# CH. 14: VECTORS and SURFACES

## 14.1: VECTORS IN 2D

### (A) Scalars

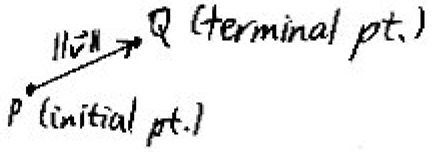
are real #s

### (B) Vectors

have magnitude and direction

Ex A vector $\vec{v}$ (or boldfaced "**v**")

$\vec{v} = \overrightarrow{PQ}$



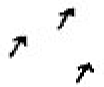

Exs Displacement, Velocity, Force

55mph

$\|\vec{v}\|$ = magnitude, length, or norm of $\vec{v}$

Equal vectors:

Harry Potter flies



(Position is irrelevant.)

### (C) $\mathbb{R}^2$

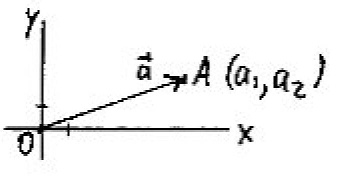
(using Cartesian / Rectangular (vs. Polar) Coordinate System)

$= \{ (x, y) \mid x \text{ and } y \text{ are real \#s} \}$

the set of all — ordered pairs — such that

"2-space"

() see Math Dictionary Borowski / Borwein




The position vector for A or $\vec{a}$

$= \langle a_1, a_2 \rangle$

horiz. — vertical — components

⟨⟩ are angle brackets

Many books simply use $\mathbb{R}^2$ instead of $V_2$.

$\vec{a}$ is in $V_2 = \{ \langle x,y \rangle \mid x \text{ and } y \text{ are real \#s} \}$

$$\boxed{\begin{aligned} \|\vec{a}\| &= \| \langle a_1, a_2 \rangle \| \\ &= \sqrt{a_1^2 + a_2^2} \end{aligned}}$$

① from Pyth. Thm. / Distance Formula
② $\|\vec{a}\| \geq 0$ always

## D Standard Unit Vectors in $V_2$

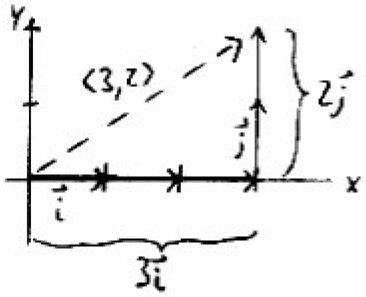
length 1

$$\boxed{\begin{aligned} &\vec{i} = \langle 1,0 \rangle \\ &\vec{j} = \langle 0,1 \rangle \\ &\langle a_1, a_2 \rangle = a_1 \vec{i} + a_2 \vec{j} \end{aligned}}$$

Physics!

<u>Ex</u> $\langle 3,2 \rangle = 3\vec{i} + 2\vec{j}$

Treasure Map




Basic Example of ...

## Ⓔ Vector Addition

Ex $\underset{\vec{v}}{\langle 1,4 \rangle} + \underset{\vec{w}}{\langle 2,3 \rangle}$

Add corresponding components.

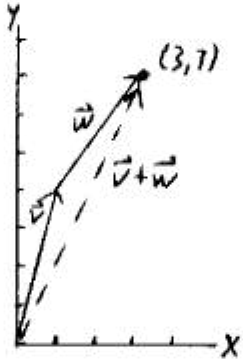
$= \langle 1+2, 4+3 \rangle$

$= \boxed{\langle 3,7 \rangle}$

### Triangle Law

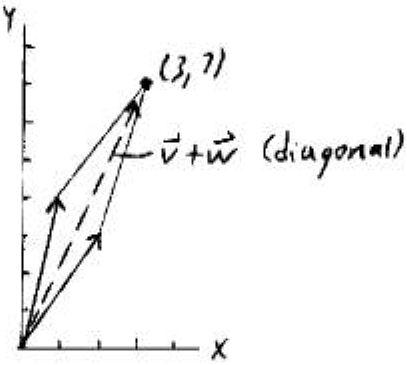
"Head-to-tail"

Treasure Map




### Parallelogram Law

"Tail-to-tail"




$\vec{v}+\vec{w}$ = <u>resultant</u>

Exs Net displacement
Net force

## (F) Scalar Multiples of $\vec{v}$

<u>Ex</u> $3\langle -1, 2\rangle$

Multiply each component by 3.

