14.6: SURFACES in 3D

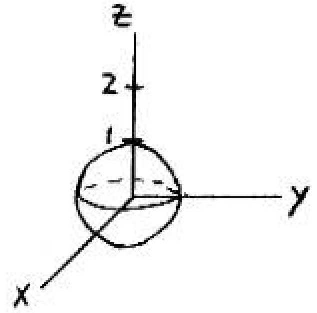
(A) Traces

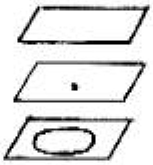
Ex Unit sphere $x^2+y^2+z^2=1$

(1) Traces in (intersections with) planes of the form $z=k$

"cross sections"

Like stacking transparent playing cards





$z=2 \Rightarrow$ empty

$z=1 \Rightarrow$ point

$z=0 \Rightarrow$ xy-trace: circle

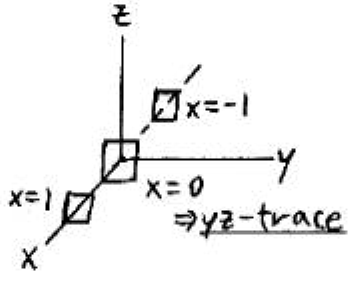
$x^2+y^2=1,\ z=0$

If $z=k \Rightarrow x^2+y^2+k^2=1$

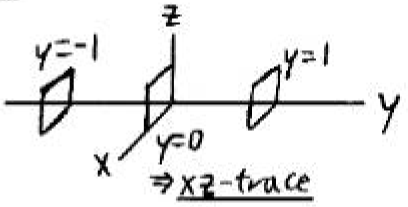
$$x^2+y^2=1-k^2$$

$$\text{family of traces}\begin{cases}\text{circle} & , |k|<1\\ \text{point} & ; |k|=1\\ \text{empty} & , |k|>1\end{cases}$$

(2) $x=k$



(3) $y=k$



(B) Spheres (14.2)

Have circles as traces

(C) Planes (14.5)

Have lines as traces

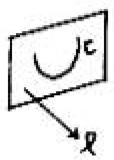
(D) Cylinders (see 14.2.5)

Have a family of "identical" traces.
maybe shifted

If a plane curve C is swept ∥ to an $\underbrace{\text{axis}}_{\not\parallel \text{ to the plane}}$,
the result is a cylinder.

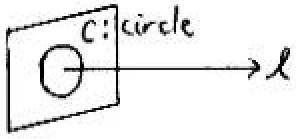
C is called the directrix or generating curve.

Ex



parabolic cylinder

Ex

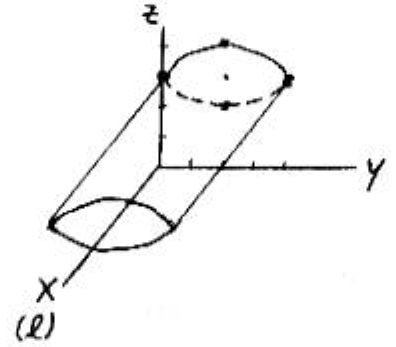


right circular cylinder
$\ell \perp$ plane

Ex Graph $\frac{(y-2)^2}{4} + \frac{(z-3)^2}{1} = 1$ ✪

yz-trace: Ellipse

Center: $y=2, z=3$
$a^2 = 4 \Rightarrow a = 2$
$b^2 = 1 \Rightarrow b = 1$

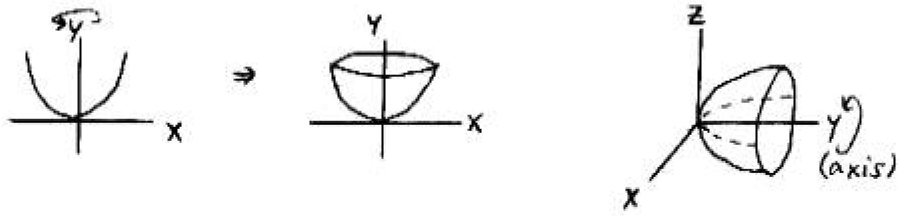


x missing in ✪
$\Rightarrow$ sweep trace ∥ to x-axis

right elliptic cylinder

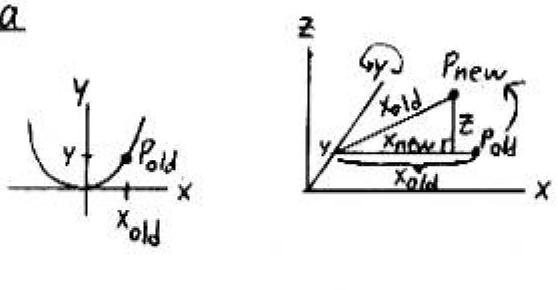
Ⓔ Surfaces of Revolution

Ex The graph of $y=x^2$ in the xy-plane is revolved about the y-axis. Find an eq. of the resulting surface.



