14.3: DOT (INNER) PRODUCT

Ⓐ $\vec{a}\cdot\vec{b}$

is a scalar. Algebraic definition: $\vec{a}\cdot\vec{b} = \sum_{i=1}^{n} a_i b_i$ $\left(\begin{matrix}\vec{a}=\langle a_1, a_2, \ldots, a_n\rangle \\ \vec{b}=\langle b_1, b_2, \ldots, b_n\rangle\end{matrix}\right)$

$= a_1b_1 + a_2b_2 + \ldots + a_nb_n$

We add products of corresponding components.

Ex $\langle 2, -3, -4\rangle \cdot \langle 1, -2, 5\rangle$

$= (2)(1) + (-3)(-2) + (-4)(5)$

Up to 5

$= \boxed{-12}$

Ⓑ Properties (p. 702)

$\vec{0}, \vec{a}, \vec{b}, \vec{c}$ in V_n ; c scalar

Prove in 39 for V_3.

(i) $\vec{a}\cdot\vec{a} = \|\vec{a}\|^2$
(ii) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
(iii) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$
(iv) $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(c\vec{b})$
(v) $\vec{0}\cdot\vec{a} = 0$

Comments

Ex $\langle 2,3\rangle\cdot\langle 2,3\rangle = (2)^2 + (3)^2 = (\overbrace{\|\langle 2,3\rangle\|}^{\sqrt{(2)^2+(3)^2}})^2$

"$\cdot$" is comm.

"$\cdot$" distributes over vector "+" (Proof p. 702)

Scalar mult. is flexible re "$\cdot$"

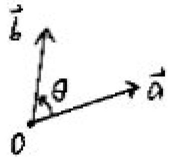
Ex Prove $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b})$ if $\vec{a}, \vec{b}$ in V_3.

Proof Let $\vec{a} = \langle a_1, a_2, a_3\rangle$
$\vec{b} = \langle b_1, b_2, b_3\rangle$
c any scalar

$$\begin{aligned}(c\vec{a})\cdot\vec{b} &= (c\langle a_1, a_2, a_3\rangle)\cdot\langle b_1, b_2, b_3\rangle \\ &= \langle ca_1, ca_2, ca_3\rangle\cdot\langle b_1, b_2, b_3\rangle \\ &= ca_1b_1 + ca_2b_2 + ca_3b_3 \\ &= c(a_1b_1 + a_2b_2 + a_3b_3) \\ &= c(\langle a_1, a_2, a_3\rangle\cdot\langle b_1, b_2, b_3\rangle) \\ &= c(\vec{a}\cdot\vec{b})\end{aligned}$$

QED

ⓒ Angle Between $\vec{a}, \vec{b}$



Let θ = smallest nonnegative angle between the position vectors for $\vec{a}, \vec{b}$.

$$0 \le \theta \le \pi$$

From the Law of Cosines for triangles,

$$\vec{a} \cdot \vec{b} = \underbrace{\|\vec{a}\| \, \|\vec{b}\|}_{\text{product of lengths}} \underbrace{\cos\theta}_{\text{times cos of angle between them}}$$

(Geometric def'n)
"polar"

$$\Rightarrow \cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \, \|\vec{b}\|} \qquad (\vec{a} \ne \vec{0}, \vec{b} \ne \vec{0})$$

Think: $\dfrac{\text{dot product}}{\text{product of lengths}}$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \, \|\vec{b}\|}\right)$$

<u>Ex</u> (#12) Find the angle between $\vec{a} = \vec{i} - 7\vec{j} + 4\vec{k}$, $\vec{b} = 5\vec{i} - \vec{k}$.

<u>Sol'n</u> $\vec{a} = \langle 1, -7, 4 \rangle$, $\vec{b} = \langle 5, 0, -1 \rangle$

$$\theta = \cos^{-1}\left(\frac{\langle 1, -7, 4 \rangle \cdot \langle 5, 0, -1 \rangle}{\|\langle 1, -7, 4 \rangle\| \, \|\langle 5, 0, -1 \rangle\|}\right)$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{66}\,\sqrt{26}}\right)$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{1716}}\right)$$

$$\approx 1.547 \text{ (radians) or } 88.6^\circ$$

$$\boxed{\vec{a} \perp \vec{b} \iff \vec{a}\cdot\vec{b} = 0}$$

orthogonal $(\cos\frac{\pi}{2} = 0)$

$\theta = 0$, acute $\qquad \theta = \frac{\pi}{2} \qquad \theta =$ obtuse, π

$\vec{a}\cdot\vec{b}$: $\oplus \qquad 0 \qquad \ominus \qquad \leftarrow$ determined by $\cos\theta$, if $\vec{a}, \vec{b}$ non-$\vec{0}$

