14.5: LINES and PLANES in $\mathbb{R}^3$

Ⓐ What Determines a Line, ℓ?

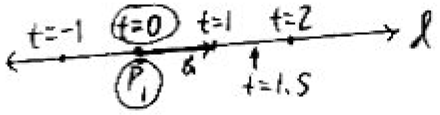
ⓐ 2 distinct points  $\Rightarrow$ ⓑ

or ⓑ 1 point : $P_1(x_1, y_1, z_1)$ and
1 direction vector : $\vec{a} = \langle a_1, a_2, a_3 \rangle$
(replaces "slope")

"$\vec{a} \parallel \ell$"

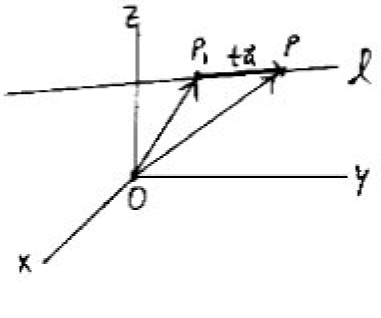
Ⓑ Parametric Equations for a Line (many possibilities)

t ("time") is our parameter. Ⓣ watch



(Different choice for P_1 or $\vec{a}$
$\Rightarrow$ Different parameterization for ℓ)

ℓ contains all points $P(x,y,z)$ such that



$\overrightarrow{OP} = \overrightarrow{OP_1} + t\vec{a}$ for some real t

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\substack{\text{Column} \\ \text{form for} \\ \text{a vector}}} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + t\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Book's approach:
$\overrightarrow{P_1P} = t\vec{a}$
$\langle x-x_1, y-y_1, z-z_1 \rangle = t\langle a_1, a_2, a_3 \rangle$

Plane:
$\overrightarrow{OP_1} + t\vec{a} + u\vec{b}$
t, u in $\mathbb{R}$

$$\boxed{\begin{cases} x = \overset{P_1}{x_1} + \overset{\vec{a}}{a_1} t \\ y = y_1 + a_2 t \\ z = z_1 + a_3 t \end{cases} , t \text{ in } \mathbb{R}}$$

(t sweeps through all real #s)

Ⓒ Symmetric Equations for a Line

Solve each eq. for t, and equate.

$$t = \boxed{\underbrace{\frac{x-x_1}{a_1}}_{\substack{\text{If } a_1=0 \\ \Rightarrow \text{"}x=x_1\text{"}}} = \frac{y-y_1}{a_2} = \frac{z-z_1}{a_3} \quad \leftarrow \text{if none are } 0}$$

etc.

Ⓓ Ex

The line ℓ contains $P_1(3,-7,2)$ and $P_2(5,-10,2)$.

ⓐ Find parametric eqs. for ℓ.

ⓑ Find symmetric eqs. for ℓ.

Point : $P_1\ (3,-7,2)$ (P_2 OK)

$$\begin{aligned}\vec{a} &= \overrightarrow{P_1P_2} \\ &= \langle 5-3, -10-(-7), 2-2\rangle \\ &= \langle 2,-3,0\rangle\end{aligned}$$

($\overrightarrow{P_2P_1}$ OK ⇒ time reversal)

ⓐ Param. eqs.:

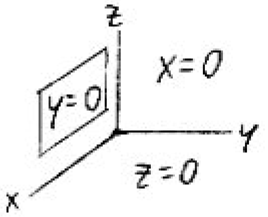
$$\boxed{\begin{aligned} x &= 3 + 2t \\ y &= -7 - 3t \\ z &= 2 \end{aligned}} \quad \begin{aligned} &\Longrightarrow t = \frac{x-3}{2} \\ &\Longrightarrow t = \frac{y+7}{-3} \\ &\xRightarrow[(a_3=0)]{} z = 2 \end{aligned}$$

} Equate

ⓑ Sym. eqs.:

$$\boxed{\frac{x-3}{2} = \frac{y+7}{-3}, \quad z=2}$$

Ⓒ Where does ℓ intersect the xz-plane?



$$y = -7 - 3t \overset{\text{set}}{=} 0$$
$$t = -\tfrac{7}{3} \quad \text{(time of "impact")}$$

$$\Rightarrow \begin{cases} x = 3 + 2\left(-\tfrac{7}{3}\right) = -\tfrac{5}{3} \\ y = 0 \\ z = 2 \end{cases}$$

Up to 9

$$\boxed{\left(-\tfrac{5}{3}, 0, 2\right)}$$

Where does ℓ intersect the xy-plane?
How do you graph $z=2$? ℓ lies on here.

