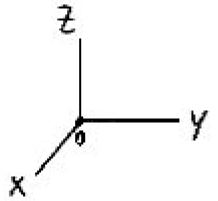
14.2: VECTORS IN 3D

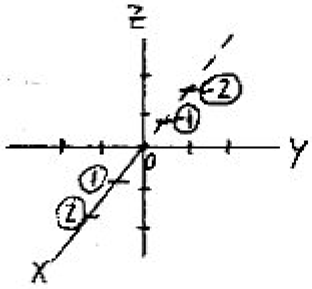
Ⓐ $\mathbb{R}^3$ (using Cartesian coords.)

$= \{ \underbrace{(x, y, z)}_{\text{ordered triples}} \mid x, y, \text{ and } z \text{ are real \#s} \}$

"3-space"



(sticks out at you)



rude

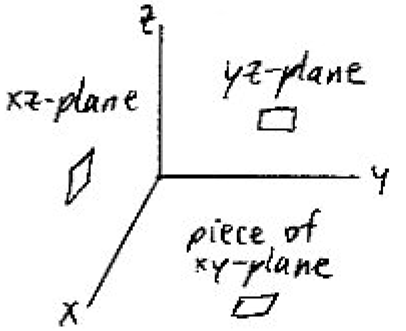
These coordinate axes are mutually perpendicular (or orthogonal, or $\perp$).

How to see ortho: upright orthodontist

(1,1,1) pt. isn't 0!

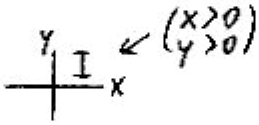
Beware of distortion! 3D → 2D paper

Coordinate planes



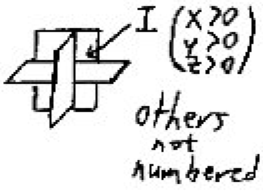
Mutually $\perp$

Ⓡ $\mathbb{R}^2$



4 quadrants

Ⓡ $\mathbb{R}^3$

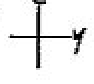


others not numbered

8 octants

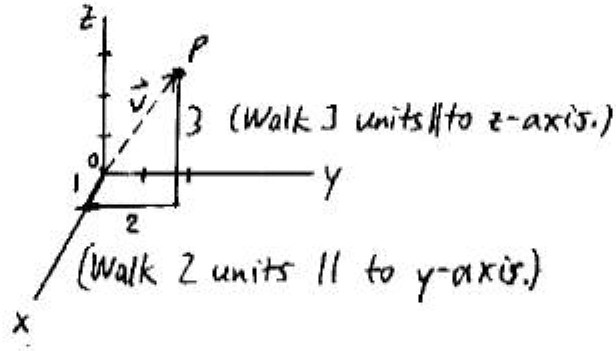
4 in front of yz-plane, (x>0)

4 behind (x<0)



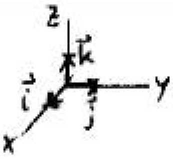
xy, yz-planes stick out at you

Ex Plot the point $P(1,2,3)$, and draw $\vec{v} = \langle 1,2,3\rangle$, a vector in V_3.



Ⓑ Standard Unit Vectors in V_3

$$\boxed{\begin{aligned} \vec{i} &= \langle 1,0,0\rangle \\ \vec{j} &= \langle 0,1,0\rangle \\ \vec{k} &= \langle 0,0,1\rangle \\ \langle a_1, a_2, a_3\rangle &= a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \end{aligned}}$$



Physics!

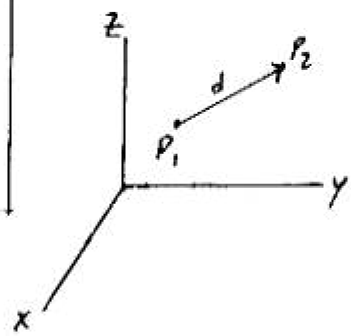
Ex $\langle 2,-3,-1\rangle = 2\vec{i} - 3\vec{j} - \vec{k}$