Book a mess

Idea



$x_{old}^2 = x_{new}^2 + z^2$

$y = x^2$

Replace x^2 with $x^2 + z^2$

"not the "axis variable"" — missing variable — (Don't touch "axis variable.")

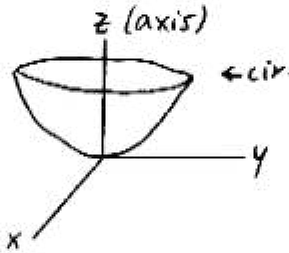
$\underbrace{\boxed{y = x^2 + z^2}}_{\Rightarrow y \geq 0}$ paraboloid with axis: y-axis (opens along)

Ⓕ Variable-Switch Trick

⇓

Switch roles

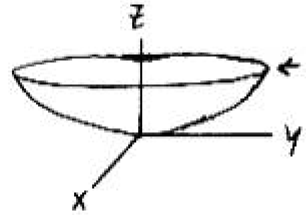
Ex $\boxed{z = x^2 + y^2}$ paraboloid with axis: z-axis (opens along)



circular paraboloid

Ⓖ Coefficients Trick (for 14.6)

Ex $\boxed{z = 4x^2 + y^2}$



elliptic paraboloid
not a surface of revol.

If you multiply or divide a term by a positive real #, you may change the shape of a graph but not its general type.
"paraboloid"

Exception: We distinguish between spheres ⊖ and ellipsoids ⊖.

(H) Quadric Surfaces (Q.S.s)

(My approach differs from the book's!)

Swokowski 734

Graphs of [nondegenerate] 2^{nd}-degree eqs. in $x, y,$ and z.
(If any are missing ⇒ cylinder?)

The trace of a "basic*" Q.S. in $x=k$, $y=k$, or $z=k$ can be:

① empty
② 1 point
③ 2 intersecting lines X (degenerate hyperbola)
or ④ a conic
E = Ellipse (or Circle)
H = Hyperbola
P = Parabola

*Note Other Q.S.s may be rotated or have single lines as traces.

We can't have different types of conics in, $z=2$ and $z=3$, say.
We can $z=2$ and $x=2$, say.

Trick Plug $z=0$ into the eq. to find the xy-trace [eq.].
If you get a conic, it must be the only conic type for the "$z=k$" family of traces.

6 Basic Q.S. Types

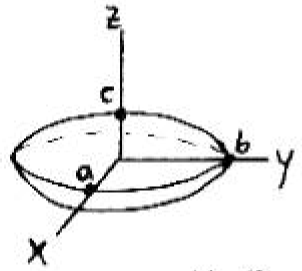
a, b, c: constants, >0

(H1) Ellipsoid

Conic Traces

$x=k$	$y=k$	$z=k$
E	E	E

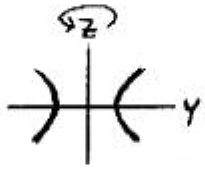
$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$$



No "axis."

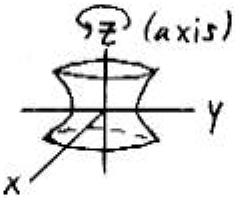
If $a=b=c \Rightarrow$ Sphere

(H2) Hyperboloid of 1 Sheet



$y^2 - z^2 = 1$

y ← Opens along y-axis
Hint: No z-ints.



$(y^2 + x^2) - z^2 = 1$

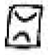
$$\boxed{x^2 + y^2 - z^2 = 1}$$ (Basic Ex)

axis is z-axis

$x=k$	$y=k$	$z=k$ (⊥ axis)
H	H	E

 $|k|<1$

 $|k|=1$

 $|k|>1$

similar

More generally, by Coefficients Trick:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

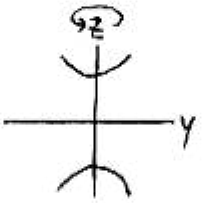
(H3) Hyperboloid of 2 Sheets

Morphing

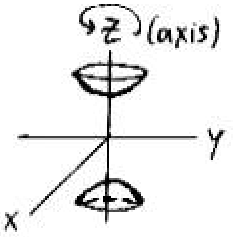
$z^2 = x^2 + y^2 + 1$

$z^2 = x^2 + y^2$

$z^2 = x^2 + y^2 - 1$



$z^2 - y^2 = 1$



$z^2 - (y^2 + x^2) = 1$

$z^2 - x^2 - y^2 = 1$

(2) "-"s $\Rightarrow$ Hyp. of (2) sheets

$$\boxed{z^2 = x^2 + y^2 + 1}$$

z^2 never 0

$\Rightarrow$ z never 0

contrast with

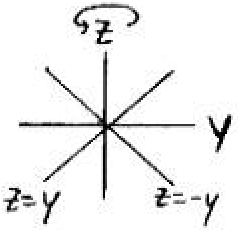
(⊥ axis)		
$x=k$	$y=k$	$z=k$
H	H	E

(no xy-trace)