D Cauchy-Schwarz Inequality

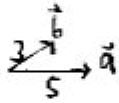
$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\underbrace{\cos\theta}_{\substack{-1 \le \cos\theta \le 1 \\ |\cos\theta| \le 1}}$$

(If I give you two sticks of fixed lengths, what are the possible dot prods.?)

$$\Rightarrow \boxed{|\vec{a}\cdot\vec{b}| \le \|\vec{a}\|\,\|\vec{b}\|}$$

The absolute value of a dot product cannot exceed the product of the lengths.

Ex



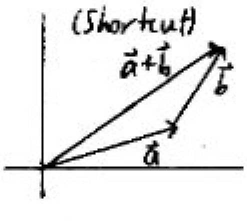
can rotate

$|\vec{a}\cdot\vec{b}| \le 15$

$-15 \le \vec{a}\cdot\vec{b} \le 15$

E Triangle Inequality

SUV



The length of one side cannot exceed the sum of the lengths of the other two sides.

$$\boxed{\|\vec{a}+\vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|}$$

Proof uses D.

Ex (#48) When is $\underbrace{\|\vec{a}+\vec{b}\| = \|\vec{a}\| + \|\vec{b}\|}$?

$$\Leftrightarrow \quad (\|\vec{a}+\vec{b}\|)^2 = (\|\vec{a}\| + \|\vec{b}\|)^2$$

because magnitudes are nonnegative

$$\Leftrightarrow \quad \underbrace{(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})}_{\substack{\text{we used:} \\ \|\vec{v}\|^2 = \vec{v}\cdot\vec{v}}} = \|\vec{a}\|^2 + 2\|\vec{a}\|\|\vec{b}\| + \|\vec{b}\|^2$$

$$\Leftrightarrow \quad \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} = \text{“} \qquad \text{”}$$

$$\Leftrightarrow \quad \cancel{\|\vec{a}\|^2} + 2(\vec{a}\cdot\vec{b}) + \cancel{\|\vec{b}\|^2} = \cancel{\|\vec{a}\|^2} + 2\|\vec{a}\|\|\vec{b}\| + \cancel{\|\vec{b}\|^2}$$

$$\Leftrightarrow \quad 2(\vec{a}\cdot\vec{b}) = 2\|\vec{a}\|\|\vec{b}\|$$

$$\Leftrightarrow \quad \vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|$$

$$\Leftrightarrow \quad \|\vec{a}\|\|\vec{b}\|\cos\theta = \|\vec{a}\|\|\vec{b}\| \qquad (0 \le \theta \le \pi)$$

$$\Leftrightarrow \quad \vec{a} = \vec{0}, \text{ or } \vec{b} = \vec{0}, \text{ or } \cos\theta = 1 \qquad \text{“} \quad \text{”}$$

$$\theta = 0$$

$$\Leftrightarrow \quad \boxed{\vec{a} = \vec{0}, \text{ or } \vec{b} = \vec{0}, \text{ or } \vec{a} \text{ and } \vec{b} \text{ have the same direction}}$$

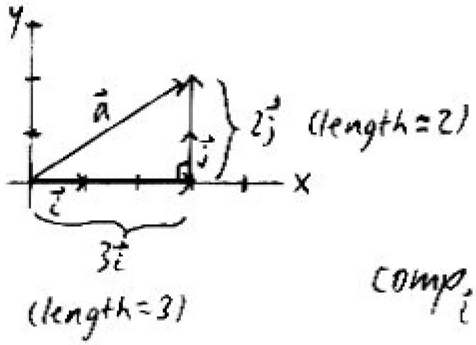
“Degenerate triangle”

Ⓕ $\text{comp}_{\vec{b}}\,\vec{a}$ = The Component of $\vec{a}$ Along $\vec{b}$ (scalar)

Larson uses this approach. Math 254-Linear Alg. approach

Review $\vec{a} = \langle 3,2\rangle = 3\vec{i} + 2\vec{j}$

We're decomposing $\vec{a}$ as a sum of 2 $\perp$ vectors. Ⓐ



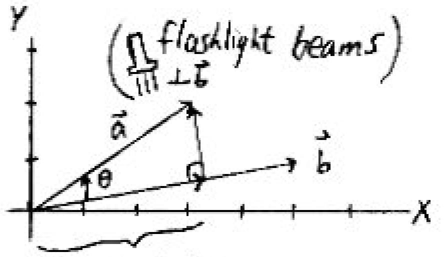
$\text{comp}_{\vec{i}}\,\vec{a} = 3$

$\text{comp}_{\vec{j}}\,\vec{a} = 2$

Other ways to do Ⓐ!

