}{\sqrt{110}}\langle -2,5,9\rangle$$

$$= \boxed{\left\langle -\frac{2}{\sqrt{110}},\ \frac{5}{\sqrt{110}},\ \frac{9}{\sqrt{110}}\right\rangle}$$

Find the opposite unit vector.

$$\vec{u}_2 = -\vec{u}_1$$

$$= \boxed{\left\langle \frac{2}{\sqrt{110}},\ -\frac{5}{\sqrt{110}},\ -\frac{9}{\sqrt{110}}\right\rangle}$$



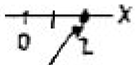
up to 17

(D) The Graph of an Equation

· consists of all points whose coords. satisfy the equation.

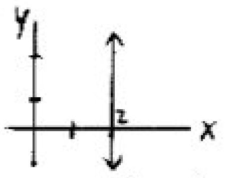
!!

$\mathbb{R}^1 \to \mathbb{R}^2$
$x=2$



sweep ∥ to y axis to get line ↗

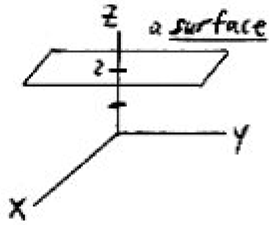
$\mathbb{R}^2$ $x=2$



line ∥ y-axis
y missing in eq.

$\{(2,y) \mid y \text{ real}\}$

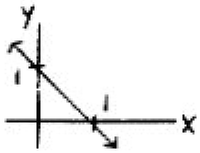
$\mathbb{R}^3$ $z=2$



plane ∥ xy plane
x, y missing in eq.

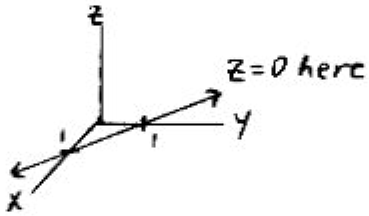
Graph $x+y=1$. In $\mathbb{R}^{what?}$

$\mathbb{R}^2$ $x+y=1$



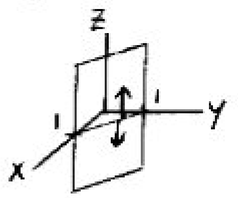
$\mathbb{R}^3$ $x+y=1$
z missing

Not the whole graph!



For any (x,y) that satisfies $x+y=1$, (x,y,z) will satisfy $x+y=1$ for <u>any</u> real z.

We "sweep" the line ∥ to the z-axis. We then pick up <u>all</u> z-coords. for any (x,y) that "works."



plane ∥ z-axis

Circles:

$(x-x_0)^2+(y-y_0)^2 = r^2$

Ⓔ Spheres

Find an eq. for the sphere with
Center $C(x_0, y_0, z_0)$;
Radius $= r$ $(r>0)$

We want all points (x,y,z) that are r units away from C.

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = r$$

$$\boxed{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = r^2}$$

Ex Find C, r for $\underbrace{2x^2+2y^2+2z^2}_{\div \text{ thru by } 2}-12x-20y+12z+68=0.$

#30 →

$$x^2+y^2+z^2-6x-10y+6z+34=0$$

Group terms.

$$(x^2-6x)+(y^2-10y)+(z^2+6z) = -34$$

Complete the Square (CTS) within groups; Balance!

$$(x^2-6x+9)+(y^2-10y+25)+(z^2+6z+9) = -34+9+25+9$$

Factor.

$$\underbrace{(x-3)^2+(y-5)^2+(z+3)^2}_{C:\text{ What makes left side } = 0?} = \underbrace{9}_{r^2=9}$$

Up to 33

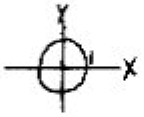
$$\boxed{\begin{array}{l} C: (3,5,-3) \\ r=3 \end{array}}$$

Ⓔ Regions in $\mathbb{R}^3$

Ex Describe $R = \{(x,y,z) \mid x^2+y^2 \le 1, \underbrace{|z| \le 4}\}$.

Extra conditions/ restrictions never grow the surface.

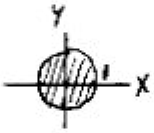
① $\mathbb{R}^2$ $x^2+y^2=1$



Unit circle

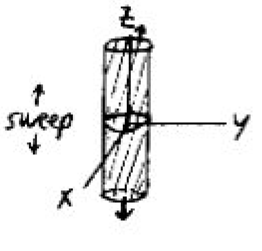
Test point method for shading - (0,0) lies inside the circle $\underbrace{x^2+y^2}_{\text{cont. in } x,y} = 1$. ↑ constant

② $\mathbb{R}^2$ $\underbrace{x^2+y^2}_{\text{squared distance from } 0} \le 1$



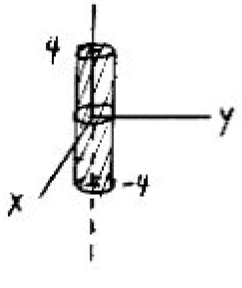
Unit disk

③ $\mathbb{R}^3$ $x^2+y^2 \le 1$



open-ended solid circular (by cross-section) cylinder

④ $\mathbb{R}^3$ $x^2+y^2 \le 1$, $\underbrace{|z| \le 4}_{-4 \le z \le 4}$



All points inside or on the closed circular cylinder with center at 0, radius 1, altitude (height) 8, and axis along the z-axis.

14.3: DOT (INNER) PRODUCT

(A) $\vec{a}\cdot\vec{b}$

is a scalar. Algebraic definition: $\vec{a}\cdot\vec{b} = \sum_{i=1}^{n} a_i b_i = a_1b_1 + a_2b_2 + \ldots + a_nb_n$ $\left(\begin{matrix}\vec{a} = \langle a_1, a_2, \ldots, a_n\rangle \\ \vec{b} = \langle b_1, b_2, \ldots, b_n\rangle\end{matrix}\right)$

We add products of corresponding components.

Ex $\langle 2, -3, -4\rangle \cdot \langle 1, -2, 5\rangle$

$= (2)(1) + (-3)(-2) + (-4)(5)$

Up to 5

$= \boxed{-12}$

(B) Properties (p. 702)

$\vec{0}, \vec{a}, \vec{b}, \vec{c}$ in V_n ; c scalar

Prove in 39 for V_3.

(i) $\vec{a}\cdot\vec{a} = \|\vec{a}\|^2$

(ii) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

(iii) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

(iv) $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(c\vec{b})$

(v) $\vec{0}\cdot\vec{a} = 0$

Comments

Ex $\langle 2,3\rangle\cdot\langle 2,3\rangle = (2)^2 + (3)^2 = \left(\overbrace{\|\langle 2,3\rangle\|}^{\sqrt{(2)^2+(3)^2}}\right)^2$

"$\cdot$" is comm.

"$\cdot$" distributes over vector "+" (Proof p. 702)

Scalar mult. is flexible re "$\cdot$"

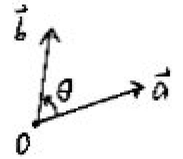
Ex Prove $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b})$ if $\vec{a}, \vec{b}$ in V_3.

Proof Let $\vec{a} = \langle a_1, a_2, a_3\rangle$
$\vec{b} = \langle b_1, b_2, b_3\rangle$
c any scalar

$$\begin{aligned}(c\vec{a})\cdot\vec{b} &= (c\langle a_1,a_2,a_3\rangle)\cdot\langle b_1,b_2,b_3\rangle \\ &= \langle ca_1, ca_2, ca_3\rangle\cdot\langle b_1,b_2,b_3\rangle \\ &= ca_1b_1 + ca_2b_2 + ca_3b_3 \\ &= c(a_1b_1 + a_2b_2 + a_3b_3) \\ &= c(\langle a_1,a_2,a_3\rangle\cdot\langle b_1,b_2,b_3\rangle) \\ &= c(\vec{a}\cdot\vec{b})\end{aligned}$$

QED

Ⓒ Angle Between $\vec{a}, \vec{b}$



Let θ = smallest nonnegative angle between the position vectors for $\vec{a}, \vec{b}$.

$$0 \le \theta \le \pi$$

From the Law of Cosines for triangles,

$$\boxed{\begin{aligned} \vec{a}\cdot\vec{b} &= \underbrace{\|\vec{a}\|\,\|\vec{b}\|}_{\text{product of lengths}}\ \underbrace{\cos\theta}_{\text{times cos of angle between them}} \\ &\text{(Geometric def'n) ``polar''} \\ \Rightarrow \cos\theta &= \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} \qquad (\vec{a}\neq\vec{0},\ \vec{b}\neq\vec{0}) \\ \Rightarrow \theta &= \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}\right) \end{aligned}}$$

Think: $\dfrac{\text{dot product}}{\text{product of lengths}}$

<u>Ex</u> (#12) Find the angle between $\vec{a} = \vec{i} - 7\vec{j} + 4\vec{k}$, $\vec{b} = 5\vec{i} - \vec{k}$.

<u>Sol'n</u> $\vec{a} = \langle 1, -7, 4\rangle$, $\vec{b} = \langle 5, 0, -1\rangle$

$$\theta = \cos^{-1}\left(\frac{\langle 1,-7,4\rangle \cdot \langle 5,0,-1\rangle}{\|\langle 1,-7,4\rangle\|\,\|\langle 5,0,-1\rangle\|}\right)$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{66}\,\sqrt{26}}\right)$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{1716}}\right)$$

$$\approx 1.547 \text{ (radians) or } 88.6^\circ$$

$$\boxed{\underset{\text{orthogonal}}{\vec{a} \perp \vec{b}} \iff \vec{a} \cdot \vec{b} = 0 \quad (\cos \tfrac{\pi}{2} = 0)}$$

	$\theta = 0$, acute	$\theta = \frac{\pi}{2}$	$\theta =$ obtuse, π	
$\vec{a} \cdot \vec{b}$:	$+$	0	$-$	$\leftarrow$ determined by $\cos\theta$, if $\vec{a}, \vec{b}$ non-$\vec{0}$

D Cauchy-Schwarz Inequality

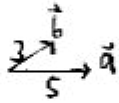
$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \underbrace{\cos\theta}_{\substack{-1 \le \cos\theta \le 1 \\ |\cos\theta| \le 1}}$$

(If I give you two sticks of fixed lengths, what are the possible dot prods.?)

$$\Rightarrow \boxed{|\vec{a} \cdot \vec{b}| \le \|\vec{a}\| \|\vec{b}\|}$$

The absolute value of a dot product cannot exceed the product of the lengths.

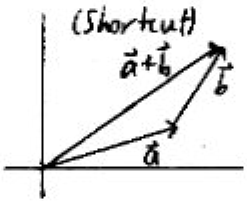
Ex



can rotate
$|\vec{a} \cdot \vec{b}| \le 15$
$-15 \le \vec{a} \cdot \vec{b} \le 15$

E Triangle Inequality

SUV



The length of one side cannot exceed the sum of the lengths of the other two sides.

$$\boxed{\|\vec{a} + \vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|}$$

Proof uses D.

<u>Ex (#48)</u> When is $\underbrace{\|\vec{a}+\vec{b}\| = \|\vec{a}\| + \|\vec{b}\|}$?

$$\Leftrightarrow \quad (\|\vec{a}+\vec{b}\|)^2 = (\|\vec{a}\| + \|\vec{b}\|)^2$$

because magnitudes are nonnegative

$$\Leftrightarrow \quad \underbrace{(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})}_{\substack{\text{we used:} \\ \|\vec{v}\|^2 = \vec{v}\cdot\vec{v}}} = \|\vec{a}\|^2 + 2\|\vec{a}\|\|\vec{b}\| + \|\vec{b}\|^2$$

$$\Leftrightarrow \quad \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} = \text{`` \qquad ''}$$

$$\Leftrightarrow \quad \cancel{\|\vec{a}\|^2} + 2(\vec{a}\cdot\vec{b}) + \cancel{\|\vec{b}\|^2} = \cancel{\|\vec{a}\|^2} + 2\|\vec{a}\|\|\vec{b}\| + \cancel{\|\vec{b}\|^2}$$

$$\Leftrightarrow \quad 2(\vec{a}\cdot\vec{b}) = 2\|\vec{a}\|\|\vec{b}\|$$

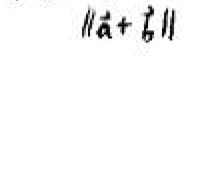
$$\Leftrightarrow \quad \vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|$$

$$\Leftrightarrow \|\vec{a}\|\|\vec{b}\|\cos\theta = \|\vec{a}\|\|\vec{b}\| \qquad (0 \le \theta \le \pi)$$

$$\Leftrightarrow \quad \vec{a} = \vec{0}, \text{ or } \vec{b} = \vec{0}, \text{ or } \cos\theta = 1 \qquad \text{`` \quad ''}$$

$$\theta = 0$$

$$\Leftrightarrow \boxed{\vec{a} = \vec{0}, \text{ or } \vec{b} = \vec{0}, \text{ or } \vec{a} \text{ and } \vec{b} \text{ have the same direction}}$$



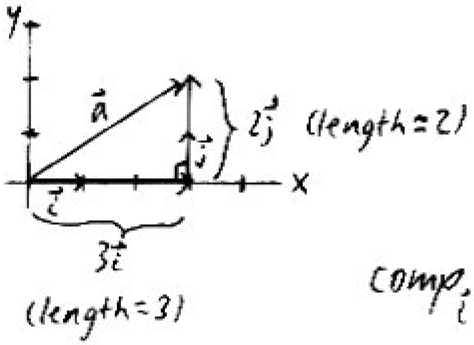
"Degenerate triangle"

(F) $\text{comp}_{\vec{b}}\,\vec{a}$ = The Component of $\vec{a}$ Along $\vec{b}$ (scalar)

Review $\vec{a} = \langle 3,2\rangle = 3\vec{i} + 2\vec{j}$

Larson uses this approach. Math 254 - Linear Alg. approach

We're decomposing $\vec{a}$ as a sum of 2 $\perp$ vectors. (A)



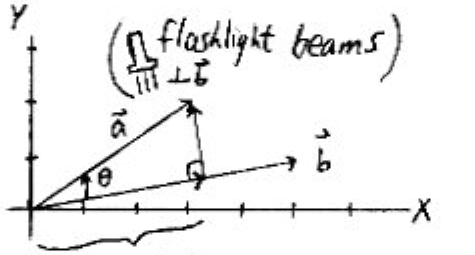
$\text{comp}_{\vec{i}}\,\vec{a} = 3$

$\text{comp}_{\vec{j}}\,\vec{a} = 2$

Other ways to do (A)!

