14.4: CROSS (VECTOR) PRODUCT

Larson 730
$\bullet$, $\times$ notation by Josiah Gibbs (U.S. phys.)

$\vec{a} \times \vec{b}$ ($\vec{a}, \vec{b}$ in V_3) is a vector.

(A) Determinant of a 2×2 Matrix
(Order 2)

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underbrace{ad - bc}_{\text{scalar}}}$$

X - butterfly

(B) Determinant of a 3×3 Matrix
(Order 3)

Method 1: Expansion by Cofactors

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

← Choose a "magic" row or column, say Row 1.

Take the corresponding signs from the sign matrix

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

"checkerboard"

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \underbrace{+}_{\substack{\text{from}\\\text{sign}\\\text{matrix}}} \underbrace{a}_{\substack{\text{1st}\\\text{magic}\\\text{entry}}} \underbrace{\begin{vmatrix} e & f \\ h & i \end{vmatrix}}_{\substack{\text{deleted row, column}\\\text{with "a"}}}$$

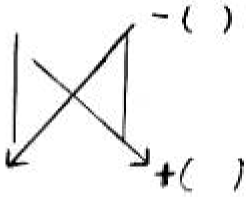
$$- b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

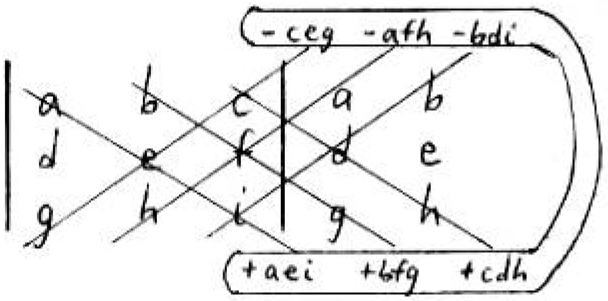
$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

Method 2 Sarrus's Rule (only for Order 3)

1. Rewrite 1^{st}, 2^{nd} columns on the right.
2. Add products along the 3 full diagonals ╲╲╲
3. Subtract ╱╱╱

Like 2×2





(equivalent to result from Method 1)

Ⓒ $\vec{a}\times\vec{b}$

If $\vec{a} = \langle a_1, a_2, a_3\rangle$,
$\vec{b} = \langle b_1, b_2, b_3\rangle$, then

$$\boxed{\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$$

← Vectors, so not technically a "determinant."

<u>Ex</u> (#2) If $\vec{a} = \langle -5, 1, -1 \rangle$, $\vec{b} = \langle 3, 6, -2 \rangle$, find $\vec{a} \times \vec{b}$.

$$\begin{array}{c} \\ \vec{a} \times \vec{b} = \end{array} \begin{array}{c} \begin{array}{ccc} \oplus & \ominus & \oplus \end{array} \\ \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & 1 & -1 \\ 3 & 6 & -2 \end{vmatrix} \end{array} \leftarrow \text{for Method 1}$$

<u>Method 1</u>

$$= \begin{vmatrix} 1 & -1 \\ 6 & -2 \end{vmatrix} \vec{i} - \begin{vmatrix} -5 & -1 \\ 3 & -2 \end{vmatrix} \vec{j} + \begin{vmatrix} -5 & 1 \\ 3 & 6 \end{vmatrix} \vec{k}$$

↑ magic entry ↑ Careful!

$$= [-2-(-6)]\vec{i} - [10-(-3)]\vec{j} + [-30-3]\vec{k}$$

$$= \boxed{4\vec{i} - 13\vec{j} - 33\vec{k} \quad \text{or} \quad \langle 4, -13, -33 \rangle}$$

<u>Method 2</u>

$$-(3\vec{k}) \; -(-6\vec{i}) \; -(10\vec{j})$$

$$\left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ -5 & 1 & -1 \\ 3 & 6 & -2 \end{array} \right| \begin{array}{cc} \vec{i} & \vec{j} \\ -5 & 1 \\ 3 & 6 \end{array}$$

$$+(-2\vec{i}) \; +(-3\vec{j}) \; +(-30\vec{k})$$

$$= -2\vec{i} - 3\vec{j} - 30\vec{k} - 3\vec{k} + 6\vec{i} - 10\vec{j}$$

Can do 1,5

$$= \boxed{4\vec{i} - 13\vec{j} - 33\vec{k} \quad \text{or} \quad \langle 4, -13, -33 \rangle}$$

Ⓓ Basic Properties

$\vec{a}, \vec{b}, \vec{c}$ in V_3,
m scalar

$\vec{a} \times \vec{0} = \vec{0} = \vec{0} \times \vec{a}$ $\vec{a} \times \vec{a} = \vec{0}$	$\}\ \vec{0}$
(i) $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$	"x" is <u>anticommutative</u>
(ii) $(m\vec{a}) \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times (m\vec{b})$	Scalar mult. is flexible re "x"
(iii) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ (iv) $(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$	} Dot, cross <u>products</u> distribute over vector "+" from either side
(v), (vi) Later	

"x" is <u>not</u> commutative or associative.