Ⓒ $\overrightarrow{P_1P_2}$

$P_1(x_1, y_1, z_1)$; $P_2(x_2, y_2, z_2)$

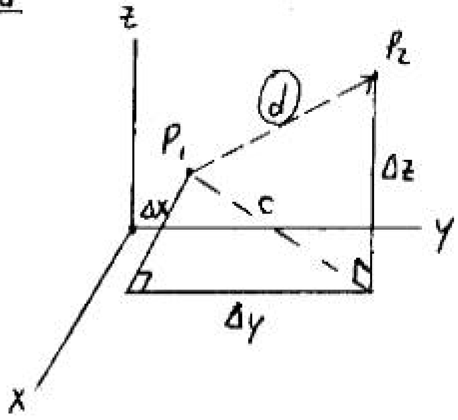
$$\boxed{\overrightarrow{P_1P_2} = \langle \underbrace{x_2 - x_1}_{\Delta x}, \underbrace{y_2 - y_1}_{\Delta y}, \underbrace{z_2 - z_1}_{\Delta z} \rangle}$$

↑ uppercase delta (change in)



$d = \|\overrightarrow{P_1P_2}\|$
$= \text{distance between } P_1, P_2$

Find d



Apply Pyth. Thm. twice!

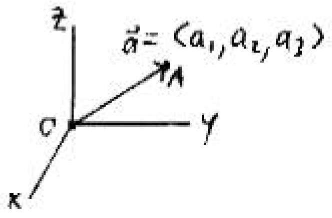
$$d^2 = c^2 + (\Delta z)^2$$

$$= (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

(OK if $\Delta x < 0$, etc.)

$$\boxed{\begin{aligned} d &= \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \end{aligned}}$$

Special Case



$$\boxed{\begin{aligned} \|\vec{a}\| &= \|\langle a_1, a_2, a_3 \rangle\| \\ &= \sqrt{a_1^2 + a_2^2 + a_3^2} \end{aligned}}$$

The midpoint of $\overline{P_1P_2}$ is $\left(\underbrace{\frac{x_1+x_2}{2}}_{\text{average of x-coords.}}, \underbrace{\frac{y_1+y_2}{2}}_{y}, \underbrace{\frac{z_1+z_2}{2}}_{z}\right)$

<u>Ex</u> Let ℓ be the line passing through $P_1(4,0,-2)$ and $P_2(2,5,7)$.
Find two unit vectors $\underbrace{\text{parallel to}}_{\text{“}\|\text{”}}$ this line.

<u>Sol'n</u>

Find $\overrightarrow{P_1P_2}$ (or $\overrightarrow{P_2P_1}$), a vector $\| \ \ell$.

$$\begin{aligned}\overrightarrow{P_1P_2} &= \langle 2-4,\ 5-0,\ 7-(-2)\rangle \\ &= \langle -2,\ 5,\ 9\rangle\end{aligned}$$

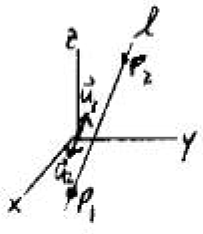
Find the unit vector in the direction of $\overrightarrow{P_1P_2}$.
<u>Normalize</u> $\overrightarrow{P_1P_2}$.

$$\begin{aligned}\|\overrightarrow{P_1P_2}\| &= \sqrt{(-2)^2+(5)^2+(9)^2} \\ &= \sqrt{110}\end{aligned}$$

$$\begin{aligned}\vec{u}_1 &= \frac{\overrightarrow{P_1P_2}}{\|\overrightarrow{P_1P_2}\|} \\ &= \frac{\langle -2,5,9\rangle}{\sqrt{110}} \qquad \text{really, } \frac{1}{\sqrt{110}}\langle -2,5,9\rangle \\ &= \boxed{\left\langle -\frac{2}{\sqrt{110}},\ \frac{5}{\sqrt{110}},\ \frac{9}{\sqrt{110}}\right\rangle}\end{aligned}$$

Find the opposite unit vector.

$$\begin{aligned}\vec{u}_2 &= -\vec{u}_1 \\ &= \boxed{\left\langle \frac{2}{\sqrt{110}},\ -\frac{5}{\sqrt{110}},\ -\frac{9}{\sqrt{110}}\right\rangle}\end{aligned}$$



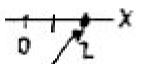
up to 17

① The Graph of an Equation

· consists of all points whose coords. satisfy the equation.

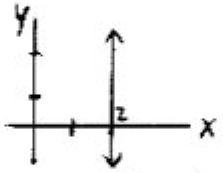
!!