<u>Another Method</u>: Use sym. eqs. Set $y=0 \Rightarrow \underbrace{\frac{x-3}{2} = \frac{0+7}{-3}}_{\Rightarrow x = -\frac{5}{3}}, z=2$

$\Rightarrow \left(-\tfrac{5}{3}, 0, 2\right)$ again

Ⓔ Do 2 Lines Intersect? Where?

<u>Ex</u>

$$\ell_1 : ☆ \begin{cases} x = 1 + 3t \\ y = 6 - 4t \\ z = -1 + 2t \end{cases} \qquad \ell_2 : ☆☆ \begin{cases} x = 2 - 5u \\ y = 2u \\ z = 4 + u \end{cases}$$

t in $\mathbb{R}$ $\qquad$ u in $\mathbb{R}$

Imagine two jets leaving trails. If the trails intersect, must the jets crash?

(x,y,z) is an intersection point $\Longleftrightarrow$

There is a "time" t at which ℓ_1 "hits" (x,y,z),
and " " u " " ℓ_2 " "

Solve the system

often used to find intersection pts.

$$\begin{cases} 1+3t = 2-5u \\ 6-4t = 2u \\ -1+2t = 4+u \end{cases}$$

$$\Leftrightarrow \begin{cases} 3t+5u = 1 \\ -4t-2u = -6 \\ 2t-u = 5 \end{cases}$$

Solve (★★), say (easiest pair?) ✓
$\Rightarrow t=2,\ u=-1$

Note If (★★) has no solution ⇒ no intersection pts.
If (★★) has ∞ many solutions ⇒ Math 254
(Projections/shadows on yz-plane might coincide.)

Subsystem (★★) has a sol'n ⇔ The projections of the lines on the yz-plane intersect. Then, ✓ to see if x-coords match.

Ideas re ∞ many sols:

$\begin{cases} x=0 \\ \vdots \end{cases}$ $\begin{cases} x=0 \\ \vdots \end{cases}$

$\begin{cases} 0=0 \\ 1=1 \end{cases}$ delete both / use $x=0$

$\begin{cases} 0=0 \\ \text{line} \end{cases}$ & combine w/remaining eq.

$\left\{\begin{matrix} \text{line} \\ \text{line} \end{matrix}\right\}$ equivalent; take one and combine w/remaining eq.

in the (not xy) plane

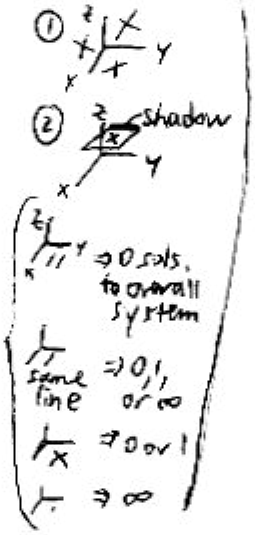
Check to see if $t=2$, $u=-1$ satisfy the remaining eq.

If so, l_1 and l_2 intersect.
If not, they don't.

$$3t+5u=1$$
$$3(2)+5(-1)=1$$
$$1=1 \checkmark \quad \Rightarrow l_1 \text{ and } l_2 \text{ intersect.}$$

Side Notes

Exs Shadows of intersecting lines



What's the intersection pt. (x,y,z)?