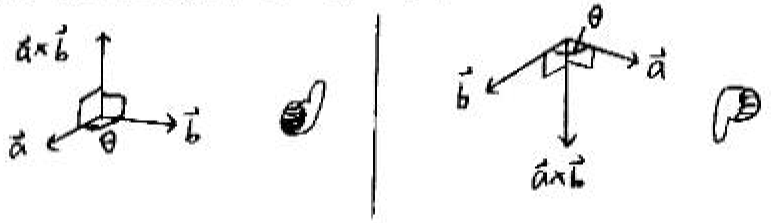
Ⓔ Geometry

Direction of $\vec{a} \times \vec{b}$

$\boxed{\begin{array}{l} \vec{a} \times \vec{b} \perp \vec{a} \\ \vec{a} \times \vec{b} \perp \vec{b} \end{array}}$ ← Show $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$.

Right-Hand Rule

If its fingers curl "through θ" $(0 < \theta < \pi)$ from $\vec{a}$ to $\vec{b}$, then its thumb points in the direction of $\vec{a} \times \vec{b}$.



Length of $\vec{a}\times\vec{b}$

$$\boxed{\underbrace{\|\underbrace{\vec{a}\times\vec{b}}_{\text{vector}}\|}_{\text{scalar}} = \|\vec{a}\|\|\vec{b}\|\sin\theta} \quad ⊛$$

Proof (Optional) - p. 713

Recall $\underbrace{\vec{a}\cdot\vec{b}}_{\text{scalar}} = \|\vec{a}\|\|\vec{b}\|\cos\theta$

$\vec{a}\parallel\vec{b} \Leftrightarrow \theta = 0$ or $\theta = \pi$ or $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$

$\Leftrightarrow \quad \sin\theta = 0$

$\Leftrightarrow \quad \|\vec{a}\|\|\vec{b}\|\sin\theta = 0$

$\underset{⊛}{\Leftrightarrow} \quad \|\vec{a}\times\vec{b}\| = 0$

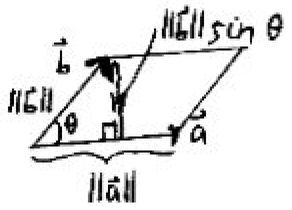
$\Leftrightarrow \quad \vec{a}\times\vec{b} = \vec{0}$

$$\boxed{\vec{a}\parallel\vec{b} \Longleftrightarrow \vec{a}\times\vec{b} = \underset{\text{vector}}{\vec{0}}}$$

Recall $\vec{a}\perp\vec{b} \Leftrightarrow \vec{a}\cdot\vec{b} = \underset{\text{scalar}}{0}$

$$\boxed{\begin{array}{l} \|\vec{a}\times\vec{b}\| = \text{area of parallelogram determined by } \vec{a},\vec{b} \\ \tfrac{1}{2}\|\vec{a}\times\vec{b}\| = \text{"} \quad \text{"} \ \text{triangle} \quad \text{"} \quad \text{"} \end{array}}$$

Proof



See 14.1.7

$\sin\theta = \frac{\text{opp.}}{H}$

$\text{Opp.} = H\sin\theta$

Area = (base)(height)

$= \|\vec{a}\| \ \|\vec{b}\|\sin\theta$

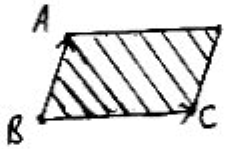
$= \|\vec{a}\times\vec{b}\|$

3 $\underbrace{\text{noncollinear}}_{\text{do not lie on the same line}}$ points in $\mathbb{R}^3$ determine

ⓐ one triangle, and

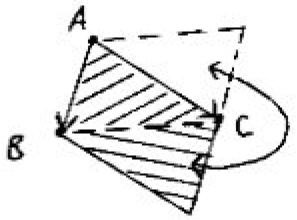
ⓑ different parallelograms with the same area. } The 3 given points are 3 of the 4 vertices.