Ex If $\vec{a} = \langle 3,2\rangle$, $\vec{b} = \langle 5,1\rangle$, find $\text{comp}_{\vec{b}}\,\vec{a}$.

$\overrightarrow{\text{proj}_{\vec{b}}\,\vec{a}}$ Think: shadow of $\vec{a}$ on line thru $\vec{b}$ when a flashlight's beams are $\perp\vec{b}$



See 14.1.7 $\cos\theta = \frac{A}{H}$ $A = H\cos\theta$

$$\text{comp}_{\vec{b}}\,\vec{a} = \|\vec{a}\|\cos\theta \qquad (\vec{b} \neq \vec{0})$$

$$= \|\vec{a}\|\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}\right)$$

$$\boxed{\text{comp}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|} \qquad (\vec{b} \neq \vec{0})}$$

Think "$\vec{b}$" for "bottom."

or $\vec{a}\cdot\left(\frac{\vec{b}}{\|\vec{b}\|}\right)$

unit vector in direction of $\vec{b}$

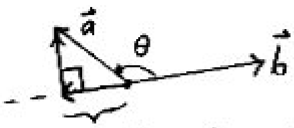
In Ex, $\text{comp}_{\vec b}\,\vec a = \dfrac{\langle 3,2\rangle \cdot \langle 5,1\rangle}{\sqrt{(5)^2+(1)^2}}$

$$= \frac{17}{\sqrt{26}}$$

≈ 3.334 (makes sense in figure)

If θ is obtuse $\Rightarrow \vec a\cdot\vec b < 0 \Rightarrow \text{comp}_{\vec b}\,\vec a < 0$

<u>Ex</u>



$\underbrace{\text{comp}_{\vec b}\,\vec a}_{<0} = -(\text{length})$

Up to 23
(All but 25, 29)

<u>Not in book</u> $\overrightarrow{\text{proj}}_{\vec b}\,\vec a$ (projection) is this "shadow" <u>vector</u>

$\vec a - \text{proj}_{\vec b}\,\vec a$ = the <u>vector component of $\vec a$ orthogonal to $\vec b$</u>

also called the <u>vector component of $\vec a$ along $\vec b$</u>

Larson

$$= \left(\text{comp}_{\vec b}\,\vec a\right)\underbrace{\left(\begin{matrix}\underline{\text{unit}}\text{ vector in the}\\ \text{direction of }\vec b\end{matrix}\right)}_{\text{length}=1\text{ to ensure } \|\overrightarrow{\text{proj}}_{\vec b}\,\vec a\| = |\text{comp}_{\vec b}\,\vec a|}$$

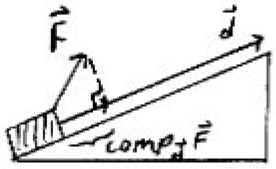
$$= \left(\frac{\vec a\cdot\vec b}{\|\vec b\|}\right)\left(\frac{\vec b}{\|\vec b\|}\right)$$

$$\boxed{\overrightarrow{\text{proj}}_{\vec b}\,\vec a = \underbrace{\frac{\vec a\cdot\vec b}{\|\vec b\|^2}}_{\text{scalar}}\,\vec b \qquad (\vec b \neq \vec 0)}$$

$(\overrightarrow{\text{proj}}_{\vec b}\,\vec a) \parallel \vec b$

Ⓖ Work

$\vec{F}$ = constant force (using Newtons, say)
$\vec{d}$ = displacement (using meters, say)



(In 1-D, Work = ([constant] force)(distance covered)
scalars
How do we extend this to 2-D, 3-D?)

$$\text{Work done "}W\text{"} = \underbrace{(\text{comp}_{\vec{d}}\vec{F})}_{\text{relevant measure of force}}\ \underbrace{(\|\vec{d}\|)}_{\text{distance}}$$

$$= \left(\frac{\vec{F}\cdot\vec{d}}{\cancel{\|\vec{d}\|}}\right)(\cancel{\|\vec{d}\|})$$

$$\boxed{W = \vec{F}\cdot\vec{d}}$$ in Newton-meters (or joules)

14.4: CROSS (VECTOR) PRODUCT

Larson 730
•, × notation by Josiah Gibbs (U.S. phys.)

$\vec{a} \times \vec{b}$ ($\vec{a}, \vec{b}$ in V_3) is a vector.

Ⓐ Determinant of a 2×2 Matrix (Order 2)

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underbrace{ad - bc}_{\text{scalar}}}$$

X - butterfly

Ⓑ Determinant of a 3×3 Matrix (Order 3)

Method 1: Expansion by Cofactors

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

← Choose a "magic" row or column, say Row 1.

Take the corresponding signs from the sign matrix

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

"checkerboard"

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \underbrace{+}_{\substack{\text{from} \\ \text{sign} \\ \text{matrix}}} \underbrace{a}_{\substack{\text{1st} \\ \text{magic} \\ \text{entry}}} \underbrace{\begin{vmatrix} e & f \\ h & i \end{vmatrix}}_{\substack{\text{deleted row, column} \\ \text{with "a"}}}$$

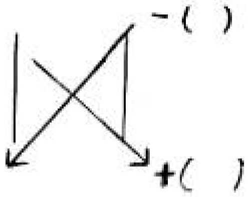
$$- b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

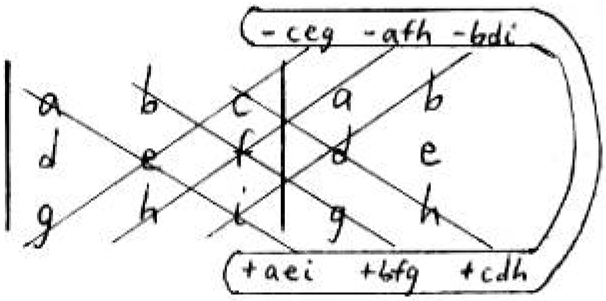
$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

Method 2 Sarrus's Rule (only for Order 3)

① Rewrite 1^{st}, 2^{nd} columns on the right.
② Add products along the 3 full diagonals ⑊⑊⑊
③ Subtract 〃 〃 ///

Like 2×2





(equivalent to result from Method 1)

Ⓒ $\vec{a} \times \vec{b}$

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$,
$\vec{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\boxed{\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$$

← Vectors, so not technically a "determinant."

<u>Ex</u> (#2) If $\vec{a} = \langle -5, 1, -1 \rangle$, $\vec{b} = \langle 3, 6, -2 \rangle$, find $\vec{a} \times \vec{b}$.

$$\vec{a} \times \vec{b} = \begin{array}{c} \begin{array}{ccc} \oplus & \ominus & \oplus \end{array} \\ \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & 1 & -1 \\ 3 & 6 & -2 \end{vmatrix} \end{array} \quad \leftarrow \text{for Method 1}$$

Method 1

$$= \begin{vmatrix} 1 & -1 \\ 6 & -2 \end{vmatrix} \vec{i} - \begin{vmatrix} -5 & -1 \\ 3 & -2 \end{vmatrix} \vec{j} + \begin{vmatrix} -5 & 1 \\ 3 & 6 \end{vmatrix} \vec{k}$$

magic entry   Careful!

$$= [-2-(-6)]\vec{i} - [10-(-3)]\vec{j} + [-30-3]\vec{k}$$

$$= \boxed{4\vec{i} - 13\vec{j} - 33\vec{k} \quad \text{or} \quad \langle 4, -13, -33 \rangle}$$

Method 2

$$-(3\vec{k}) - (-6\vec{i}) - (10\vec{j})$$

$$\left| \begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ -5 & 1 & -1 & -5 & 1 \\ 3 & 6 & -2 & 3 & 6 \end{array} \right.$$

$$+(-2\vec{i}) \; +(-3\vec{j}) \; +(-30\vec{k})$$

$$= -2\vec{i} - 3\vec{j} - 30\vec{k} - 3\vec{k} + 6\vec{i} - 10\vec{j}$$

Can do 1,5

$$= \boxed{4\vec{i} - 13\vec{j} - 33\vec{k} \quad \text{or} \quad \langle 4, -13, -33 \rangle}$$

(D) Basic Properties

$\vec{a}, \vec{b}, \vec{c}$ in V_3,
m scalar

$\vec{a} \times \vec{0} = \vec{0} = \vec{0} \times \vec{a}$ $\vec{a} \times \vec{a} = \vec{0}$	$\Big\} \vec{0}$
(i) $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$	"x" is <u>anticommutative</u>
(ii) $(m\vec{a}) \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times (m\vec{b})$	Scalar mult. is flexible re "x"
(iii) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ (iv) $(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$	$\Big\}$ Dot, cross <u>products</u> distribute over vector "+" from either side
(v),(vi) Later	

"x" is <u>not</u> commutative or associative.

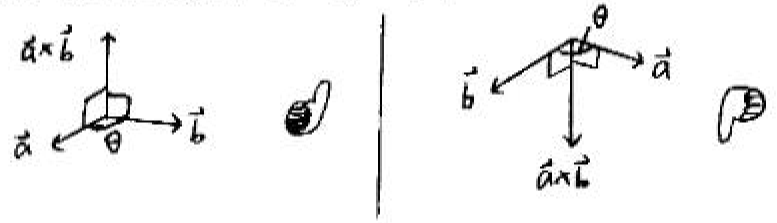
(E) Geometry

Direction of $\vec{a} \times \vec{b}$

$$\boxed{\begin{array}{l} \vec{a} \times \vec{b} \perp \vec{a} \\ \vec{a} \times \vec{b} \perp \vec{b} \end{array}} \leftarrow \text{Show } (\vec{a} \times \vec{b}) \cdot \vec{a} = 0.$$

Right-Hand Rule

If its fingers curl "through θ" $(0 < \theta < \pi)$ from $\vec{a}$ to $\vec{b}$, then its thumb points in the direction of $\vec{a} \times \vec{b}$.



Length of $\vec{a} \times \vec{b}$

$$\boxed{\underbrace{\| \underbrace{\vec{a} \times \vec{b}}_{\text{vector}} \|}_{\text{scalar}} = \|\vec{a}\| \|\vec{b}\| \sin\theta} \quad (\star)$$

Proof (Optional) - p. 713

Recall $\underbrace{\vec{a} \cdot \vec{b}}_{\text{scalar}} = \|\vec{a}\| \|\vec{b}\| \cos\theta$

$$\vec{a} \parallel \vec{b} \Leftrightarrow \theta = 0 \text{ or } \theta = \pi \text{ or } \vec{a} = \vec{0} \text{ or } \vec{b} = \vec{0}$$

$$\Leftrightarrow \sin\theta = 0$$

$$\Leftrightarrow \|\vec{a}\| \|\vec{b}\| \sin\theta = 0$$

$$\underset{(\star)}{\Leftrightarrow} \|\vec{a} \times \vec{b}\| = 0$$

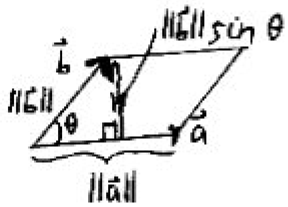
$$\Leftrightarrow \vec{a} \times \vec{b} = \vec{0}$$

$$\boxed{\vec{a} \parallel \vec{b} \Longleftrightarrow \vec{a} \times \vec{b} = \underset{\text{vector}}{\vec{0}}}$$

Recall $\vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = \underset{\text{scalar}}{0}$

$$\boxed{\begin{aligned} \|\vec{a}\times\vec{b}\| &= \text{area of parallelogram determined by } \vec{a},\vec{b} \\ \tfrac{1}{2}\|\vec{a}\times\vec{b}\| &= \quad '' \quad \text{triangle} \quad '' \end{aligned}}$$

Proof



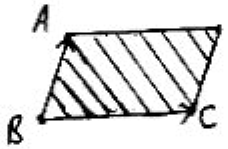
See 14.1.7
$\sin\theta = \frac{\text{opp.}}{H}$
$\text{Opp.} = H\sin\theta$

$$\begin{aligned} \text{Area} &= (\text{base})(\text{height}) \\ &= \|\vec{a}\|\ \|\vec{b}\|\sin\theta \\ &= \|\vec{a}\times\vec{b}\| \end{aligned}$$

3 $\underbrace{\text{noncollinear}}_{\text{do not lie on the same line}}$ points in $\mathbb{R}^3$ determine

ⓐ one triangle, and
ⓑ different parallelograms with the same area. } The 3 given points are 3 of the 4 vertices.

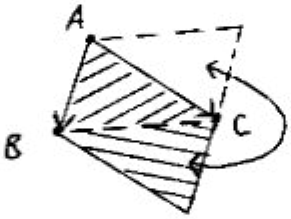
How are $\vec{BA}\times\vec{BC}$, $\vec{BC}\times\vec{BA}$ related?



$$\text{Area} = \|\vec{BA}\times\vec{BC}\|$$

{ opposite vectors have same lengths

$$= \|\vec{BC}\times\vec{BA}\|$$



$$= \|\vec{AB}\times\vec{AC}\|$$

etc.

Area of triangle ABC = $\frac{1}{2}$ (any of these)

Use all 3 points!

<u>Ex</u> Find the area of the triangle determined by $A(8,-3,2)$, $B(3,-2,1)$, and $C(11,3,0)$.