Plug $t=2$ into (★) or $u=-1$ into (★★).

$$(★)\begin{cases} x = 1+3(2) = 7 \\ y = 6-4(2) = -2 \\ z = -1+2(2) = 3 \end{cases}$$

$$\boxed{(7,-2,3)}$$

1/p to 13

(F) Angles Between Lines

Ex from (E)

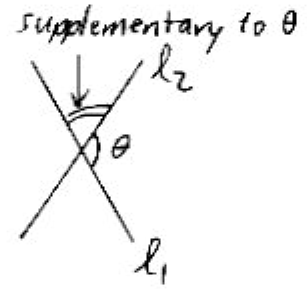
$$\ell_1 : \begin{cases} x = 1 + 3t \\ y = 6 - 4t \\ z = -1 + 2t \end{cases} \qquad \ell_2 : \begin{cases} x = 2 - 5u \\ y = \phantom{2 -{}} 2u \\ z = 4 + u \end{cases}$$

direction vector $\vec{a} = \langle 3, -4, 2 \rangle$ $\qquad$ $\vec{b} = \langle -5, 2, 1 \rangle$

$$\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right)$$

$$= \cos^{-1}\left(\frac{-21}{\sqrt{29}\sqrt{30}}\right)$$

$$\approx \boxed{2.36 \text{ radians, or } 135^\circ}$$



$\pi - \theta$ $\Downarrow$ Get supp. angle. $\Downarrow$ $180^\circ - \theta$

$$\boxed{0.78 \text{ radians, or } 45^\circ}$$

Applies to nonintersecting lines, also.
Translate lines without rotating $\Rightarrow$ same angles.

If $\vec{a}$ is any direction vector for ℓ_1, and $\vec{b}$ for ℓ_2, then

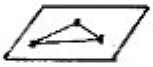
$$\ell_1 \parallel \ell_2 \iff \vec{a} \parallel \vec{b}$$
$$\ell_1 \perp \ell_2 \iff \vec{a} \perp \vec{b}$$

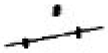
Up to 15

Ⓖ What Determines a Plane?

3 points if they don't...
2 skew lines
$\nexists$ plane

Truneh:
imagine 3 legs of a slanted stool
||| Tips over:
|||

ⓐ 3 noncollinear points  $\Rightarrow$ ⓒ

or ⓑ 1 line and a point not on the line  $\Rightarrow$ ⓒ

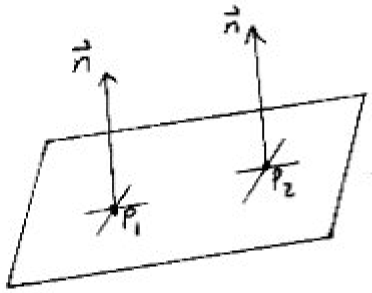
or ⓒ 1 point and 2 non-$\parallel$ vectors  $\Rightarrow$ ⓓ

or ⓓ 1 point and a <u>normal vector</u> $\vec{n}$

$\vec{n} \neq \vec{0}$

$\vec{n} \perp$ plane

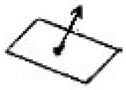
(i.e., $\vec{n} \perp$ [direction vectors for] every line in the plane)



$\vec{n} \perp$ every line in the plane containing P_1,

$\vec{n} \perp$ every line in the plane containing P_2

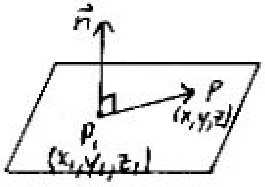
How to Ace: "joystick on a flying carpet"



(H) Equation Forms for a Plane

Given: Point $P_1(x_1, y_1, z_1)$ in the plane
$\vec{n} = \langle a, b, c \rangle$

A point $P(x, y, z)$ is in the plane



$$\iff \vec{n} \cdot \overrightarrow{P_1P} = 0$$

$$\iff \langle a, b, c \rangle \cdot \langle x - x_1, y - y_1, z - z_1 \rangle = 0$$

Larson

$$\iff \boxed{a(x - x_1) + b(y - y_1) + c(z - z_1) = 0}$$

Standard Form

Larson

$$\iff \boxed{ax + by + cz + \underset{\substack{\uparrow \\ \text{a real \#} \\ = -ax_1 - by_1 - cz_1}}{d} = 0}$$

General Form

If a, b, c are not all 0,

The graph of $ax + by + cz + d = 0$ is a plane with normal $\vec{n} = \langle a, b, c \rangle$

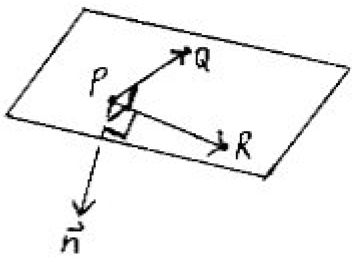
Ex Find an eq. for the plane determined by $P(2,-3,4)$, $Q(-3,-2,3)$, and $R(5,3,2)$. ← noncollinear