How are $\vec{BA}\times\vec{BC}$, $\vec{BC}\times\vec{BA}$ related?



Area $= \|\vec{BA}\times\vec{BC}\|$ { opposite vectors have same lengths

$= \|\vec{BC}\times\vec{BA}\|$



$= \|\vec{AB}\times\vec{AC}\|$

etc.

Area of triangle ABC $= \frac{1}{2}$ (any of these)

Use all 3 points!

<u>Ex</u> Find the area of the triangle determined by $A(8,-3,2)$, $B(3,-2,1)$, and $C(11,3,0)$.

<u>Sol'n</u>

$$\vec{AB} = \langle 3-8, -2-(-3), 1-2 \rangle = \langle -5, 1, -1 \rangle$$

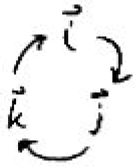
$$\vec{AC} = \langle 11-8, 3-(-3), 0-2 \rangle = \langle 3, 6, -2 \rangle$$

From Ⓒ, $\vec{AB} \times \vec{AC} = \langle 4, -13, -33 \rangle$

$$\begin{aligned} \text{Area} &= \tfrac{1}{2} \| \langle 4,-13,-33 \rangle \| \\ &= \tfrac{1}{2} \sqrt{(4)^2 + (-13)^2 + (-33)^2} \\ &= \tfrac{1}{2}\sqrt{1274} \\ &\approx \boxed{17.8} \end{aligned}$$

Ⓕ $\vec{i}, \vec{j}, \vec{k}$

<u>Wheel:</u>



face board

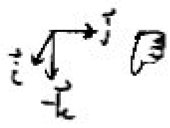
(vector) $\times$ (successor) = (3^{rd})

$\vec{i} \times \vec{j} = \vec{k}$:

and $\|\vec{i} \times \vec{j}\| = 1$ (Why? Use 14.4.5Ⓐ)

(vector) $\times$ (predecessor) = $-$(3^{rd})

⊖ $\vec{j} \times \vec{i} = -\vec{k}$:



and $\|\vec{j} \times \vec{i}\| = 1$

<u>$\vec{a} \times \vec{b}$ Table</u>

"×"

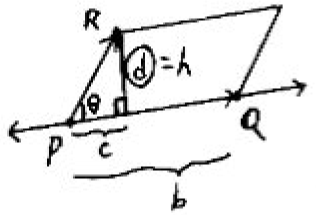
$\vec{a} \backslash \vec{b}$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	$\vec{0}$	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	$\vec{0}$	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	$\vec{0}$

Like a skew-sym matrix

"×" is anticommutative

Ⓖ The Distance Between a Point and a Line
(d) (R) ($\overleftrightarrow{PQ}$)

Proof on Ex 3, p. 715 uses sin, not ▱



Area of ▱ $= bh$

$$h = \frac{\text{Area}}{b}$$

(Use all 3 points)

$$\boxed{d = \frac{\|\overrightarrow{PQ} \times \overrightarrow{PR}\|}{\|\overrightarrow{PQ}\|} = \frac{\|\overrightarrow{PR} \times \overrightarrow{PQ}\|}{\|\overrightarrow{PQ}\|}}$$

Up to 19

"$\vec a$" · "$\vec b$" can be defined as $\|\vec a\|\|\vec b\|\cos\theta$ / $\|\vec a \times \vec b\|$ sin

<u>Recall</u> $\quad c = \text{comp}_{\overrightarrow{PQ}} \overrightarrow{PR} = \dfrac{\overrightarrow{PR} \cdot \overrightarrow{PQ}}{\|\overrightarrow{PQ}\|}$ (Point off line omitted)

Ⓗ Triple Scalar Product (TSP, Box Product)

Not Traveling Salesman Problem

Prop. (v) $\boxed{TSP = (\vec a \times \vec b) \cdot \vec c = \vec a \cdot (\vec b \times \vec c)}$ (scalar)

i.e., You can switch "×" and "·"
i.e., Cross 2 successive vectors in order, and dot the result with the 3rd ∋ some #

Nonsense: $\underbrace{(\vec a \cdot \vec b)}_{\text{scalar}} \times \underbrace{\vec c}_{\text{vector}}$

#21 (unassigned?)