<u>Sol'n</u>

$$\vec{AB} = \langle 3-8, -2-(-3), 1-2 \rangle = \langle -5, 1, -1 \rangle$$

$$\vec{AC} = \langle 11-8, 3-(-3), 0-2 \rangle = \langle 3, 6, -2 \rangle$$

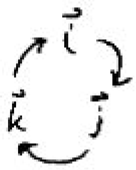
From Ⓒ, $\vec{AB} \times \vec{AC} = \langle 4, -13, -33 \rangle$

$$\begin{aligned} \text{Area} &= \tfrac{1}{2} \|\langle 4, -13, -33 \rangle\| \\ &= \tfrac{1}{2}\sqrt{(4)^2 + (-13)^2 + (-33)^2} \\ &= \tfrac{1}{2}\sqrt{1274} \\ &\approx \boxed{17.8} \end{aligned}$$

Ⓕ $\vec{i}, \vec{j}, \vec{k}$

face board

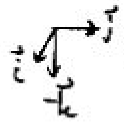
<u>Wheel:</u>



(vector) × (successor) = (3rd)

$\vec{i} \times \vec{j} = \vec{k}$:   and $\|\vec{i} \times \vec{j}\| = 1$ (Why? Use 14.4.5Ⓐ)

(vector) × (predecessor) = −(3rd)

$\Rightarrow \ominus$ $\vec{j} \times \vec{i} = -\vec{k}$:   and $\|\vec{j} \times \vec{i}\| = 1$

<u>$\vec{a} \times \vec{b}$ Table</u>

"×"

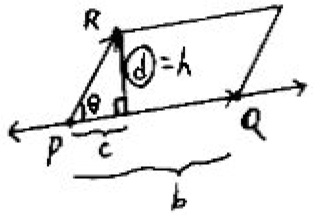
$\vec{a}$ \ $\vec{b}$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	$\vec{0}$	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	$\vec{0}$	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	$\vec{0}$

Like a skew-sym matrix

"×" is anticommutative

(G) The Distance Between a Point and a Line
(d) (R) ($\overleftrightarrow{PQ}$)

Proof on Ex 3, p. 715 uses sin, not ▱



Area of ▱ $= bh$

$$h = \frac{\text{Area}}{b}$$

(Use all 3 points)

$$\boxed{d = \frac{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|}{\|\overrightarrow{PQ}\|} = \frac{\|\overrightarrow{PR} \times \overrightarrow{PQ}\|}{\|\overrightarrow{PQ}\|}}$$

Up to 19

"$\vec{a}$"·"$\vec{b}$" can be defined as $\|\vec{a}\|\|\vec{b}\|\cos\theta$
$\|\vec{a}$"×"$\vec{b}\|$ sin

Recall $\quad c = \text{comp}_{\overrightarrow{PQ}} \overrightarrow{PR} = \dfrac{\overrightarrow{PR} \cdot \overrightarrow{PQ}}{\|\overrightarrow{PQ}\|}$ (Point off line omitted)

(H) Triple Scalar Product (TSP, Box Product)

Not Traveling Salesman Problem

Prop. (v) $\boxed{TSP = (\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})}$ (scalar)

i.e., You can switch "×" and "·"
i.e., Cross 2 successive vectors in order, and dot the result with the 3rd ∋ some #

Nonsense: $\underbrace{(\vec{a} \cdot \vec{b})}_{\text{scalar}} \times \underbrace{\vec{c}}_{\text{vector}}$

#21 (unassigned?)

For computation:

$$\boxed{TSP = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{matrix} \leftarrow \vec{a} \\ \leftarrow \vec{b} \\ \leftarrow \vec{c} \end{matrix}}$$

Can do: $TSP = \underbrace{|\vec{a}\ \vec{b}\ \vec{c}|}_{\text{by columns}}$; $\underset{\uparrow\ \text{matrix}}{|A|} = |\underbrace{A^T}_{\text{A-transpose (rows} \leftrightarrow \text{cols)}}|$

$$\boxed{\underset{\text{abs. value}}{|TSP|} = \text{volume of box (parallelepiped) determined by } \vec{a}, \vec{b}, \vec{c}}$$

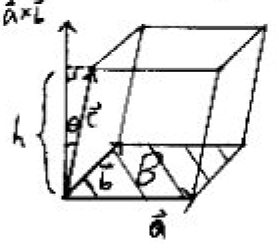
Larson 734 uses proj

$\vec{a}=\vec{0}$, or $\vec{b}=\vec{0}$ $\Rightarrow$ $\vec{a} \parallel \vec{b}$

$\vec{a} \parallel \vec{c} \Rightarrow \vec{a}\times\vec{b} \perp \vec{c} \Rightarrow TSP = 0$

dot com

<u>Proof</u> (Optional) p. 716 or:



$$\begin{aligned} V &= Bh \\ &= \|\vec{a}\times\vec{b}\| \, |\text{comp}_{\vec{a}\times\vec{b}}\,\vec{c}| \\ &= \cancel{\|\vec{a}\times\vec{b}\|} \left| \frac{\vec{c}\cdot(\vec{a}\times\vec{b})}{\underbrace{\cancel{\|\vec{a}\times\vec{b}\|}}_{>0}} \right| \quad \leftarrow (\vec{a} \nparallel \vec{b}) \text{ otherwise, } TSP = 0 \\ &= |(\vec{a}\times\vec{b})\cdot\vec{c}| \quad (\text{"}\cdot\text{" comm.}) \\ &= |TSP| \end{aligned}$$

$$\boxed{TSP = 0 \Leftrightarrow [\text{position vectors for}]\ \vec{a}, \vec{b}, \vec{c} \text{ are } \underbrace{\text{coplanar}}_{\text{lie in same plane}}}$$

	[Position] Vectors are	Think:
$\vec{a}\times\vec{b}=\vec{0}$	collinear	Area = 0
$TSP = 0$	coplanar	Vol. = 0

<u>(I) Triple Vector Product</u>

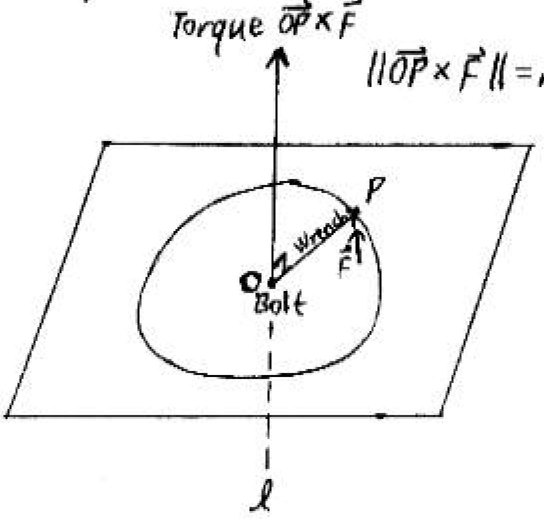
Prop. (vi) $\boxed{\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}}$ (I'll give)

"×" not associative:
Often, $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$

14.4.10

(J) Torque (Optional)

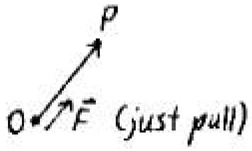
or Moment of $\vec{F}$ about O. P moves



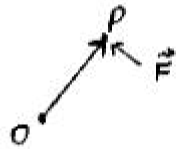
$\|\vec{OP}\times\vec{F}\|$ = magnitude of rotational effect about ℓ

$\|\vec{OP}\|$ fixed (constant).
Let's say $\|\vec{F}\|$ is fixed.

When is $\|\vec{OP}\times\vec{F}\|$ maximized?



$\|\vec{OP}\times\vec{F}\| = 0$



$\|\vec{OP}\times\vec{F}\| = \|\vec{OP}\|\,\|\vec{F}\|\underbrace{\sin\theta}_{\text{max'ed } (=1) \text{ when } \theta=\frac{\pi}{2}}$

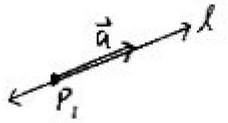
i.e., when $\vec{F}\perp\vec{OP}$

14.5: LINES and PLANES in $\mathbb{R}^3$

(A) What Determines a Line, ℓ?

(a) 2 distinct points  $\Rightarrow$ (b)

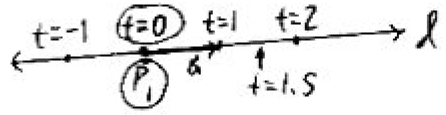
or (b) 1 point: $P_1(x_1, y_1, z_1)$ and
1 direction vector: $\vec{a} = \langle a_1, a_2, a_3 \rangle$
(replaces "slope")



"$\vec{a} \parallel \ell$"

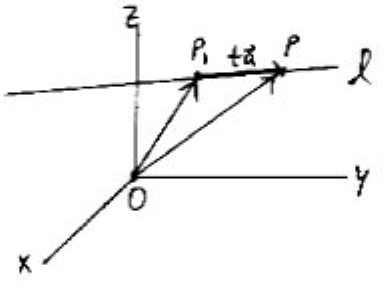
(B) Parametric Equations for a Line (many possibilities)

t ("time") is our parameter. (t) watch



(Different choice for P_1 or $\vec{a}$
$\Rightarrow$ Different parameterization for ℓ)

ℓ contains all points $P(x,y,z)$ such that

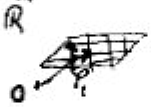


$\overrightarrow{OP} = \overrightarrow{OP_1} + t\vec{a}$ for some real t

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\text{Column form for a vector}} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + t\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Book's approach:
$\overrightarrow{P_1P} = t\vec{a}$
$\langle x-x_1, y-y_1, z-z_1 \rangle = t\langle a_1, a_2, a_3 \rangle$

Plane:
$\overrightarrow{OP_1} + t\vec{a} + u\vec{b}$
t, u in $\mathbb{R}$



$$\boxed{\begin{cases} x = \overset{P_1}{x_1} + \overset{\vec{a}}{a_1}t \\ y = y_1 + a_2 t \\ z = z_1 + a_3 t \end{cases} ,\ t \text{ in } \mathbb{R}}$$

(t sweeps through all real #s)

Ⓒ Symmetric Equations for a Line

Solve each eq. for t, and equate.

$$t = \boxed{\underbrace{\frac{x-x_1}{a_1}}_{\substack{\text{If } a_1=0 \\ \Rightarrow \text{“}x=x_1\text{”}}} = \frac{y-y_1}{a_2} = \frac{z-z_1}{a_3} \quad \leftarrow \text{if none are } 0}$$

etc.

Ⓓ Ex

The line ℓ contains $P_1(3,-7,2)$ and $P_2(5,-10,2)$.

ⓐ Find parametric eqs. for ℓ.
ⓑ Find symmetric eqs. for ℓ.

Point: $P_1\ (3,-7,2)$ $\quad$ (P_2 OK)

$\vec{a} = \overrightarrow{P_1P_2}$ $\quad$ ($\overrightarrow{P_2P_1}$ OK $\Rightarrow$ time reversal)

$= \langle 5-3, -10-(-7), 2-2 \rangle$

$= \langle 2, -3, 0 \rangle$

ⓐ Param. eqs.:

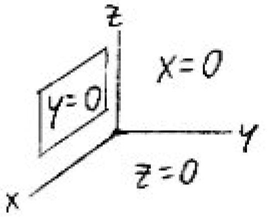
$$\boxed{\begin{aligned} x &= 3 + 2t \\ y &= -7 - 3t \\ z &= 2 \end{aligned}} \quad \begin{aligned} &\Longrightarrow t = \frac{x-3}{2} \\ &\Longrightarrow t = \frac{y+7}{-3} \\ &\xrightarrow[(a_3=0)]{} z = 2 \end{aligned}$$

$\}$ Equate

ⓑ Sym. eqs.:

$$\boxed{\frac{x-3}{2} = \frac{y+7}{-3}, \quad z=2}$$

Ⓒ Where does ℓ intersect the xz-plane?



$$y = -7 - 3t \overset{\text{set}}{=} 0$$

$$t = -\tfrac{7}{3} \quad \text{(time of "impact")}$$

$$\Rightarrow \begin{cases} x = 3 + 2\left(-\tfrac{7}{3}\right) = -\tfrac{5}{3} \\ y = 0 \\ z = 2 \end{cases}$$

Up to 9

$$\boxed{\left(-\tfrac{5}{3}, 0, 2\right)}$$

Where does ℓ intersect the xy-plane?
How do you graph $z=2$? ℓ lies on here.

Another Method: Use sym. eqs. Set $y=0 \Rightarrow \underbrace{\frac{x-3}{2} = \frac{0+7}{-3}}_{\Rightarrow x = -\frac{5}{3}}, z=2$

$\Rightarrow \left(-\tfrac{5}{3}, 0, 2\right)$ again

Ⓔ Do 2 Lines Intersect? Where?

Ex

Imagine two jets leaving trails. If the trails intersect, must the jets crash?

$$\ell_1 : ☆ \begin{cases} x = 1 + 3t \\ y = 6 - 4t \\ z = -1 + 2t \end{cases} \quad t \text{ in } \mathbb{R} \qquad \ell_2 : ☆☆ \begin{cases} x = 2 - 5u \\ y = 2u \\ z = 4 + u \end{cases} \quad u \text{ in } \mathbb{R}$$

(x,y,z) is an intersection point $\Longleftrightarrow$

There is a "time" t at which ℓ_1 "hits" (x,y,z),
and " " u " " ℓ_2 " "

Solve the system — often used to find intersection pts.

$$\begin{cases} 1+3t = 2-5u \\ 6-4t = 2u \\ -1+2t = 4+u \end{cases}$$

$$\Leftrightarrow \begin{cases} 3t+5u = 1 \\ -4t-2u = -6 \\ 2t-u = 5 \end{cases}$$ Solve (★★★), say (easiest pair?) ✓
$\Rightarrow t=2, u=-1$

<u>Note</u> If (★★★) has no solution $\Rightarrow$ no intersection pts.
If (★★★) has ∞ many solutions $\Rightarrow$ Math 254
(Projections/shadows on yz-plane might coincide.)

Subsystem (★★★) has a sol'n $\Leftrightarrow$ The projections of the lines on the yz-plane intersect. Then, ✓ to see if x-coords match.

Ideas re ∞ many sols:

$\begin{cases} x=0 \\ \vdots \end{cases}$ $\begin{cases} x=0 \\ \vdots \end{cases}$

$\begin{cases} 0=0 \\ 1=1 \end{cases}$ or delete both / use $x=0$

$\begin{cases} 0=0 \\ \text{line} \end{cases}$ ← combine w/remaining eq.

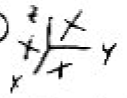
$\begin{cases} \text{line} \\ \text{line} \end{cases}$ equivalent; take one and combine w/remaining eq.
↑ in the (not xy) plane

Check to see if $t=2, u=-1$ satisfy the remaining eq.

If so, l_1 and l_2 intersect.
If not, they don't.

$$3t+5u=1$$
$$3(2)+5(-1)=1$$
$$1=1 \checkmark \quad \Rightarrow l_1 \text{ and } l_2 \text{ intersect.}$$

<u>Side Notes</u>
<u>Exs</u> Shadows of intersecting lines

① 

② 

 $\Rightarrow$ 0 sols. to overall system

 same line $\Rightarrow$ 0, 1, or ∞

 $\Rightarrow$ 0 or 1

 $\Rightarrow \infty$

What's the intersection pt. (x,y,z)?