Sol'n

$$\left.\begin{aligned}\overrightarrow{PQ} &= \langle -5, 1, -1\rangle \\ \overrightarrow{PR} &= \langle 3, 6, -2\rangle\end{aligned}\right\}\text{ There are other possibs.}$$

Find a normal vector for the plane.

noncollinear
$\Rightarrow \overrightarrow{PQ} \nparallel \overrightarrow{PR}$
$\Rightarrow \overrightarrow{PQ} \times \overrightarrow{PR} \neq \vec{0}$



$$\left.\begin{aligned}\vec{n} &\perp \overrightarrow{PQ} \\ \vec{n} &\perp \overrightarrow{PR}\end{aligned}\right\} \Rightarrow \vec{n} \perp \text{plane}$$

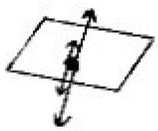
Math 254
See 14.3, HW #44

Let $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$

$\vdots$ (see 14.4.3)

$= \langle 4, -13, -33\rangle$

(or any non-$\vec{0}$ scalar multiple) ☆



Standard Form: (using P, say)	$4(x-2) - 13\underbrace{(y-(-3))}_{(y+3)} - 33(z-4) = 0$

☆ OK to mult. through by any non-0 scalar.

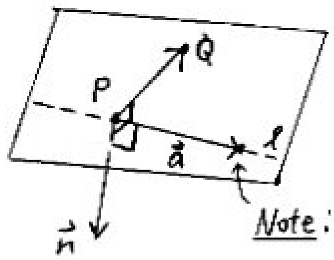
General Form:	$4x - 13y - 33z + 85 = 0$

<u>Ex</u> Find an eq. for the plane containing the point $Q(-3,-2,3)$ and the line

$$\ell : \begin{cases} x = 2 + 3t \\ y = -3 + 6t \\ z = 4 - 2t \end{cases}, \ t \text{ in } \mathbb{R}$$

<u>Sol'n</u>

$P(2,-3,4)$ $(t=0)$ is on ℓ $\qquad \vec{a} = \langle 3,6,-2 \rangle$ direction vector for ℓ



<u>Note</u>: $t=1 \Rightarrow R(5,3,2)$

Let $\vec{n} = \overrightarrow{PQ} \times \vec{a}$

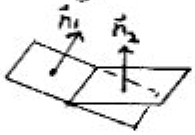
$\vdots$

(same as in previous Ex)

(I) Angles Between Planes

Let $\vec{n}_1$ be a normal vector for Plane #1.
Let $\vec{n}_2$ be a normal vector for Plane #2.
Let θ be the angle between $\vec{n}_1, \vec{n}_2$.

$\Rightarrow$ The angles between the planes are θ and its supplement.



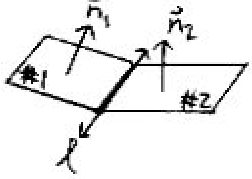
The planes are $\parallel \iff \vec{n}_1 \parallel \vec{n}_2$
$\perp \iff \perp$

Up to 45

(J) The Line of Intersection of 2 Planes

In How to Ace, mentioned in Larson 740

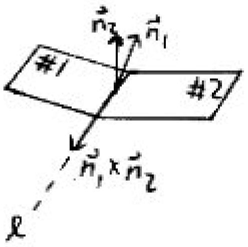
Not in book!



Find a direction vector for ℓ

$\vec{n}_1 \perp$ Plane #1 (including ℓ)
$\vec{n}_2 \perp$ Plane #2

If planes $\nparallel$, $\vec{n}_1$ and $\vec{n}_2$ determine a family of $\parallel$ planes. (Together w/ pt. $\Rightarrow$ 1 plane)
Only vectors $\parallel$ $\vec{n}_1 \times \vec{n}_2$ are $\perp$ these planes (i.e., $\perp \vec{n}_1$, $\perp \vec{n}_2$)
$\Rightarrow \vec{n}_1 \times \vec{n}_2$ = a direc. vector for ℓ



$$\Rightarrow \underbrace{\vec{n}_1 \times \vec{n}_2}_{\text{Take this!}} \parallel \ell$$

(What if the planes are $\parallel$?)