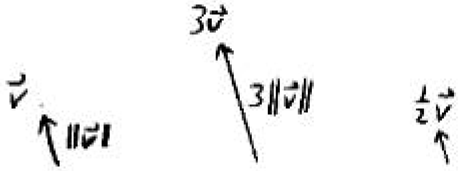
$= \langle 3(-1), 3(2)\rangle$

$= \boxed{\langle -3, 6\rangle}$

$c\vec{v}$

scalar can (1) rescale $\vec{v}$

(2) flip direction (if $c<0$)

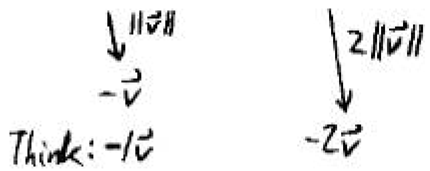



$c>0$

$\Rightarrow$ same direction as $\vec{v}$

• $\vec{0} = \langle 0,0\rangle$ has <u>every</u> direction

$c=0$




$c<0$

$\Rightarrow$ opposite direction

$\boxed{\|c\vec{v}\| = |c|\,\|\vec{v}\|}$ never negative

$\vec{w}$ is parallel to $\vec{v}$ $(\vec{w} \parallel \vec{v})$ $\Leftrightarrow$ (if and only if) $\vec{w} = c\vec{v}$ for some scalar $c$ (or $\vec{v} = \vec{0}$)

i.e., The vectors parallel to $\vec{v}$ are the scalar multiples of $\vec{v}$. $(\vec{v} \neq \vec{0})$

$\boxed{\underline{\text{Def'n}} \quad \frac{\vec{v}}{c} = \frac{1}{c}\vec{v} \quad (c \neq 0)}$

Divide each component by c.

$\boxed{\underline{\text{Def'n}} \quad \vec{v} - \vec{w} = \vec{v} + (-\vec{w})}$

$\underline{\text{Ex}}$ $\langle 3,2 \rangle - \langle 5,1 \rangle = \langle 3-5, 2-1 \rangle$

$= \langle -2, 1 \rangle$



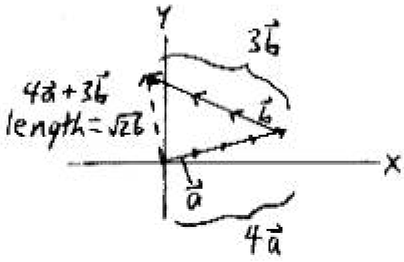
$\vec{w} + (\text{what?}) = \vec{v}$
$\vec{v} - \vec{w}$

⑥ $\underline{\text{Ex}}$

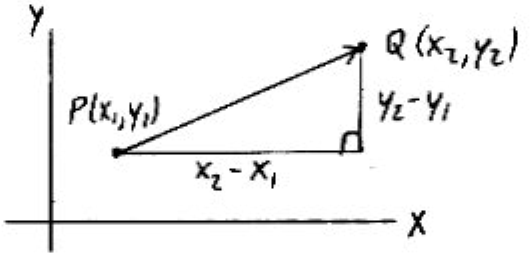
If $\vec{a} = \langle 2, \frac{1}{2} \rangle$, $\vec{b} = \langle -3, 1 \rangle$, find $\|4\vec{a} + 3\vec{b}\|$.

$$\begin{aligned} \|4\vec{a} + 3\vec{b}\| &= \|4\langle 2, \tfrac{1}{2} \rangle + 3\langle -3, 1 \rangle\| \\ &= \|\langle 8, 2 \rangle + \langle -9, 3 \rangle\| \\ &= \|\langle 8+(-9), 2+3 \rangle\| \\ &= \|\langle -1, 5 \rangle\| \\ &= \sqrt{(-1)^2 + (5)^2} \\ &= \boxed{\sqrt{26}} \end{aligned}$$

up to 19



## Ⓗ An Initial Point and a Terminal Point Determine a Vector




$$\boxed{\begin{gathered}\overrightarrow{PQ} = \langle x_2 - x_1,\ y_2 - y_1 \rangle \\ \Rightarrow \|\overrightarrow{PQ}\| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \\ \text{Distance Formula!}\end{gathered}}$$

$$\overrightarrow{QP} = -\overrightarrow{PQ}$$

## Ⓘ The Unit Vector in the Direction of $\vec{v}$ ($\vec{v} \neq \vec{0}$)

$$\boxed{\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} \quad \text{or} \quad \frac{\vec{v}}{\|\vec{v}\|}}$$

Normalizing $\vec{v}$

$\vec{v}$ $\vec{u}$ length 1

## Ⓙ Ex

$P(1, -2)$; $Q(-3, -1)$. Find the unit vector in the direction of $\overrightarrow{PQ}$.

Sol'n

$$\begin{aligned}\overrightarrow{PQ} &= \langle -3-1,\ -1-(-2) \rangle \\ &= \langle -4, 1 \rangle\end{aligned}$$

Think: "Q−P"

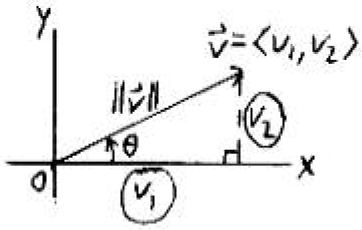
$$\begin{aligned}\|\overrightarrow{PQ}\| &= \sqrt{(-4)^2 + (1)^2} \\ &= \sqrt{17}\end{aligned}$$

$$\vec{u} = \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \frac{\langle -4, 1 \rangle}{\sqrt{17}} = \boxed{\left\langle -\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \right\rangle}$$

can rat'lize the denom.