Plug $t=2$ into (★) or $u=-1$ into (★★).

$$(★)\begin{cases} x = 1+3(2) = 7 \\ y = 6-4(2) = -2 \\ z = -1+2(2) = 3 \end{cases}$$

$$\boxed{(7,-2,3)}$$

1/p to 13

Ⓕ Angles Between Lines

Ex from Ⓔ

$$\ell_1 : \begin{cases} x = 1 + 3t \\ y = 6 - 4t \\ z = -1 + 2t \end{cases} \qquad \ell_2 : \begin{cases} x = 2 - 5u \\ y = 2u \\ z = 4 + u \end{cases}$$

direction vector

$\vec{a} = \langle 3, -4, 2 \rangle$ $\qquad$ $\vec{b} = \langle -5, 2, 1 \rangle$

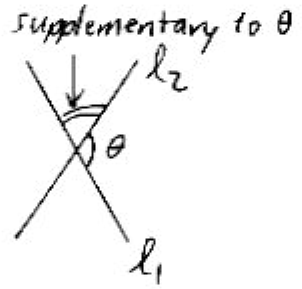
$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right)$$

$$= \cos^{-1}\left(\frac{-21}{\sqrt{29}\sqrt{30}}\right)$$

$$\approx \boxed{2.36 \text{ radians, or } 135^\circ}$$

$\pi - \theta$ $\quad$ Get supp. angle. $\quad$ $180^\circ - \theta$

$$\boxed{0.78 \text{ radians, or } 45^\circ}$$



Applies to nonintersecting lines, also.
Translate lines without rotating ⇒ same angles.

If $\vec{a}$ is any direction vector for ℓ_1, and $\vec{b}$ … ℓ_2, then

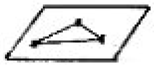
$$\ell_1 \parallel \ell_2 \iff \vec{a} \parallel \vec{b}$$
$$\ell_1 \perp \ell_2 \iff \vec{a} \perp \vec{b}$$

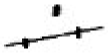
Up to 15

Ⓖ What Determines a Plane?

3 points if they don't...
2 skew lines ∄ plane

Truneh: Imagine 3 legs of a slanted stool
||| Tips over: |||

ⓐ 3 noncollinear points  $\Rightarrow$ ⓒ

or ⓑ 1 line and a point not on the line  $\Rightarrow$ ⓒ

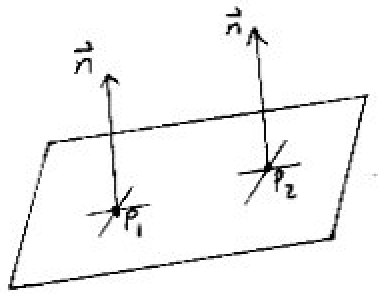
or ⓒ 1 point and 2 non-$\parallel$ vectors  $\Rightarrow$ ⓓ

or ⓓ 1 point and a normal vector $\vec{n}$

$\vec{n} \neq \vec{0}$

$\vec{n} \perp$ plane

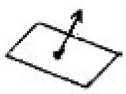
(i.e., $\vec{n} \perp$ [direction vectors for] every line in the plane)



$\vec{n} \perp$ every line in the plane containing P_1, P_2

$\vec{n} \perp$ "

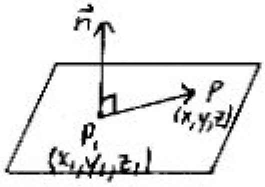
How to Ace: "joystick on a flying carpet"



(H) Equation Forms for a Plane

Given: Point $P_1(x_1, y_1, z_1)$ in the plane
$\vec{n} = \langle a, b, c \rangle$

A point $P(x, y, z)$ is in the plane



$$\iff \vec{n} \cdot \overrightarrow{P_1P} = 0$$

$$\iff \langle a, b, c \rangle \cdot \langle x - x_1, y - y_1, z - z_1 \rangle = 0$$

$$\iff \boxed{a(x - x_1) + b(y - y_1) + c(z - z_1) = 0}$$

<u>Standard Form</u>

Larson

$$\iff \boxed{ax + by + cz + d = 0}$$

d — a real # $= -ax_1 - by_1 - cz_1$

<u>General Form</u>

Larson

If a, b, c are not all 0,

The graph of $ax + by + cz + d = 0$ is a plane with normal $\vec{n} = \langle a, b, c \rangle$

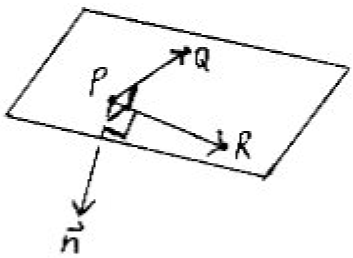
Ex Find an eq. for the plane determined by $P(2,-3,4)$, $Q(-3,-2,3)$, and $R(5,3,2)$. ← noncollinear

Sol'n

$$\left.\begin{aligned}\vec{PQ} &= \langle -5, 1, -1\rangle \\ \vec{PR} &= \langle 3, 6, -2\rangle\end{aligned}\right\} \text{There are other possibs.}$$

Find a normal vector for the plane.

noncollinear
$\Rightarrow \vec{PQ} \not\parallel \vec{PR}$
$\Rightarrow \vec{PQ} \times \vec{PR} \neq \vec{0}$

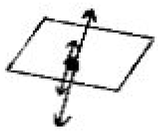


$$\left.\begin{aligned}\vec{n} &\perp \vec{PQ} \\ \vec{n} &\perp \vec{PR}\end{aligned}\right\} \Rightarrow \vec{n} \perp \text{plane}$$

Math 254
See 14.3, HW #44

Let $\vec{n} = \vec{PQ} \times \vec{PR}$
$\quad\vdots$ (see 14.4.3)
$\quad = \langle 4, -13, -33\rangle$

(or any non-$\vec{0}$ scalar multiple) ✪



$$\boxed{\text{Standard Form (using } P\text{, say)}: \quad 4(x-2) - 13\underbrace{(y-(-3))}_{(y+3)} - 33(z-4) = 0}$$

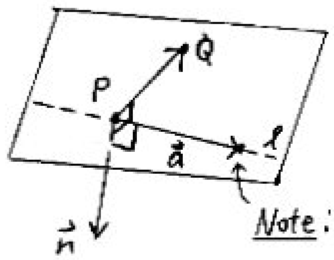
✪ OK to mult. through by any non-0 scalar.

$$\boxed{\text{General Form}: \quad 4x - 13y - 33z + 85 = 0}$$

Ex Find an eq. for the plane containing the point $Q(-3,-2,3)$ and the line

$$\ell: \begin{cases} x = 2 + 3t \\ y = -3 + 6t \\ z = 4 - 2t \end{cases}, \quad t \text{ in } \mathbb{R}$$

Sol'n $P(2,-3,4)$ $(t=0)$ is on ℓ $\vec{a} = \langle 3,6,-2 \rangle$ direction vector for ℓ



Let $\vec{n} = \overrightarrow{PQ} \times \vec{a}$

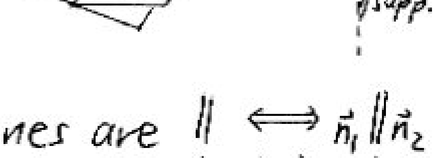
⋮

(same as in previous Ex)

(I) Angles Between Planes

Let $\vec{n}_1$ be a normal vector for Plane #1.
Let $\vec{n}_2$ be a normal vector for Plane #2.
Let θ be the angle between $\vec{n}_1, \vec{n}_2$.

$\Rightarrow$ The angles between the planes are θ and its supplement.



The planes are $\parallel \iff \vec{n}_1 \parallel \vec{n}_2$
$\perp \iff \perp$

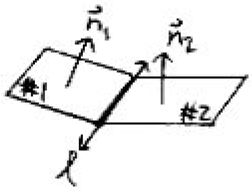
Up to 45

14.5.10

(J) The Line of Intersection of 2 Planes

In How to Ace, mentioned in Larson 740

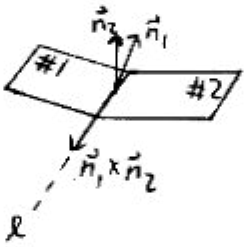
Not in book!



Find a direction vector for ℓ

$\vec{n}_1 \perp$ Plane #1 (including ℓ)
$\vec{n}_2 \perp$ Plane #2

If planes ∦, $\vec{n}_1$ and $\vec{n}_2$ determine a family of ∥ planes. (Together w/ 1 pt. ⇒ 1 plane) Only vectors ∥ $\vec{n}_1 \times \vec{n}_2$ are ⊥ these planes (i.e., $\perp \vec{n}_1$, $\perp \vec{n}_2$) ⇒ $\vec{n}_1 \times \vec{n}_2$ = a direc. vector for ℓ



$$\Rightarrow \underbrace{\vec{n}_1 \times \vec{n}_2}_{\text{Take this!}} \parallel \ell$$

(What if the planes are ∥?)

Ex 12 in book (pp. 725-6) - but new way!

$$\text{☆}\left\{\begin{array}{ll} 2x - y + 4z = 4 & \text{(Plane \#1)} \\ x + 3y - 2z = 1 & \text{(Plane \#2)} \end{array}\right\} \Rightarrow \text{line } \ell$$

Find parametric eqs. and symmetric eqs. for ℓ.

Sol'n

Find $\vec{a}$, a direction vector for ℓ.

$$\begin{aligned} \text{Let } \vec{a} &= \vec{n}_1 \times \vec{n}_2 \\ &= \langle 2, -1, 4 \rangle \times \langle 1, 3, -2 \rangle \\ &\vdots \\ &= \langle -10, 8, 7 \rangle \end{aligned}$$

Where does ℓ hit the xy-plane?

A line must intersect $x=a$, $y=b$, or $z=c$.

Ex $\begin{cases} x+y+z=1 \\ x+y+4z=2 \end{cases}$

$z=0 \Rightarrow$ No soln

$\Rightarrow$ line of intersection, xy-plane ($z=0$) do not intersect

Find a point on ℓ.

Plug $z=0$, say, into ⓐ.

(← A line in xyz-space must intersect at least one coord. plane ($x=0$, $y=0$, or $z=0$). Think: Electric fences)

$$\Rightarrow ⓐ \begin{cases} 2x - y = 4 \\ x + 3y = 1 \end{cases}$$

$$\xrightarrow{\text{Solve!}} x = \frac{13}{7},\ y = -\frac{2}{7},\ (z=0)$$

$$\left(\frac{13}{7}, -\frac{2}{7}, 0\right)$$

Parametric Eqs. for ℓ

$$\begin{cases} x = \frac{13}{7} - 10t \\ y = -\frac{2}{7} + 8t \\ z = 7t \end{cases},\ t \text{ in } \mathbb{R}$$

(Why is the book's answer on p. 726 also OK?)

Symmetric Eqs. for ℓ

$$\boxed{\frac{x - \frac{13}{7}}{-10} = \frac{y + \frac{2}{7}}{8} = \frac{z}{7}}$$

14.5.12

Ⓚ The Distance (h) Between a Point: $P_0(x_0, y_0, z_0)$ and a Plane: $\underbrace{ax + by + cz + d}_{\star} = 0$

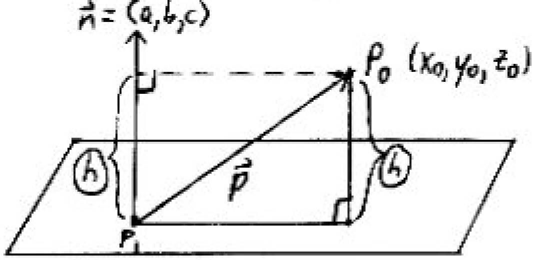
Key for Distance

$$\begin{pmatrix}\text{A normal } \vec{n} \\ \text{for a plane}\end{pmatrix} \perp \text{ The plane}$$

↓ implies "shortest distance"

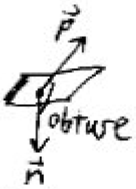
Think of $\vec{p}$ as a tether or connector, we really want $|comp_{\vec{n}}\vec{p}|$. We prefer directions $\perp$ plane.

Figure



(x_1, y_1, z_1), an arbitrary point in the plane, satisfies $\star = 0$

$\Rightarrow \underbrace{d = -ax_1 - by_1 - cz_1}_{\star\star}$



$\Rightarrow \vec{p}\cdot\vec{n} < 0$

$\Rightarrow comp_{\vec{n}}\vec{p} < 0$

$$\boxed{\begin{aligned} h &= |comp_{\vec{n}}\vec{p}| \\ &= \frac{|\vec{p}\cdot\vec{n}|}{\|\vec{n}\|} \end{aligned}}$$

We'll use in Ⓜ.

$$= \frac{|\langle x_0 - x_1, y_0 - y_1, z_0 - z_1 \rangle \cdot \langle a,b,c \rangle|}{\|\langle a,b,c \rangle\|}$$

by $\star\star$

$$\boxed{h = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

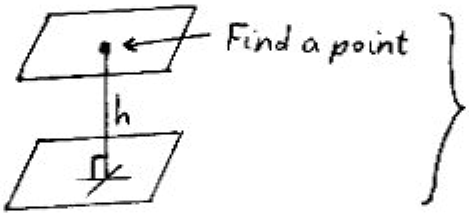
← |Result from "plugging P_0" into $\star$|

← $\|\vec{n}\|$

No need to find P_1!! See Ex in Ⓛ.

(L) The Distance (h) Between ∥ Planes

The dist. bet. $x+y=1$, $x+y=2$ is not 1; it's $\frac{\sqrt{2}}{2}$. Why? True in 2-D and in 3-D



$\Rightarrow$ Use (K)

What if $4x-2y+10z+6=0$? →

Ex Find the distance between [the graphs of] $2x-y+5z+3=0$ (Plane #1)
and $4x-2y+10z+8=0$ (Plane #2)

Sol'n

Show that the planes are ∥.

We can let $\vec{n}_1 = \langle 2,-1,5 \rangle$,
$\vec{n}_2 = \langle 4,-2,10 \rangle$.

$\Rightarrow \vec{n}_2 = 2\vec{n}_1$
$\Rightarrow \vec{n}_2 \parallel \vec{n}_1$
$\Rightarrow$ The planes are ∥.

Find a point on, say, Plane #2 ("uglier eq.")

$$4x-2y+10z+8=0$$

Choose $y=0$, $z=0$, say.