Ex 12 in book (pp. 725-6) - but new way!

$$☆ \begin{cases} 2x - y + 4z = 4 & \text{(Plane \#1)} \\ x + 3y - 2z = 1 & \text{(Plane \#2)} \end{cases} \Rightarrow \text{line } \ell$$

Find parametric eqs. and symmetric eqs. for ℓ.

Sol'n

Find $\vec{a}$, a direction vector for ℓ.

$$\begin{aligned} \text{Let } \vec{a} &= \vec{n}_1 \times \vec{n}_2 \\ &= \langle 2, -1, 4 \rangle \times \langle 1, 3, -2 \rangle \\ &\vdots \\ &= \langle -10, 8, 7 \rangle \end{aligned}$$

Where does ℓ hit the xy-plane?

A line must intersect $x=a$, $y=b$, or $z=c$.

Ex $\begin{cases} x+y+z=1 \\ x+y+4z=2 \end{cases}$

$z=0 \Rightarrow$ No sol'n

$\Rightarrow$ line of intersection, xy-plane ($z=0$) do not intersect

Find a point on ℓ.

Plug $z=0$, say, into $\circledast$.

$$\Rightarrow \circledast\circledast \begin{cases} 2x - y = 4 \\ x + 3y = 1 \end{cases}$$

$$\xRightarrow{\text{Solve!}} x = \frac{13}{7},\ y = -\frac{2}{7},\ (z=0)$$

$$\left(\frac{13}{7}, -\frac{2}{7}, 0\right)$$

(← A line in xyz-space must intersect at least one coord. plane ($x=0$, $y=0$, or $z=0$). Think: Electric fences)

Parametric Eqs. for ℓ

$$\begin{cases} x = \frac{13}{7} - 10t \\ y = -\frac{2}{7} + 8t \\ z = 7t \end{cases}, \quad t \text{ in } \mathbb{R}$$

(Why is the book's answer on p.726 also OK?)

Symmetric Eqs. for ℓ

$$\boxed{\frac{x - \frac{13}{7}}{-10} = \frac{y + \frac{2}{7}}{8} = \frac{z}{7}}$$

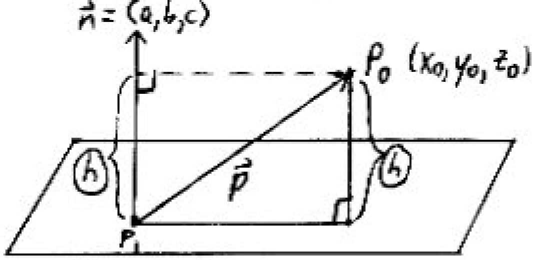
Ⓚ The Distance (h) Between a Point: $P_0(x_0, y_0, z_0)$ and a Plane: $\underbrace{ax + by + cz + d}_{\textcircled{\star}} = 0$

Key for Distance

$\begin{pmatrix} \text{A normal } \vec{n} \\ \text{for a plane} \end{pmatrix} \perp$ The plane

↓ implies "shortest distance"

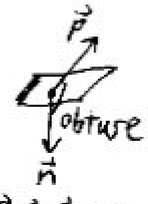
Figure



(x_1, y_1, z_1), an arbitrary point in the plane, satisfies ⭑ $= 0$

$\Rightarrow \underbrace{d = -ax_1 - by_1 - cz_1}_{\star\star}$

Think of $\vec{p}$ as a tether or connector, we really want $|\text{comp}_{\vec{n}}\vec{p}|$. We prefer directions ⊥ plane.



$\Rightarrow \vec{p}\cdot\vec{n} < 0$

$\Rightarrow \text{comp}_{\vec{n}}\vec{p} < 0$

$$\boxed{\begin{aligned} h &= |\text{comp}_{\vec{n}}\,\vec{p}\,| \\ &= \frac{|\vec{p}\cdot\vec{n}|}{\|\vec{n}\|} \end{aligned}}$$

We'll use in Ⓜ.

$$= \frac{|\langle x_0 - x_1,\ y_0 - y_1,\ z_0 - z_1 \rangle \cdot \langle a,b,c \rangle|}{\|\langle a,b,c \rangle\|}$$

by ⭑⭑

$$\boxed{h = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

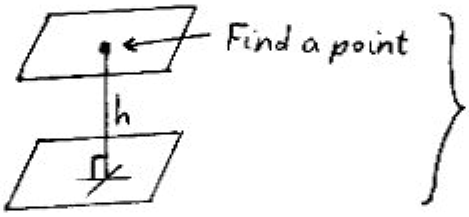
← |Result from "plugging P_0" into ⭑|

← $\|\vec{n}\|$

No need to find P_1!! See <u>Ex</u> in Ⓛ.