Ex If $\vec{a} = \langle 3,2\rangle$, $\vec{b} = \langle 5,1\rangle$, find $\text{comp}_{\vec{b}}\,\vec{a}$.

$\overrightarrow{\text{proj}_{\vec{b}}\,\vec{a}}$ Think: shadow of $\vec{a}$ on line thru $\vec{b}$ when a flashlight's beams are $\perp\vec{b}$



signed length = $\text{comp}_{\vec{b}}\,\vec{a}$
the vector = $\overrightarrow{\text{proj}_{\vec{b}}\,\vec{a}}$

See 14.1.7
$\cos\theta = \frac{A}{H}$
$A = H\cos\theta$

$$\text{comp}_{\vec{b}}\,\vec{a} = \|\vec{a}\|\cos\theta \qquad (\vec{b} \neq \vec{0})$$

$$= \cancel{\|\vec{a}\|}\left(\frac{\vec{a}\cdot\vec{b}}{\cancel{\|\vec{a}\|}\,\|\vec{b}\|}\right)$$

$$\boxed{\text{comp}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|} \qquad (\vec{b} \neq \vec{0})}$$

Think "b" for "bottom."

or $\vec{a}\cdot\dfrac{\vec{b}}{\|\vec{b}\|}$

unit vector in direction of $\vec{b}$

In Ex, $\displaystyle comp_{\vec b}\,\vec a = \frac{\langle 3,2\rangle \cdot \langle 5,1\rangle}{\sqrt{(5)^2+(1)^2}}$

$$= \frac{17}{\sqrt{26}}$$

$$\approx 3.334 \quad \text{(makes sense in figure)}$$

If θ is obtuse $\Rightarrow \vec a \cdot \vec b < 0 \Rightarrow comp_{\vec b}\,\vec a < 0$

Ex

$\underbrace{comp_{\vec b}\,\vec a}_{<0} = -(\text{length})$

Up to 23
(All but 25, 29)

Not in book $\overrightarrow{proj}_{\vec b}\,\vec a$ (projection) is this vector ("shadow")

$\vec a - proj_{\vec b}\,\vec a$ = the vector component of $\vec a$ orthogonal to $\vec b$

also called the vector component of $\vec a$ along $\vec b$

Larson

$$= \left(comp_{\vec b}\,\vec a\right)\underbrace{\left(\begin{array}{c}\text{unit vector in the}\\ \text{direction of } \vec b\end{array}\right)}_{\text{length}=1 \text{ to ensure } \|\overrightarrow{proj}_{\vec b}\,\vec a\| = |comp_{\vec b}\,\vec a|}$$

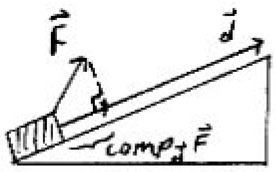
$$= \left(\frac{\vec a \cdot \vec b}{\|\vec b\|}\right)\left(\frac{\vec b}{\|\vec b\|}\right)$$

$$\boxed{\overrightarrow{proj}_{\vec b}\,\vec a = \frac{\vec a \cdot \vec b}{\|\vec b\|^2}\,\vec b \qquad (\vec b \neq \vec 0)}$$

$\underbrace{\frac{\vec a \cdot \vec b}{\|\vec b\|^2}}_{\text{scalar}}$ $\quad (\overrightarrow{proj}_{\vec b}\,\vec a) \parallel \vec b$

G Work

$\vec{F}$ = constant force (using Newtons, say)
$\vec{d}$ = displacement (using meters, say)



(In 1-D, Work = ([constant] force)(distance covered)   scalars
How do we extend this to 2-D, 3-D?)

$$\text{Work done "}W\text{"} = \underbrace{(\text{comp}_{\vec{d}}\vec{F})}_{\text{relevant measure of force}}\underbrace{(\|\vec{d}\|)}_{\text{distance}}$$

$$= \left(\frac{\vec{F}\cdot\vec{d}}{\cancel{\|\vec{d}\|}}\right)(\cancel{\|\vec{d}\|})$$

$$\boxed{W = \vec{F}\cdot\vec{d}}$$ in Newton-meters (or joules)