$\Rightarrow 4x+8=0$
$\Rightarrow x=-2$

$(-2,0,0)$

If you had $4x-2y+8=0$, you could still choose $z=0$.
$\vec{n} = \langle 4,-2,0 \rangle$ anyway when do numerator of h formula

The other ("nicer") plane has eq. $ax+by+cz+d=0$.

$$\underbrace{2x - y + 5z + 3 = 0}_{\text{☆}}$$

Use Ⓚ

$$h = \frac{|2(-2) - (0) + 5(0) + 3|}{\sqrt{(2)^2 + (-1)^2 + (5)^2}} \quad \leftarrow \text{Plug } (-2,0,0) \text{ into ☆}$$

$$= \frac{1}{\sqrt{30}}$$

$$\approx \boxed{0.18} \text{ [units]}$$

Ⓜ The [Shortest] Distance (h) Between Skew Lines

Think: Edges of box or room.

neither ∥, nor intersecting

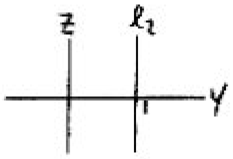
noncoplanar

Ex

$$l_1 : \begin{cases} x = t \\ y = 0 \\ z = 0 \end{cases} \qquad l_2 : \begin{cases} x = 0 \\ y = 1 \\ z = u \end{cases} \qquad 0 \neq 1 \Rightarrow l_1, l_2 \text{ don't intersect.}$$

t, u in $\mathbb{R}$

$$\vec{a}_1 = \langle 1,0,0 \rangle \nparallel \vec{a}_2 = \langle 0,0,1 \rangle \quad \Rightarrow l_1 \nparallel l_2$$

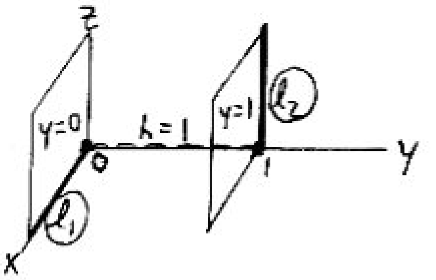


l_1 (x-axis) sticks out at you.

14.5.15

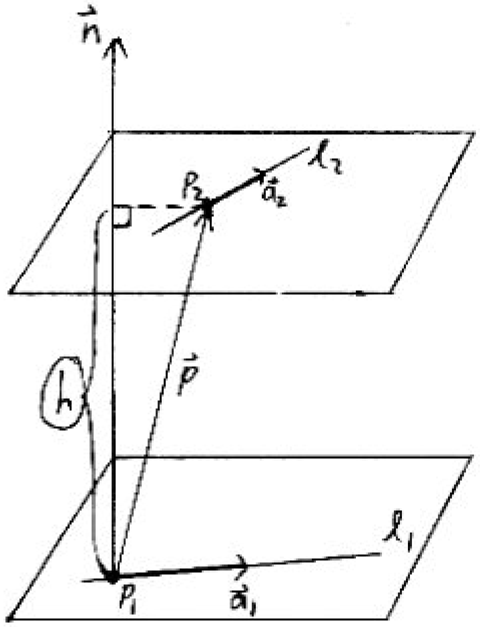
We want the distance between ∥ planes containing skew lines.

In Ex



Book more complicated.
$\vec{n}$: Remember Right-Hand Rule

Fall back to a more fundamental formula.



We don't have eqs. for these planes!

Use: $h = |\text{comp}_{\vec{n}}\,\vec{p}|$

① Find a point P_1 on ℓ_1,
" P_2 on ℓ_2.

② Let $\vec{p} = \overrightarrow{P_1P_2}$.

③ Let $\vec{a}_1$ be a direction vector for ℓ_1.
Let $\vec{a}_2$ " " ℓ_2.

④ Let $\vec{n} = \vec{a}_1 \times \vec{a}_2$.

$$\left(\begin{array}{l}\Rightarrow \vec{n} \perp \vec{a}_1,\ \vec{n} \perp a_2 \\ \Rightarrow \vec{n} \perp \text{both planes}\end{array}\right)$$

⑤ $h = |\text{comp}_{\vec{n}}\,\vec{p}|$

$$= \frac{|\vec{p}\cdot\vec{n}|}{\|\vec{n}\|}$$

Do HW #55

(N) Graphs

Ex The plane $x - 2y + 3z = 6$

Find x-intercept (if any)

Plug in $y=0, z=0$

$\Rightarrow x=6$ or $(6,0,0)$ is x-int.

y-int. $= -3$ or $(0,-3,0)$

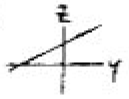
z-int. $= 2$ or $(0,0,2)$

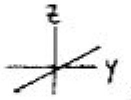
Note

ⓐ $-2y+3z=6$ has no x-int.

ⓑ $-2y+3z=0$ includes the entire x-axis.

Sweep // x-axis:

ⓐ 

ⓑ 

Find xy-trace

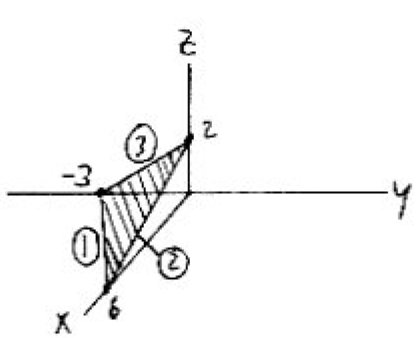
intersection with the xy-plane

Plug in $z=0$

$\Rightarrow x - 2y = 6,\ z=0$ (line) ①

xz-trace: $x + 3z = 6,\ y=0$ ②

yz-trace: $-2y+3z = 6,\ x=0$ ③



<u>Ex</u> Line l_1 from (E) $\begin{cases} x = 1+3t \\ y = 6-4t \\ z = -1+2t \end{cases}$

has symmetric eqs.

$$\frac{x-1}{3} = \frac{y-6}{-4} = \frac{z+1}{2}$$

(1)

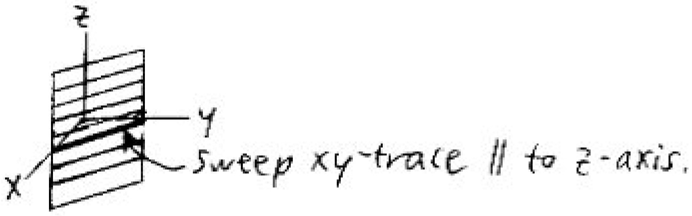
(2)

(3)

l_1 is the intersection of, say, (1) and (2). ((3) also contains l_1.)

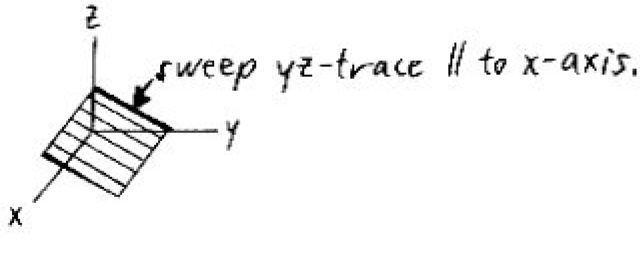
(1) Plane ⊥ xy-plane

xy-trace
$y = -\frac{4}{3}x + 7\frac{1}{3}$



(2) Plane ⊥ yz-plane

yz-trace
$z = -\frac{1}{2}y + 2$



14.6: SURFACES in 3D

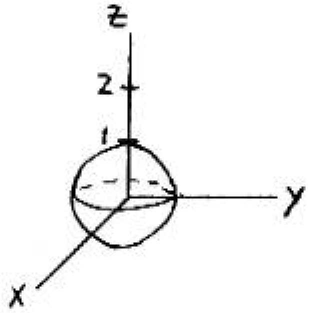
(A) Traces

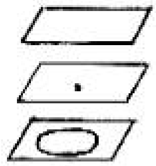
Ex Unit sphere $x^2+y^2+z^2=1$

(1) Traces in (intersections with) planes of the form $z=k$

"cross sections"

Like stacking transparent playing cards





$z=2 \Rightarrow$ empty

$z=1 \Rightarrow$ point

$z=0 \Rightarrow$ xy-trace: circle

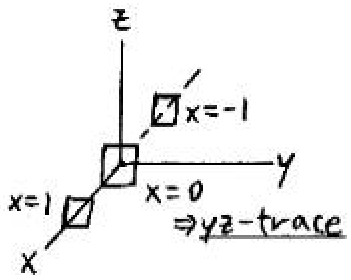
$x^2+y^2=1,\ z=0$

If $z=k \Rightarrow x^2+y^2+k^2=1$

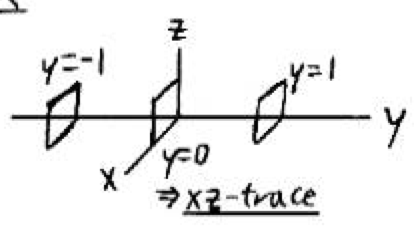
$$x^2+y^2=1-k^2$$

$$\text{family of traces}\begin{cases}\text{circle}, & |k|<1\\ \text{point}, & |k|=1\\ \text{empty}, & |k|>1\end{cases}$$

(2) $x=k$



(3) $y=k$



Ⓑ Spheres (14.2)

Have circles as traces

Ⓒ Planes (14.5)

Have lines as traces

Ⓓ Cylinders (see 14.2.5)

Have a family of "identical" traces.
maybe shifted

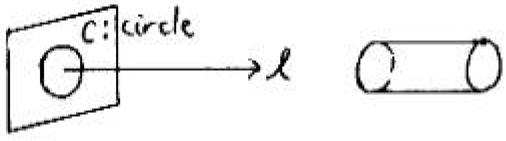
If a plane curve C is swept // to an $\underbrace{\text{axis}}_{\nparallel \text{ to the plane}}$,
the result is a cylinder.

C is called the directrix or generating curve.

Ex

parabolic cylinder

Ex

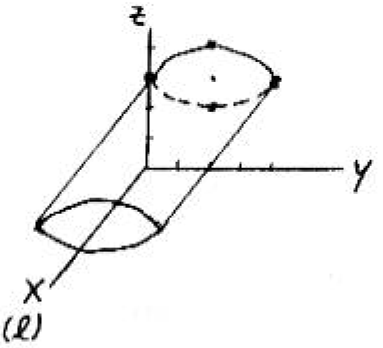


right circular cylinder
ℓ ⊥ plane

<u>Ex</u> Graph $\frac{(y-2)^2}{4} + \frac{(z-3)^2}{1} = 1$ (☆)

yz-trace: Ellipse

Center: $y=2$, $z=3$
$a^2 = 4 \Rightarrow a = 2$
$b^2 = 1 \Rightarrow b = 1$

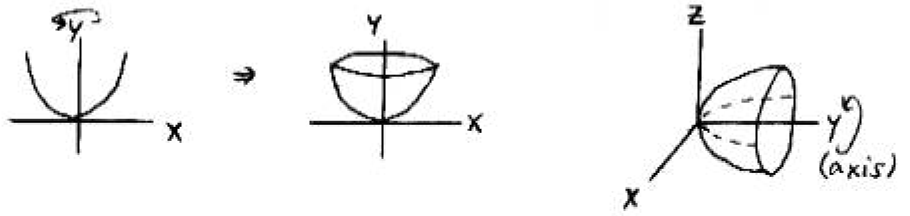


x missing in (☆)
$\Rightarrow$ sweep trace $\parallel$ to x-axis

<u>right elliptic cylinder</u>

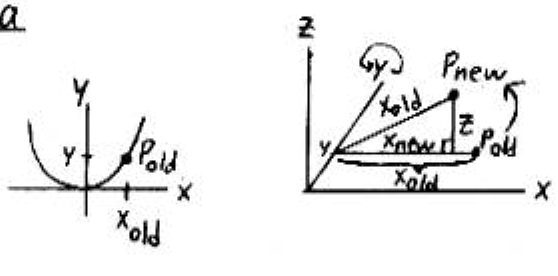
Ⓔ Surfaces of Revolution

<u>Ex</u> The graph of $y=x^2$ in the xy-plane is revolved about the y-axis. Find an eq. of the resulting surface.



Book a mess

<u>Idea</u>



$x_{old}^2 = x_{new}^2 + z^2$

$y = x^2$

Replace x^2 with $x^2 + z^2$

not the "axis variable" — missing variable — (Don't touch "axis variable.")

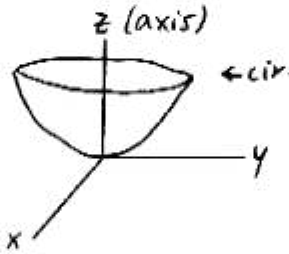
$\underbrace{\boxed{y = x^2 + z^2}}_{\Rightarrow y \geq 0}$ paraboloid with axis: y-axis
(opens along)

Ⓕ Variable-Switch Trick

⇓

Switch roles

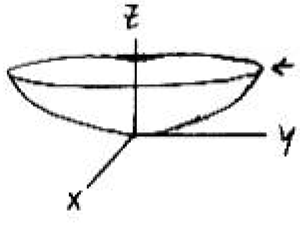
Ex $\boxed{z = x^2 + y^2}$ paraboloid with axis: z-axis
(opens along)



circular paraboloid

Ⓖ Coefficients Trick (for 14.6)

Ex $\boxed{z = 4x^2 + y^2}$



elliptic paraboloid
not a surface of revol.

If you multiply or divide a term by a positive real #, you may change the shape of a graph but not its general type.
"paraboloid"

Exception: We distinguish between spheres and ellipsoids.

(H) Quadric Surfaces (Q.S.s)

(My approach differs from the book's!)

Swokowski 734

Graphs of [nondegenerate] 2^{nd}-degree eqs. in x, y, and z.
(If any are missing $\Rightarrow$ cylinder?)

The trace of a "basic*" Q.S. in $x=k$, $y=k$, or $z=k$ can be:

1. empty
2. 1 point
3. 2 intersecting lines X (degenerate hyperbola)

or 4. a conic
- E = Ellipse (or Circle)
- H = Hyperbola
- P = Parabola

*Note Other Q.S.s may be rotated or have single lines as traces.

We can't have different types of conics in, $z=2$ and $z=3$, say.
We can $z=2$ and $x=2$, say.

Trick Plug $z=0$ into the eq. to find the xy-trace [eq.].
If you get a conic, it must be the only conic type for the "$z=k$" family of traces.

6 Basic Q.S. Types

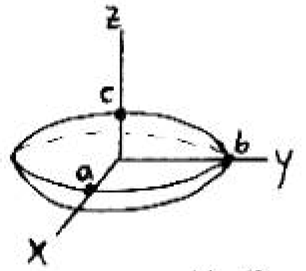
a,b,c : constants, >0

(H1) Ellipsoid

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1}$$

Conic Traces

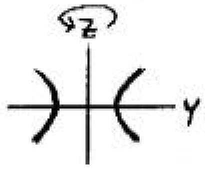
$x=k$	$y=k$	$z=k$
E	E	E



No "axis."

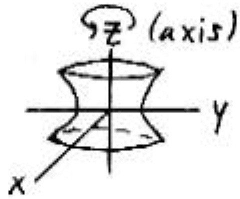
If $a=b=c \Rightarrow$ Sphere

(H2) Hyperboloid of 1 Sheet



$y^2-z^2=1$

Opens along y-axis
Hint: No z-ints.