(L) The Distance (h) Between ∥ Planes

The dist. bet. $x+y=1$, $x+y=2$ is not 1; it's $\frac{\sqrt{2}}{2}$. Why? True in 2-D and in 3-D



$\Rightarrow$ Use (K)

What if $4x-2y+10z+6=0$? →

<u>Ex</u> Find the distance between [the graphs of] $2x-y+5z+3=0$ (Plane #1) and $4x-2y+10z+8=0$ (Plane #2)

<u>Sol'n</u>

<u>Show that the planes are ∥.</u>

We can let $\vec{n}_1 = \langle 2,-1,5\rangle$,
$\vec{n}_2 = \langle 4,-2,10\rangle$.

$\Rightarrow \vec{n}_2 = 2\vec{n}_1$
$\Rightarrow \vec{n}_2 \parallel \vec{n}_1$
$\Rightarrow$ The planes are ∥.

<u>Find a point on, say, Plane #2 ("uglier eq.")</u>

$$4x-2y+10z+8=0$$

Choose $y=0$, $z=0$, say.

$\Rightarrow 4x+8=0$
$\Rightarrow x=-2$

$$(-2,0,0)$$

If you had $4x-2y+8=0$, you could still choose $z=0$.
$\vec{n} = \langle 4,-2,0\rangle$ anyway when do numerator of h formula

The other ("nicer") plane has eq. $ax+by+cz+d=0$.

$$\underbrace{2x - y + 5z + 3 = 0}_{\text{☆}}$$

Use (K)

$$h = \frac{|2(-2) - (0) + 5(0) + 3|}{\sqrt{(2)^2 + (-1)^2 + (5)^2}} \quad \leftarrow \text{Plug } (-2,0,0) \text{ into ☆}$$

$$= \frac{1}{\sqrt{30}}$$

$$\approx \boxed{0.18} \text{ [units]}$$

(M) The [Shortest] Distance (h) Between Skew Lines

Think: Edges of box or room. 

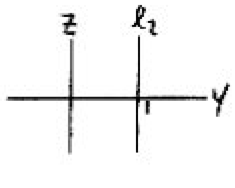
neither ∥, nor intersecting
noncoplanar

Ex

$$l_1 : \begin{cases} x = t \\ y = 0 \\ z = 0 \end{cases} \qquad l_2 : \begin{cases} x = 0 \\ y = 1 \\ z = u \end{cases} \qquad 0 \neq 1 \Rightarrow l_1, l_2 \text{ don't intersect.}$$

t, u in $\mathbb{R}$

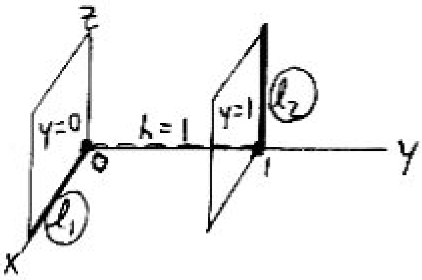
$$\vec{a}_1 = \langle 1,0,0 \rangle \;\not\parallel\; \vec{a}_2 = \langle 0,0,1 \rangle \quad \Rightarrow l_1 \not\parallel l_2$$



l_1 (x-axis) sticks out at you.

We want the distance between $\parallel$ planes containing skew lines.