(H4) [Circular/Elliptic] Cone

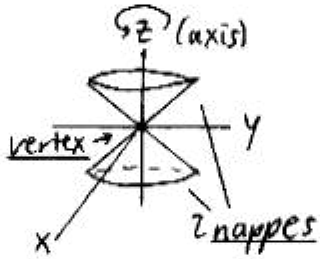
What do I revolve about z?

How can I write as 1 eq.?



$z = \pm y$

$z^2 = y^2$

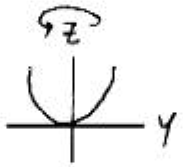


$$\boxed{z^2 = x^2 + y^2}$$

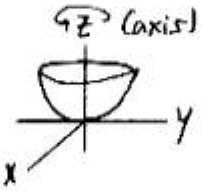
contains (0,0,0)

(⊥ axis)		
$x=k$	$y=k$	$z=k$
H	H	E

(H5) [Circular/Elliptic] Paraboloid



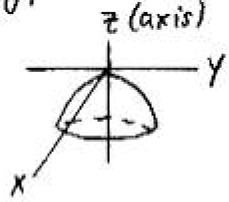
$z = y^2$



$\boxed{z = x^2 + y^2}$

$x=k$	$y=k$	$z=k$ (⊥ axis)
P	P	E

Also:



$\boxed{-z = x^2 + y^2}$

(H6) Hyperbolic Paraboloid

$\boxed{z = y^2 - x^2}$

"saddle"

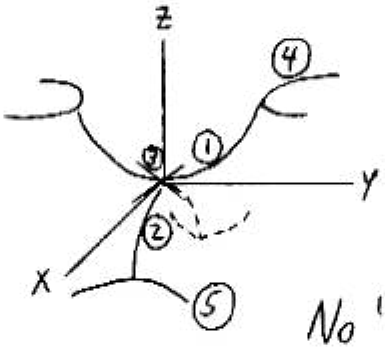
$x=k$	$y=k$	$z=k$
P	P	H

① $x=0 \Rightarrow z=y^2$  ② $y=0 \Rightarrow z=-x^2$  ③ $z=0 \Rightarrow y^2=x^2$ 

④ $z>0 \Rightarrow y^2-x^2=(\# >0)$ 

⑤ $z<0 \Rightarrow y^2-x^2=(\# <0)$

$\Rightarrow x^2-y^2=(\# >0)$ 



No "axis"

Also: $\boxed{\begin{matrix}-z = y^2 - x^2 \Leftrightarrow \\ z = x^2 - y^2\end{matrix}}$

Extensions

<u>Note 1</u> Recall the Variable-Switch and Coefficients Tricks

<u>Note 2</u> If x is replaced by, $(x-x_0)$,
y 〃 , $(y-y_0)$,
z 〃 , $(z-z_0)$

$\Rightarrow$ Translation in which $(0,0,0)$ is moved to (x_0, y_0, z_0)

<u>Note 3</u> Rotations may lead to cross-terms such as xy.

Memorize

step-by-step changes

	Basic Eqs.	$\Rightarrow$ Axis	$x=k$	$y=k$	$z=k$
			(Can figure out)		
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	None	E	E	E
Hyp.-1 Sheet	$x^2+y^2-z^2=1$	z	H	H	E
Hyp.-2 Sheets	$z^2=x^2+y^2+1$	z	〃		〃
Cone	$z^2=x^2+y^2$	z	〃		〃
Paraboloid	$z=x^2+y^2$, $-z=x^2+y^2$	z	P	P	E
Hyp. Paraboloid	$z=y^2-x^2$, $z=x^2-y^2$	None ($z=k \Rightarrow H$)	P	P	H

<u>Ex</u> Identify the surface $x^2 = \frac{y^2}{6} - 3z^2 + 4$. (☆)

<u>Sol'n</u> By Coefficients Trick, consider $x^2 = y^2 - z^2 + 1$
$\Leftrightarrow x^2 - y^2 + z^2 = 1$ (☆☆)

Looks like $x^2+y^2-z^2=1$, except switch $z \leftrightarrow y$.

$\boxed{\text{Hyperboloid of 1 sheet with the } y\text{-axis as its axis.}}$

<u>Note</u> (☆☆) is a surface of revolution, but (☆) is not.