(axis) $(y^2+x^2)-z^2=1$

$$\boxed{x^2+y^2-z^2=1}$$ (Basic Ex)

axis is z-axis

$x=k$	$y=k$	$z=k$ (⊥ axis)
H	H	E

↓
)($|k|<1$
X $|k|=1$
$|k|>1$
similar

More generally, by Coefficients Trick:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

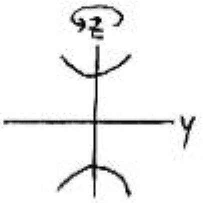
(H3) Hyperboloid of 2 Sheets

Morphing

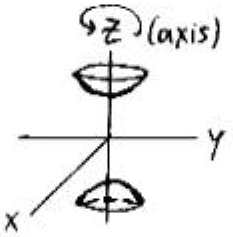
$z^2 = x^2+y^2+1$

$z^2 = x^2+y^2$

$z^2 = x^2+y^2-1$



$$z^2 - y^2 = 1$$



$$z^2 - (y^2+x^2) = 1$$
$$z^2 - x^2 - y^2 = 1$$

(2) "-"s ⇒ Hyp. of (2) sheets

$$\boxed{z^2 = x^2+y^2+1}$$

z^2 never 0
⇒ z never 0
contrast with

(⊥ axis)

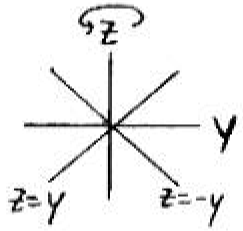
$x=k$	$y=k$	$z=k$
H	H	E

(no xy-trace)

(H4) [Circular/Elliptic] Cone

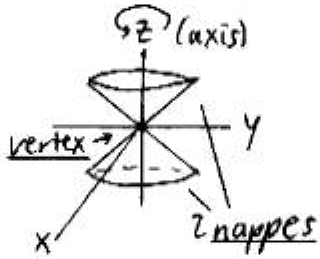
What do I revolve about z?

How can I write as 1 eq.?



$$z = \pm y$$

$$z^2 = y^2$$



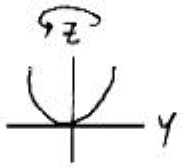
$$\boxed{z^2 = x^2+y^2}$$

contains (0,0,0)

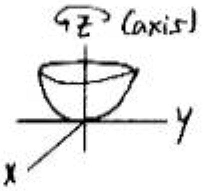
(⊥ axis)

$x=k$	$y=k$	$z=k$
H	H	E

(H5) [Circular / Elliptic] Paraboloid



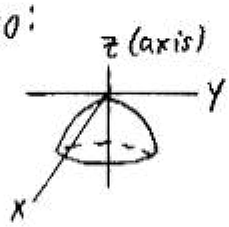
$z = y^2$



$\boxed{z = x^2 + y^2}$

$x = k$	$y = k$	$z = k$ (⊥ axis)
P	P	E

Also:

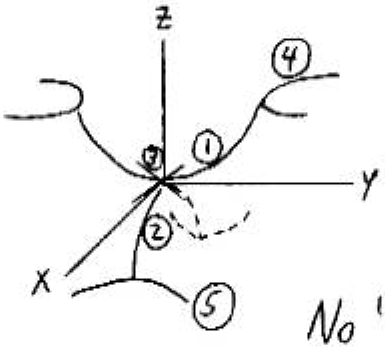


$\boxed{-z = x^2 + y^2}$

(H6) Hyperbolic Paraboloid

$\boxed{z = y^2 - x^2}$

"saddle"



No "axis"

$x = k$	$y = k$	$z = k$
P	P	H

① $x = 0 \Rightarrow z = y^2$

② $y = 0 \Rightarrow z = -x^2$

③ $z = 0 \Rightarrow y^2 = x^2$

④ $z > 0 \Rightarrow y^2 - x^2 = (\# > 0)$

⑤ $z < 0 \Rightarrow y^2 - x^2 = (\# < 0)$
$\Rightarrow x^2 - y^2 = (\# > 0)$

Also: $\boxed{\begin{array}{l} -z = y^2 - x^2 \Leftrightarrow \\ z = x^2 - y^2 \end{array}}$

Extensions

Note 1 Recall the Variable-Switch and Coefficients Tricks

Note 2 If x is replaced by $(x-x_0)$,
y " " $(y-y_0)$,
z " " $(z-z_0)$

$\Rightarrow$ Translation in which $(0,0,0)$ is moved to (x_0, y_0, z_0)

Note 3 Rotations may lead to cross-terms such as xy.

Memorize

step-by-step changes

	Basic Eqs.	$\Rightarrow$ Axis	$x=k$	$y=k$	$z=k$
			(Can figure out)		
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	None	E	E	E
Hyp.-1 Sheet	$x^2+y^2-z^2=1$	z	H	H	E
Hyp.-2 Sheets	$z^2=x^2+y^2+1$	z			
Cone	$z^2=x^2+y^2$	z			
Paraboloid	$z=x^2+y^2,\ -z=x^2+y^2$	z	P	P	E
Hyp. Paraboloid	$z=y^2-x^2,\ z=x^2-y^2$	None ($z=k \Rightarrow H$)	P	P	H

Ex Identify the surface $x^2 = \frac{y^2}{6} - 3z^2 + 4$. (☆)

Sol'n By Coefficients Trick, consider $x^2 = y^2 - z^2 + 1$
$\Leftrightarrow x^2 - y^2 + z^2 = 1$ (☆☆)

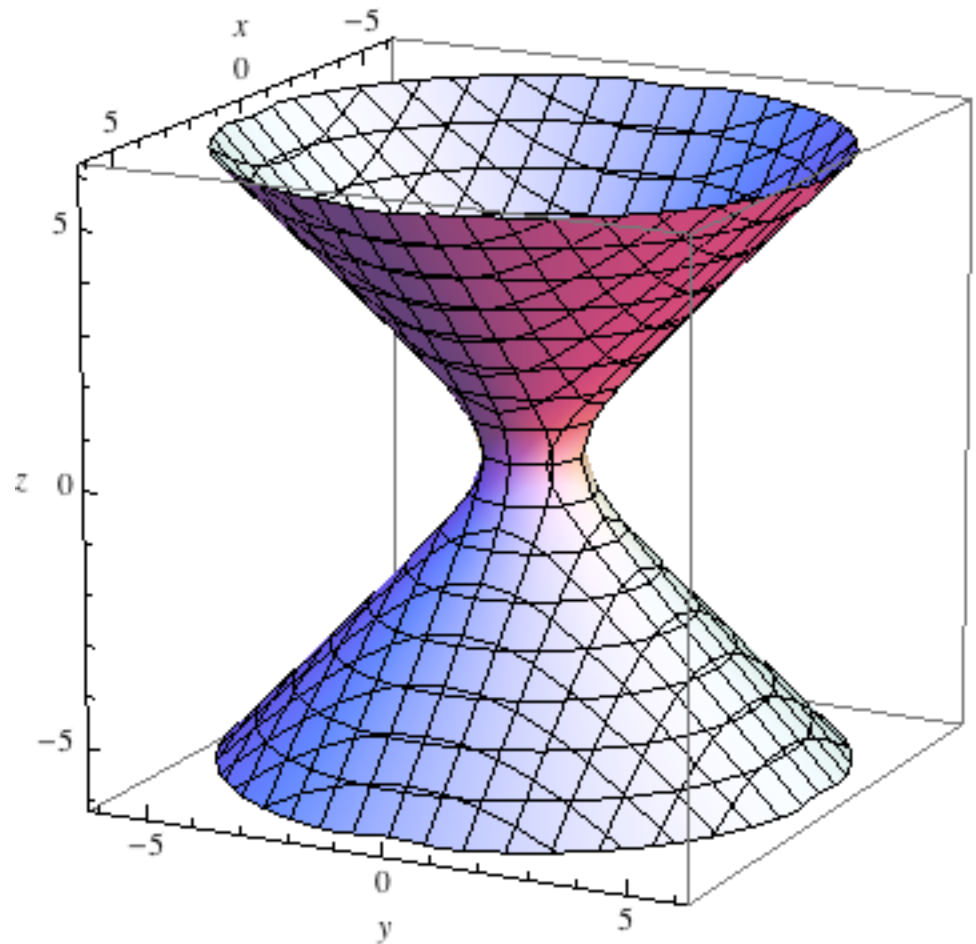
Looks like $x^2+y^2-z^2=1$, except switch $z \leftrightarrow y$.

$\boxed{\text{Hyperboloid of 1 sheet with the y-axis as its axis.}}$

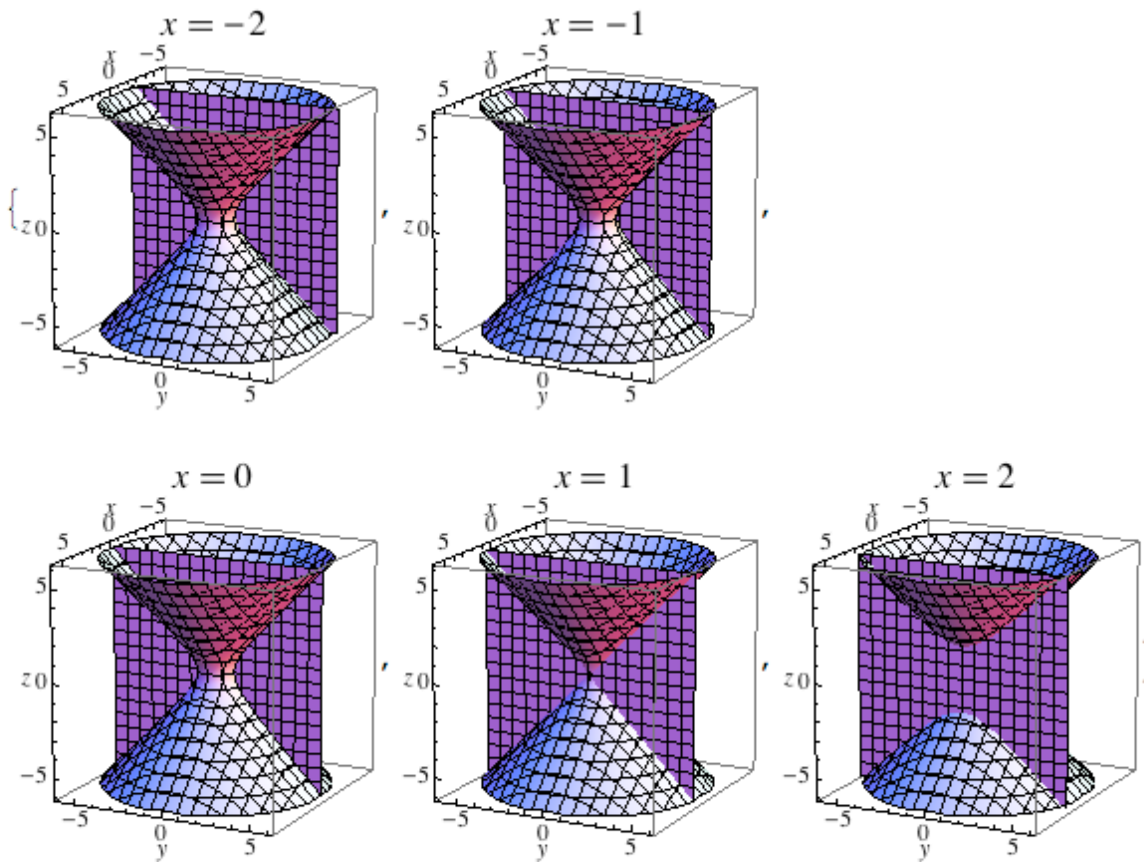
Note (☆☆) is a surface of revolution, but (☆) is not.