For computation:

$$\boxed{TSP = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{array}{l} \leftarrow \vec a \\ \leftarrow \vec b \\ \leftarrow \vec c \end{array}}$$

Can do: $TSP = \underbrace{|\vec a\ \vec b\ \vec c|}_{\text{by columns}}$; $\underset{\text{matrix}}{|A|} = \underbrace{|A^T|}_{\text{A-transpose (rows}\leftrightarrow\text{cols)}}$

$\boxed{|TSP| = \text{volume of box (parallelepiped) determined by } \vec a, \vec b, \vec c}$

abs. value

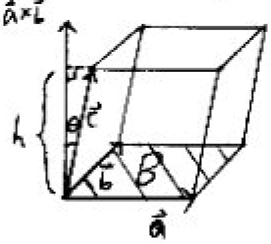
Proof (Optional) p. 716 or:

Larson 734 uses proj

$\vec a = \vec 0$, or $\vec b = \vec 0$ $\Rightarrow \vec a \parallel \vec b$

$\vec a \parallel \vec c \Rightarrow \vec a \times \vec b \perp \vec c$ $\Rightarrow TSP = 0$

dot com



$$
\begin{aligned}
V &= Bh \\
&= \|\vec a \times \vec b\| \, |\text{comp}_{\vec a \times \vec b} \vec c| \\
&= \cancel{\|\vec a \times \vec b\|} \left| \frac{\vec c \cdot (\vec a \times \vec b)}{\underbrace{\cancel{\|\vec a \times \vec b\|}}_{>0}} \right| \leftarrow (\vec a \nparallel \vec b) \text{ otherwise, } TSP = 0 \\
&= |(\vec a \times \vec b) \cdot \vec c| \quad (\text{"}\cdot\text{" comm.}) \\
&= |TSP|
\end{aligned}
$$

$\boxed{TSP = 0 \Leftrightarrow \text{[position vectors for] } \vec a, \vec b, \vec c \text{ are } \underbrace{\text{coplanar}}_{\text{lie in same plane}}}$

	[Position] Vectors are	Think:
$\vec a \times \vec b = \vec 0$	collinear	Area = 0
$TSP = 0$	coplanar	Vol. = 0

(I) Triple Vector Product

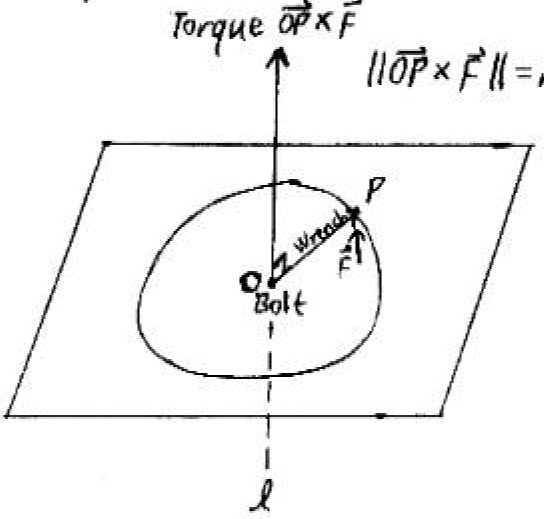
Prop. (vi) $\boxed{\vec a \times (\vec b \times \vec c) = (\vec a \cdot \vec c)\vec b - (\vec a \cdot \vec b)\vec c}$ (I'll give)

"x" not associative:

Often, $\vec a \times (\vec b \times \vec c) \neq (\vec a \times \vec b) \times \vec c$

(J) Torque (Optional)

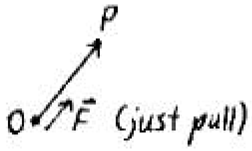
or Moment of $\vec{F}$ about O.
P moves



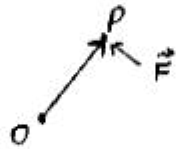
$\|\vec{OP}\times\vec{F}\|$ = magnitude of rotational effect about ℓ

$\|\vec{OP}\|$ fixed (constant).
Let's say $\|\vec{F}\|$ is fixed.

When is $\|\vec{OP}\times\vec{F}\|$ maximized?



$\|\vec{OP}\times\vec{F}\| = 0$



$$\|\vec{OP}\times\vec{F}\| = \|\vec{OP}\|\,\|\vec{F}\|\underbrace{\sin\theta}_{\text{max'ed } (=1) \text{ when } \theta=\frac{\pi}{2}}$$

i.e., when $\vec{F}\perp\vec{OP}$