$\mathbb{R}^1 \to \mathbb{R}^2$
$x = 2$



sweep ∥ to y-axis to get line

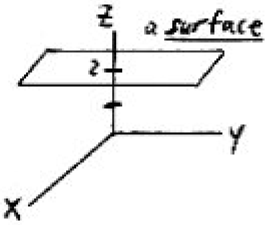
$\mathbb{R}^2$ $x = 2$



$\{(2, y) \mid y \text{ real}\}$

line ∥ y-axis
y missing in eq.

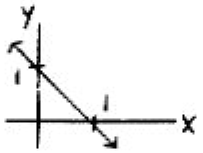
$\mathbb{R}^3$ $z = 2$



plane ∥ $\underbrace{xy}$ plane
x, y missing in eq.

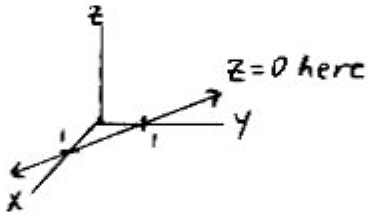
Graph $x + y = 1$. In $\mathbb{R}^{what}$?

$\mathbb{R}^2$ $x + y = 1$



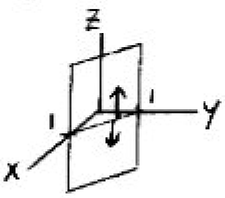
$\mathbb{R}^3$ $x + y = 1$
z missing

Not the whole graph!



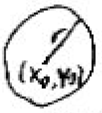
For any (x, y) that satisfies $x + y = 1$, (x, y, z) will satisfy $x + y = 1$ for <u>any</u> real z.

We "sweep" the line ∥ to the z-axis. We then pick up <u>all</u> z-coords. for any (x, y) that "works."



plane ∥ z-axis

Circles:



$(x-x_0)^2+(y-y_0)^2 = r^2$

Ⓔ Spheres

Find an eq. for the sphere with
Center $C(x_0, y_0, z_0)$;
Radius $= r$ $(r>0)$

We want all points (x,y,z) that are r units away from C.

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = r$$

$$\boxed{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = r^2}$$

Ex Find C, r for $\underbrace{2x^2+2y^2+2z^2-12x-20y+12z+68=0}_{\div \text{ thru by } 2}$.

#30 →

$$x^2+y^2+z^2-6x-10y+6z+34=0$$

Group terms.

$$(x^2-6x)+(y^2-10y)+(z^2+6z) = -34$$

Complete the Square (CTS) within groups; Balance!

$$(x^2-6x+9)+(y^2-10y+25)+(z^2+6z+9) = -34+9+25+9$$

Factor.

$$\underbrace{(x-3)^2+(y-5)^2+(z+3)^2}_{C:\text{ What makes left side } = 0?} = \underbrace{9}_{r^2=9}$$

Up to 33

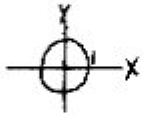
$$\boxed{\begin{array}{l} C: (3,5,-3) \\ r=3 \end{array}}$$

(E) Regions in $\mathbb{R}^3$

Ex Describe $R = \{(x,y,z) \mid x^2+y^2 \le 1, \underbrace{|z| \le 4}\}$.

Extra conditions/ restrictions never grow the surface.

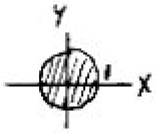
(1) ($\mathbb{R}^2$) $x^2+y^2=1$



Unit circle

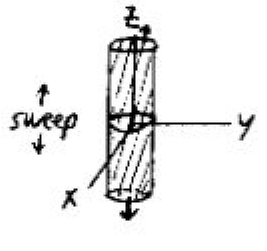
Test point method for shading - (0,0) lies inside the circle $\underbrace{x^2+y^2}_{\text{cont. in } x,y} = \underset{\text{constant}}{1}$.

(2) ($\mathbb{R}^2$) $\underbrace{x^2+y^2}_{\text{squared distance from } O} \le 1$



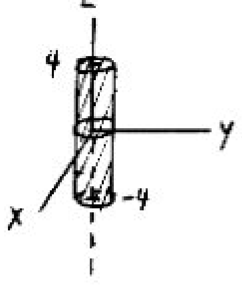
Unit disk

(3) ($\mathbb{R}^3$) $x^2+y^2 \le 1$



open-ended solid circular (by cross-section) cylinder

(4) ($\mathbb{R}^3$) $x^2+y^2 \le 1$, $\underbrace{|z| \le 4}_{-4 \le z \le 4}$



All points inside or on the closed circular cylinder with center at O, radius 1, altitude (height) 8, and axis along the z-axis.