$$x^2 + y^2 - z^2 = 1$$

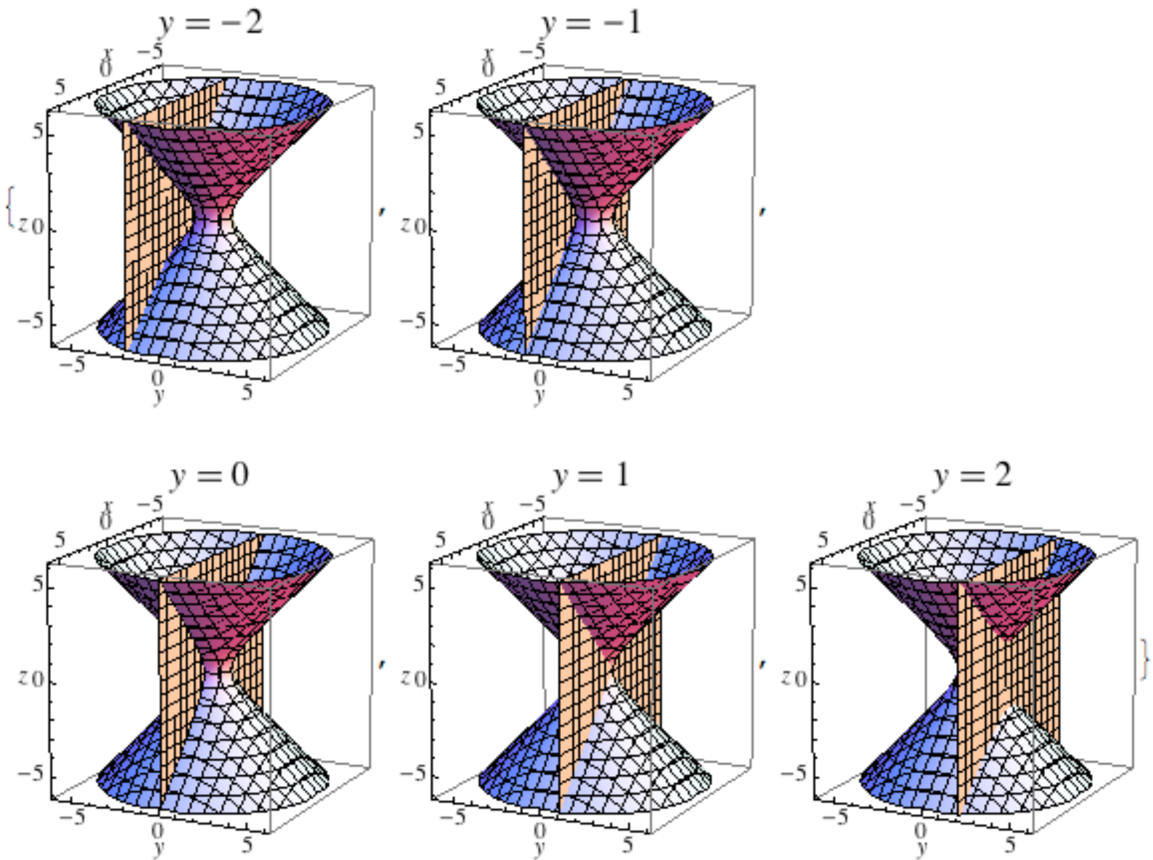
A Hyperboloid of One Sheet



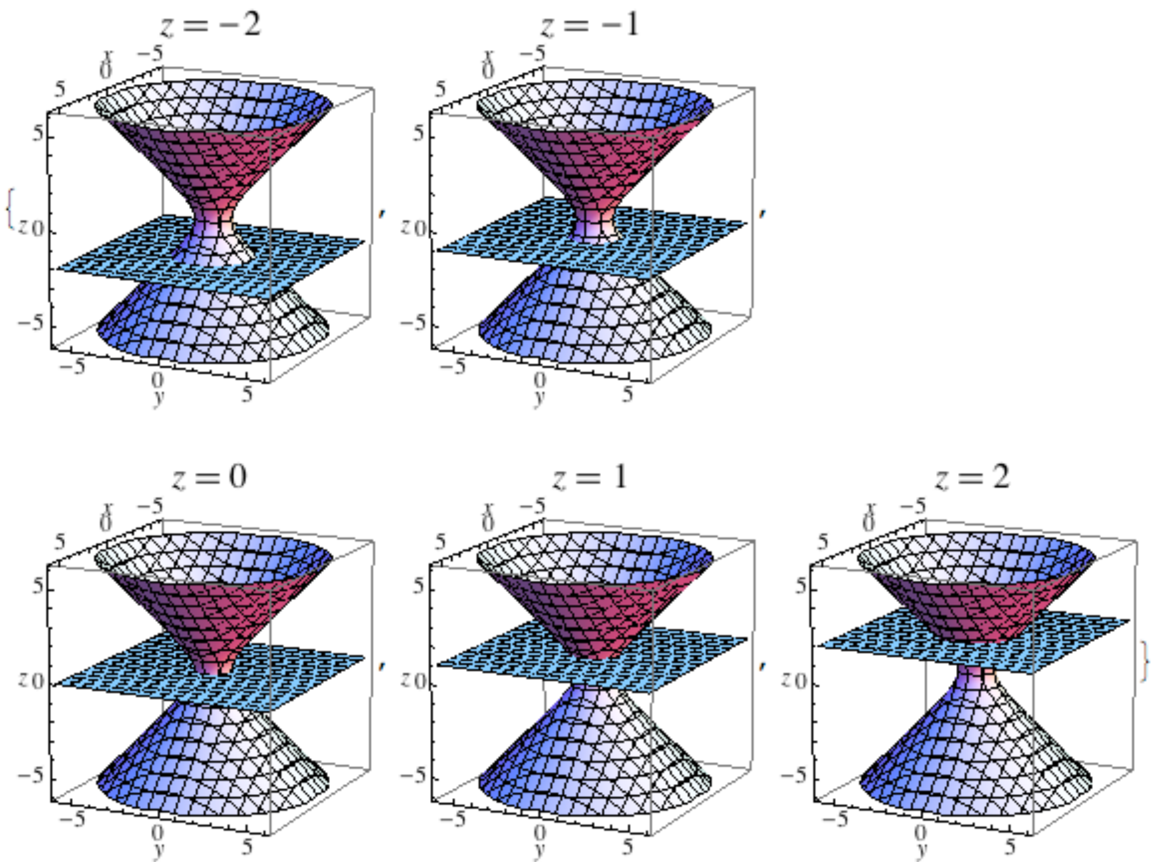
Traces in $x = k$ (Hyperbola class):



Traces in $y = k$ (Hyperbola class):

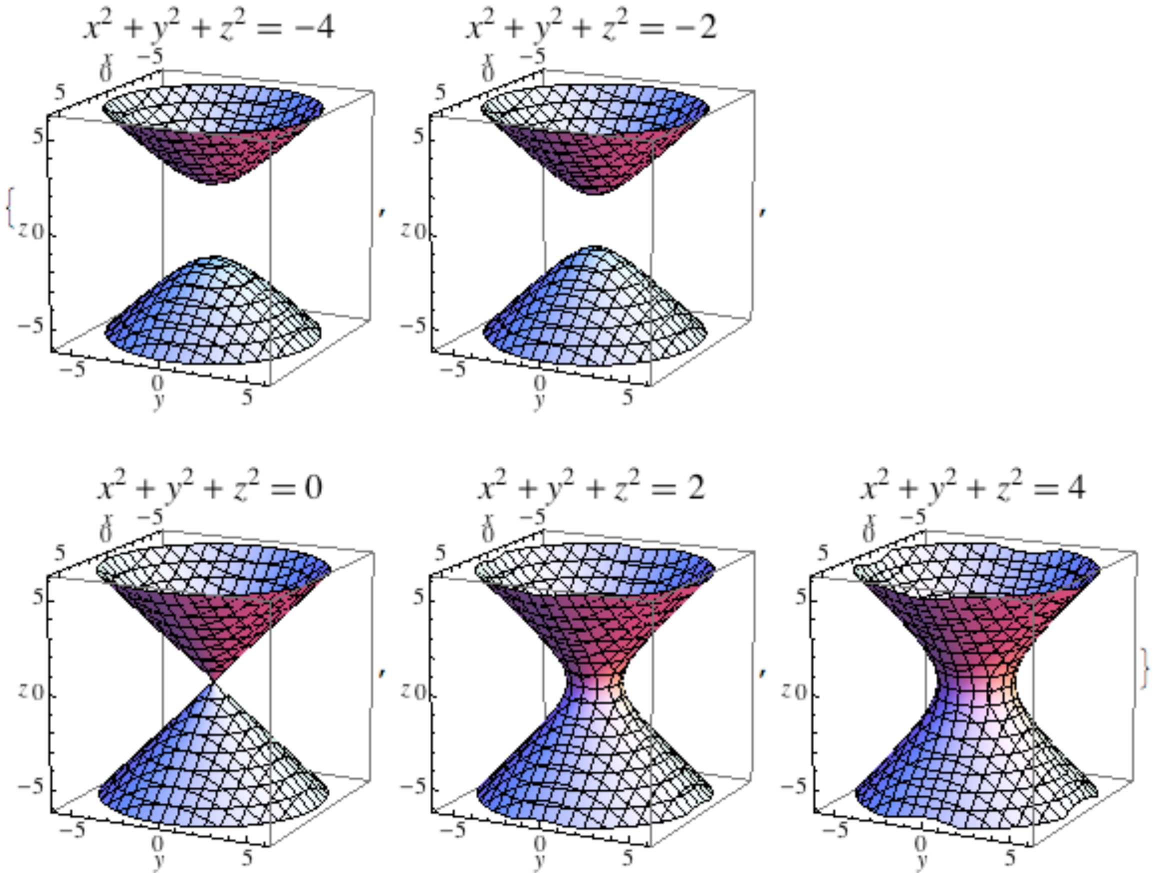


Traces in $z = k$ (Ellipse / Circle class):



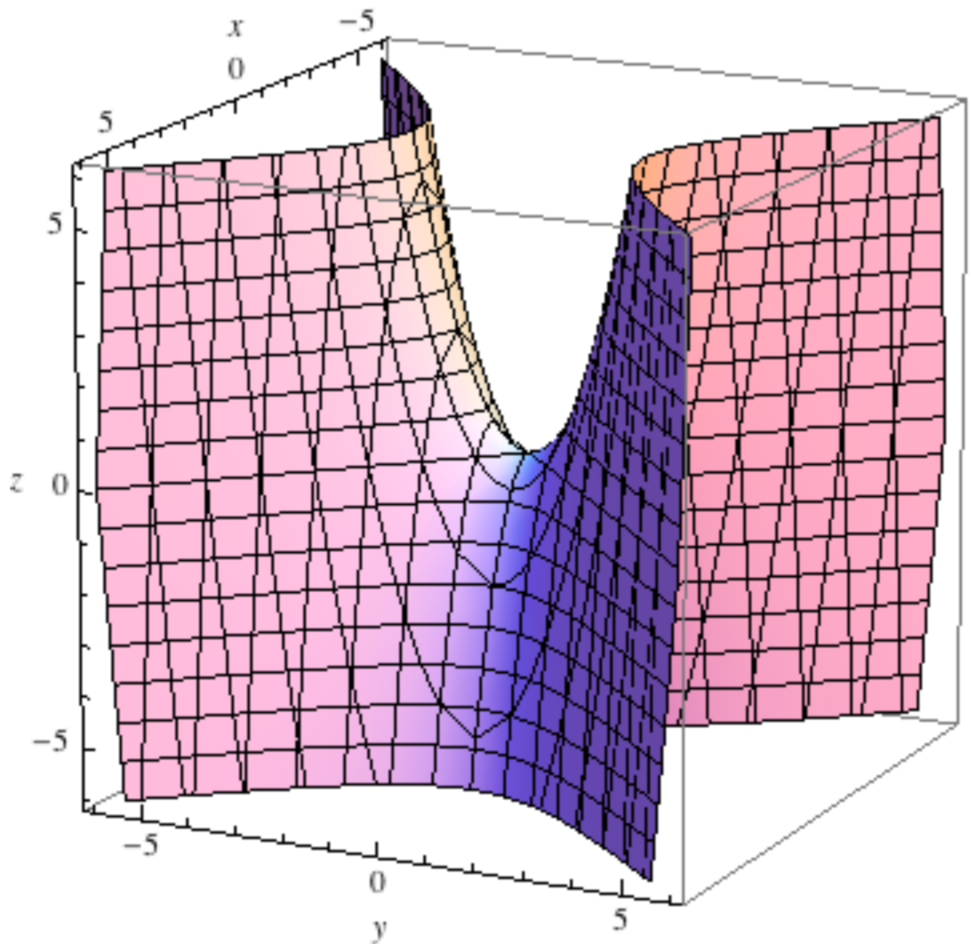
$$x^2 + y^2 - z^2 = k$$

Morphing from a Hyperboloid of Two Sheets to a Cone to a Hyperboloid of One Sheet

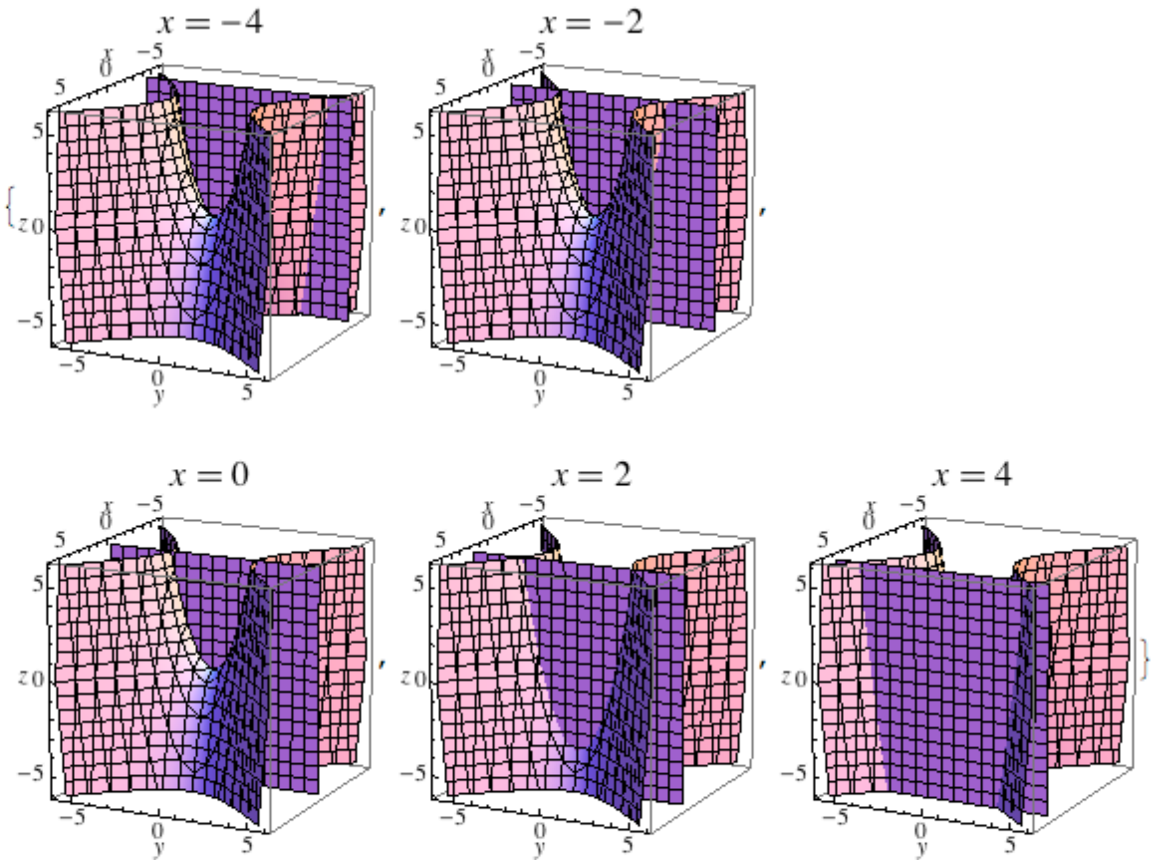


$$z = y^2 - x^2$$

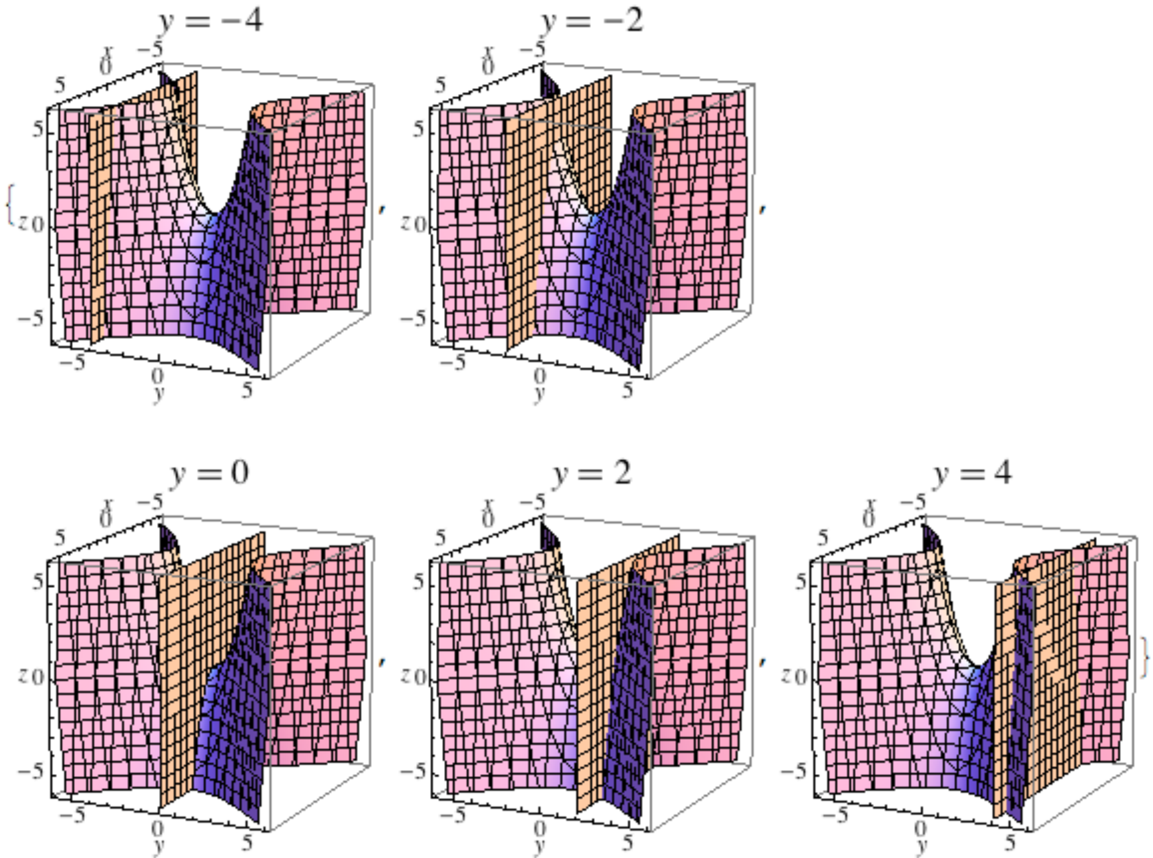
A Hyperbolic Paraboloid (“Saddle”)