Up to 31

## (K) Finding Horiz., Vertical Components of a Vector in $V_2$




$\theta$ = <u>direction angle</u> for $\vec{v}$

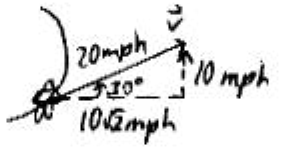
### <u>Given $\|\vec{v}\|, \theta \Rightarrow$ find $v_1, v_2$</u>

$$\cos\theta = \frac{v_1}{\|\vec{v}\|} \qquad \sin\theta = \frac{v_2}{\|\vec{v}\|}$$

$$v_1 = \|\vec{v}\|\cos\theta \qquad v_2 = \|\vec{v}\|\sin\theta$$

$$\boxed{\vec{v} = \langle \underbrace{\|\vec{v}\|\cos\theta}_{\substack{v_1 \\ = \text{horiz. comp. of } \vec{v}}}, \underbrace{\|\vec{v}\|\sin\theta}_{\substack{v_2 \\ = \text{vertical}}} \rangle}$$

<u>Ex</u> (Slow) car on a track
tank!




$$\begin{aligned}\vec{v} &= \langle \|\vec{v}\|\cos\theta, \|\vec{v}\|\sin\theta \rangle \\ &= \langle 20\cos 30°, 20\sin 30° \rangle \\ &= \langle 20(\tfrac{\sqrt{3}}{2}), 20(\tfrac{1}{2}) \rangle \\ &= \langle 10\sqrt{3}, 10 \rangle\end{aligned}$$

Up to 47

### <u>Given $v_1, v_2 \Rightarrow$ find $\|\vec{v}\|, \theta$</u>

$$\boxed{\begin{aligned}&\|\vec{v}\| = \sqrt{v_1^2 + v_2^2} \\ &\tan\theta = \frac{v_2}{v_1}, \text{ Watch Quadrant! (I or III? II or IV?)}\end{aligned}}$$

## (L) Vector Properties (Swokowski p.687)

$\vec{a}, \vec{b}, \vec{c}, \vec{0}$ are vectors in $V_n$ (n is a fixed natural #),
c, d are scalars.

| | | | |
|---|---|---|---|
| Vector "+" | ☆(i) | $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ | Vector "+" is commutative |
| | (ii) | $\vec{a}+(\vec{b}+\vec{c})=(\vec{a}+\vec{b})+\vec{c}$ | associative |
| | (iii) | $\vec{a}+\vec{0}=\vec{a}$ | $\vec{0}$ is the additive identity |
| | (iv) | $\vec{a}+(-\vec{a})=\vec{0}$ | $-\vec{a}$ is the additive inverse of $\vec{a}$ |
| Scalar Mult. | ☆(v) | $c(\vec{a}+\vec{b})=c\vec{a}+c\vec{b}$ | Scalar mult. distributes over vector "+" |
| | (vi) | $(c+d)\vec{a}=c\vec{a}+d\vec{a}$ | Vector , scalar "+" |
| | (vii) | $(cd)\vec{a}=c(d\vec{a})=d(c\vec{a})$ | Mult. by scalars is flexible |
| | (viii) | $1\vec{a}=\vec{a}$ | 1 |
| | (ix) | $0\vec{a}=\vec{0}=c\vec{0}$ | 0 |

☆ Proof p.687

Like #54 but with c, not p.

Ex Prove $c(\vec{a}-\vec{b})=c\vec{a}-c\vec{b}$. ($\vec{a},\vec{b}$ in $V_2$, c scalar)

Let $\vec{a}=\langle a_1, a_2\rangle$
$\vec{b}=\langle b_1, b_2\rangle$

Trick: Boil things down to properties of real #s; rebuild up.

$$\begin{aligned}
c(\vec{a}-\vec{b}) &= c(\langle a_1,a_2\rangle - \langle b_1,b_2\rangle)\\
&= c(\langle a_1-b_1,\ a_2-b_2\rangle)\\
&= \langle c(a_1-b_1),\ c(a_2-b_2)\rangle\\
&= \langle ca_1-cb_1,\ ca_2-cb_2\rangle\\
&= \langle ca_1, ca_2\rangle - \langle cb_1, cb_2\rangle\\
&= c\langle a_1,a_2\rangle - c\langle b_1,b_2\rangle\\
&= c\vec{a}-c\vec{b}
\end{aligned}$$

QED (end of proof)
Quod Erat Demonstrandum