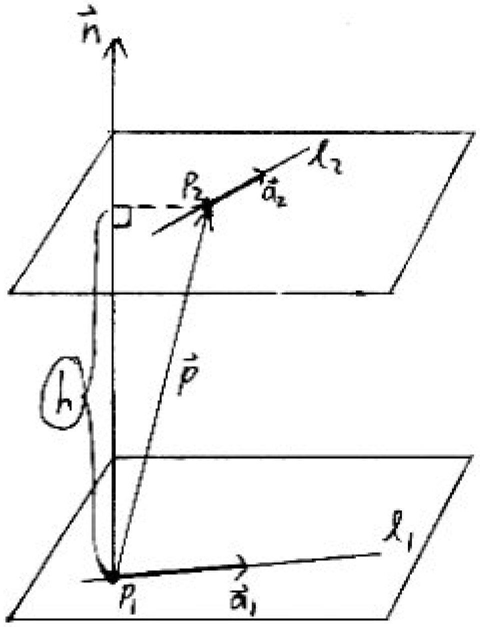
<u>In Ex</u>



Book more complicated.

$\vec{n}$: Remember Right-Hand Rule

Fall back to a more fundamental formula.



We don't have eqs. for these planes!

Use: $h = |\text{comp}_{\vec{n}}\, \vec{p}|$

① Find a point P_1 on ℓ_1.
" P_2 on ℓ_2.

② <u>Let $\vec{p} = \overrightarrow{P_1P_2}$.</u>

③ Let $\vec{a}_1$ be a direction vector for ℓ_1.
Let $\vec{a}_2$ " " ℓ_2.

④ <u>Let $\vec{n} = \vec{a}_1 \times \vec{a}_2$.</u>

$\left(\begin{array}{l} \Rightarrow \vec{n} \perp \vec{a}_1,\ \vec{n} \perp a_2 \\ \Rightarrow \vec{n} \perp \text{both planes} \end{array} \right)$

⑤ $h = |\text{comp}_{\vec{n}}\, \vec{p}|$

$$= \frac{|\vec{p} \cdot \vec{n}|}{\|\vec{n}\|}$$

Do HW #55

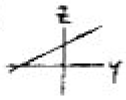
(N) Graphs

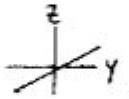
Ex The plane $x - 2y + 3z = 6$

Find x-intercept (if any)

Plug in $y=0, z=0$
$\Rightarrow x = 6$ or $(6,0,0)$ is x-int.

y-int. $= -3$ or $(0,-3,0)$
z-int. $= 2$ or $(0,0,2)$

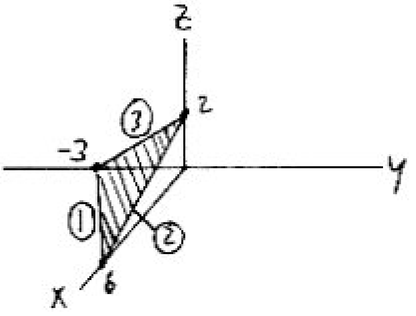
Note
ⓐ $-2y+3z=6$ has no x-int.
ⓑ $-2y+3z=0$ includes the entire x-axis.
Sweep ∥ x-axis:
ⓐ 

ⓑ 

Find xy-trace
intersection with the xy-plane

Plug in $z=0$
$\Rightarrow x - 2y = 6,\ z=0$ (line) ①

xz-trace: $x + 3z = 6,\ y=0$ ②
yz-trace: $-2y+3z=6,\ x=0$ ③



<u>Ex</u> Line l_1 from (E) $\begin{cases} x = 1+3t \\ y = 6-4t \\ z = -1+2t \end{cases}$

has symmetric eqs.

$$\frac{x-1}{3} = \frac{y-6}{-4} = \frac{z+1}{2}$$

(1)

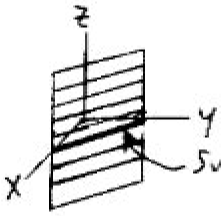
(2)

(3)

l_1 is the intersection of, say, (1) and (2). ((3) also contains l_1.)

(1) Plane $\perp$ xy-plane

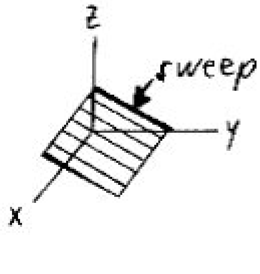
xy-trace
$y = -\frac{4}{3}x + 7\frac{1}{3}$



sweep xy-trace $\parallel$ to z-axis.

(2) Plane $\perp$ yz-plane

yz-trace
$z = -\frac{1}{2}y + 2$



sweep yz-trace $\parallel$ to x-axis.