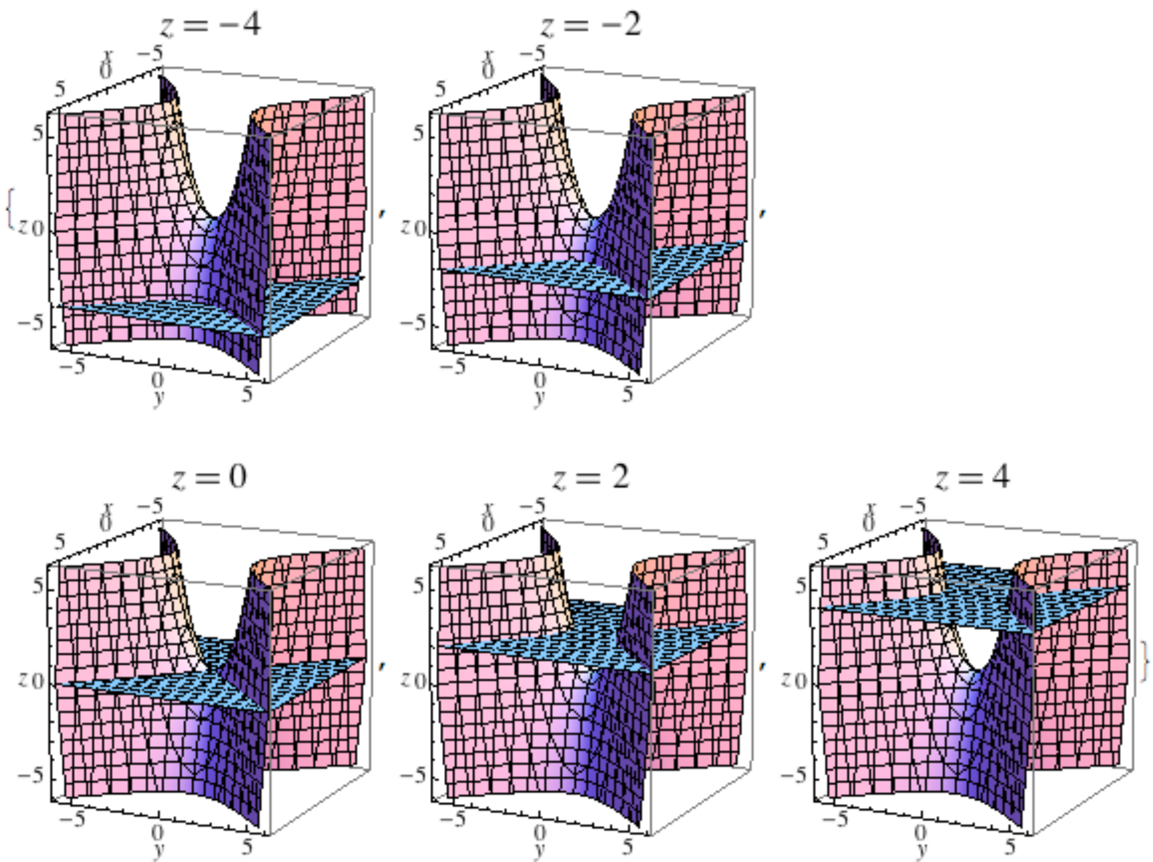
Traces in $x = k$ (Parabola class):



Traces in $y = k$ (Parabola class):



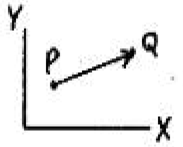
Traces in $z = k$ (Hyperbola class):



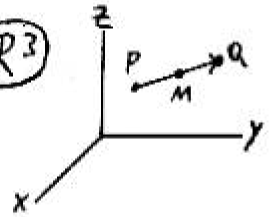
REVIEW: CH.14

VECTORS

$\mathbb{R}^2$



$\mathbb{R}^3$



$\overrightarrow{PQ} = \langle \Delta x, \Delta y \rangle$
$\vec{i}, \vec{j}$

$\overrightarrow{PQ} = \langle \Delta x, \Delta y, \Delta z \rangle$
$\vec{i}, \vec{j}, \vec{k}$
Midpoint M: avg. coords.

$\nearrow \vec{v}$

$\|\vec{v}\| = \sqrt{\sum v_i^2}$

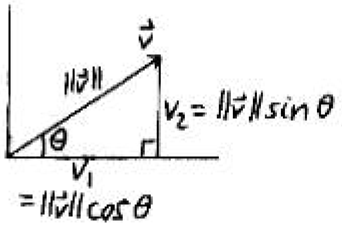
$\vec{v} + \vec{w}$, $c\vec{v}$
Compute, Draw

$\vec{w} \parallel \vec{v} \Leftrightarrow \underset{\text{there exists}}{\exists} c: \vec{w} = c\vec{v}$, or $\vec{v} = \vec{0}$

$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$

Horiz., Vert. Components

$\mathbb{R}^2$



$\tan\theta = \frac{v_2}{v_1}$ $(\neq 0)$

$0 \le \theta \le \pi$

Vector Properties, Proofs

GRAPHS in $\mathbb{R}^3$

Spheres

CTS $\Rightarrow (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$

Graph, Describe Region R in words $\Leftrightarrow$ Describe R using symbols

$\vec{a}\cdot\vec{b}$ (scalar) — Properties, Proofs — $\vec{a}\times\vec{b}$ (vector)

$\vec{a}\cdot\vec{b} = \sum a_i b_i$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \overset{\ominus}{\vec{j}} & \vec{k} \\ & & \\ & & \end{vmatrix} \begin{matrix} \\ \leftarrow \vec{a} \\ \leftarrow \vec{b} \end{matrix}$$

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$ ✪

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}\right) \leftarrow$ if $\neq 0$

$\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\sin\theta$

= Area of 

Direction of $\vec{a}\times\vec{b}$

$\perp \vec{a}, \vec{b}$

Right-Hand Rule

$\vec{a}\perp\vec{b} \Leftrightarrow \vec{a}\cdot\vec{b} = 0$

$\vec{a}\parallel\vec{b} \Leftrightarrow \vec{a}\times\vec{b} = \vec{0}$

Physics

Work $W = \vec{F}\cdot\vec{d}$

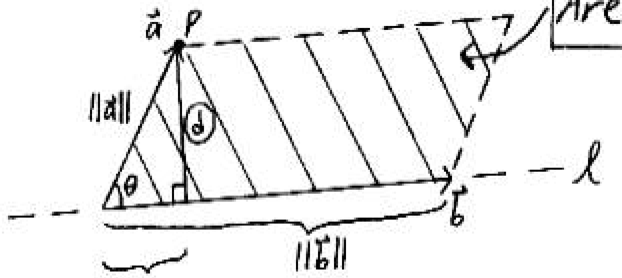
Torque uses "×" (Optional)

Inequalities

Cauchy-Schwarz $|\vec{a}\cdot\vec{b}| \le \|\vec{a}\|\,\|\vec{b}\|$ (from ⊛)

Triangle $\|\vec{a}+\vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|$

Geometry



Area $= \|\vec{a}\times\vec{b}\|$ ($\frac{1}{2}\Rightarrow$ triangle)

(d) = distance bet. P, ℓ $= \dfrac{\|\vec{a}\times\vec{b}\|}{\|\vec{b}\|}$

$\text{comp}_{\vec{b}}\vec{a}$ (scalar) $= \|\vec{a}\|\cos\theta$

$= \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|}$ from ⊛ (- <0)

$\text{proj}_{\vec{b}}\vec{a}$ (vector) $= (\text{comp}_{\vec{b}}\vec{a})\dfrac{\vec{b}}{\|\vec{b}\|}$

$= \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\,\vec{b}$

TSP $= (\vec{a}\times\vec{b})\cdot\vec{c}$

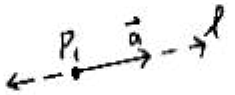
$= \vec{a}\cdot(\vec{b}\times\vec{c})$

$= \left|\begin{array}{c} \\ \\ \\ \end{array}\right| \begin{array}{l}\leftarrow\vec{a}\\ \leftarrow\vec{b}\\ \leftarrow\vec{c}\end{array}$

$|TSP|$ = Volume of box (= 0 ⇔ coplanar)

TVP (I'll give)

LINES



Vector Eq. for ℓ:

$$\overrightarrow{OP} = \overrightarrow{OP_1} + t\vec{a}, \ t \text{ in } \mathbb{R}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} t \quad \text{" "}$$

$\Rightarrow$ Parametric Eqs.

$$\begin{cases} x = x_1 + a_1 t \\ y = y_1 + a_2 t \\ z = z_1 + a_3 t \end{cases}, \ t \text{ in } \mathbb{R}$$

$\Rightarrow$ Symmetric Eqs.

$$\underbrace{\frac{x - x_1}{a_1}}_{\substack{\text{If } a_1 = 0 \\ \Rightarrow x = x_1}} = \frac{y - y_1}{a_2} = \frac{z - z_1}{a_3}$$

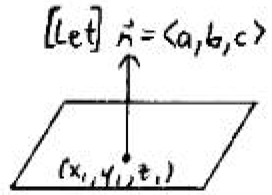
 Solve system for (t, u)

$\Downarrow$ Use either

(x, y, z)

Use direction vectors for angles, $\parallel$, $\perp$.

PLANES

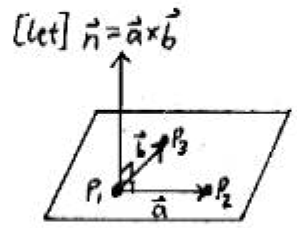


$$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$$

$$ax + by + cz + d = 0 \Longrightarrow \vec{n} = \langle a, b, c \rangle$$

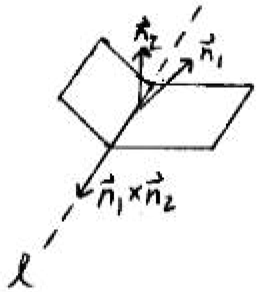
(or any non-$\vec{0}$ scalar mult.)

Use normal vectors for angles, $\parallel$, $\perp$.



Use $\vec{n}$ and P_1, say, to construct an eq. for the plane.

Plane 2 looks like $z=k$; plug in $x=0$ here?



Find ℓ

Find a point.
Plug $x=0$, say, into the system.

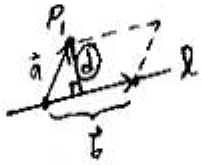
Find a direction vector.
$\vec{n}_1 \times \vec{n}_2$ works.

DISTANCES

2 Points

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

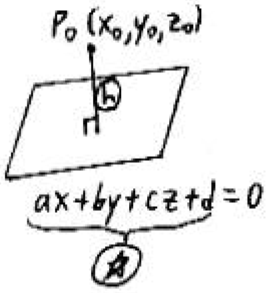
Point, Line



$$d = \frac{\|\vec{a}\times\vec{b}\|}{\|\vec{b}\|}$$ $\leftarrow \vec{b} \parallel \ell$ (Careful! I sometimes do $\vec{b}\times\vec{a}$ to ensure $\vec{n}$ points "up.")

Point, Plane

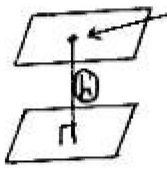
If know eq. of plane



$$h = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$ $\leftarrow$ Plug P_0 into A

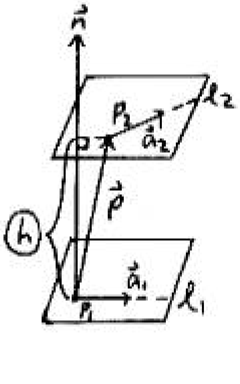
If not, find one!

2 ∥ Planes



Find a point.
Plug in $x=0, y=0$, say.

Skew Lines



Let $\vec{n} = \vec{a}_1 \times \vec{a}_2$.

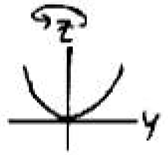
$$h = |\mathrm{comp}_{\vec{n}}\,\vec{p}| = \frac{|\vec{p}\cdot\vec{n}|}{\|\vec{n}\|}$$

SURFACES

Traces

Spheres
Planes
Cylinders
Ex 1 variable missing ⇒ sweep (∥ axis)

Surfaces of Revolution

Ex 

$z = y^2$ in yz-plane

Replace y^2 with $y^2 + x^2$.
missing variable

Don't touch "axis variable" (z, here).

Quadric Surfaces

Know Basic Eqs., Axes for 6 Basic Types
Surfaces of Revolution may help!
Traces

Tricks : Variable-Switch
⇒ may change axis
Coefficients
⇒ makes it easier to identify