

CH. 15: VECTOR-VALUED FUNCTIONS (VVFs)

15.1.1

15.1: VVFs and SPACE CURVES

Ⓐ Exs

Parametric eqs. for a line segment C.

$$x = \underbrace{1 + 3t}_{f(t)}, \text{ a scalar (or real-valued) function of } t$$

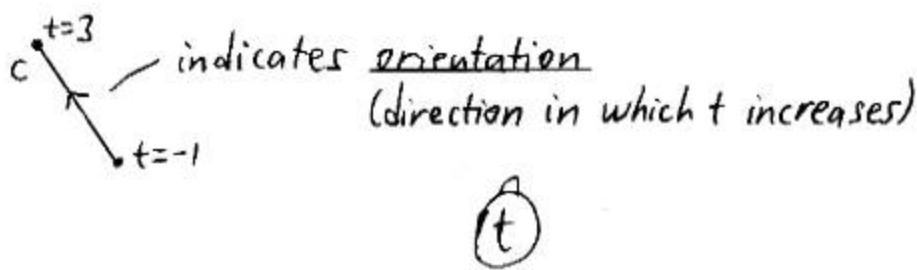
$$y = \underbrace{6 - 4t}_{g(t)}$$

$$z = \underbrace{-1 + 2t}_{h(t)}$$

$$t \text{ in } \underbrace{[-1, 3]}$$

Domain "D"

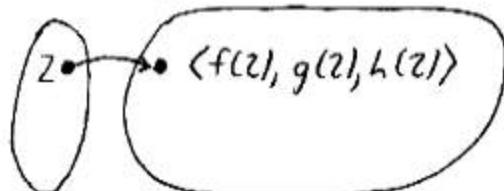
No jumps \Rightarrow We can call this
"I" for Interval.



$$\begin{aligned} \text{Let } \vec{r}(t) &= \langle f(t), g(t), h(t) \rangle \\ &= \langle 1+3t, 6-4t, -1+2t \rangle \\ t \text{ in } [-1, 3] \end{aligned}$$

Then, \vec{r} is a VVF that maps D into V_3 .

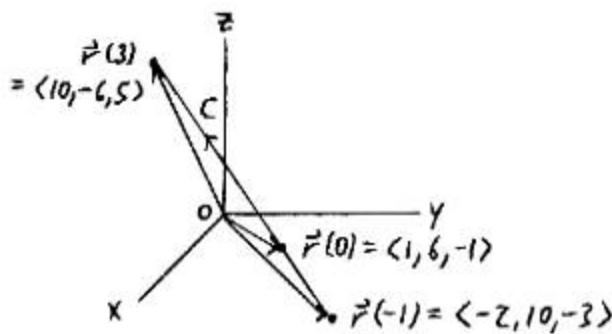
$$\vec{r}: D \rightarrow V_3$$



We'll typically assume:

The domain "D" is an interval ("I").
 f, g, h are continuous on I.

The curve C determined by \vec{r} consists of the endpoints of the position vectors for $\vec{r}(t)$ for all t in I.



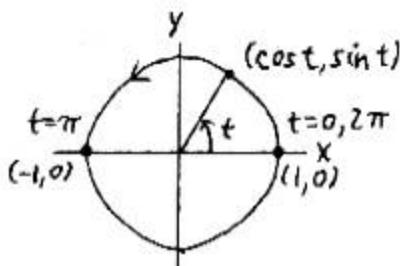
C is a smooth curve (has no breaks, corners, cusps)
 if it has a smooth parameterization on I
 in which f', g', h' are continuous on I , and
 there is no t -value in I at which all are 0,
 except maybe at endpoints.
 i.e., $\vec{r}'(t) = \vec{0}$

Ex $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, t in \mathbb{R} . Draw C .

Book →

$$\text{i.e., } \vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j} + t\hat{k}$$

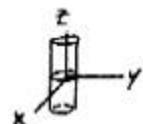
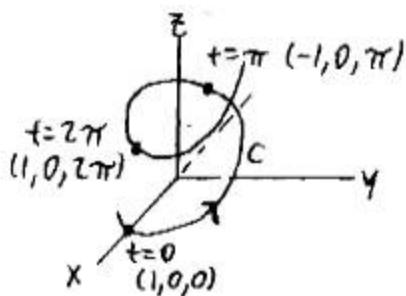
(R²) Consider $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$



Slinky?

(R³) $z = t \Rightarrow$ spring effect along cylinder $x^2 + y^2 = 1$ in \mathbb{R}^3

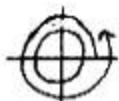
Circular helix



(B) Arc Length

Let C have a smooth parameterization
 $\langle f(t), g(t), h(t) \rangle$, $a \leq t \leq b$.

NO self-overlaps



Then, C has length " L ":

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt \\ &= \int_a^b \underbrace{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}}_{\frac{ds}{dt}}, \text{ where } s \text{ represents arc length.} dt \end{aligned}$$

from Pyth. Thm.

Ex (#24) Find the length of C if

$$\begin{array}{c} \text{Sol'n} \quad \vec{r}(t) = \langle 1 - 2t^2, 4t, 3 + 2t^2 \rangle, \quad 0 \leq t \leq 2. \\ \downarrow \quad \downarrow \quad \downarrow \\ -4t \quad 4 \quad 4t \end{array}$$

$$L = \int_0^2 \sqrt{(-4t)^2 + (4)^2 + (4t)^2} dt$$

$$= \int_0^2 \sqrt{32t^2 + 16} dt$$

$$= \int_0^2 \sqrt{16(2t^2 + 1)} dt$$

$$= \int_0^2 4\sqrt{2t^2 + 1} dt$$

$$= \int_0^2 4\sqrt{1+2t^2} dt$$

Method 1 (Trig Sub)

Want: $u^2 = 2t^2$

Let $u = \sqrt{2}t$

Form $\sqrt{1+u^2} \Rightarrow$ Use $u = \sqrt{2}t = \tan \theta$, etc.

Method 2 (Table of Integrals - p.A21)

Want: $u^2 = 2t^2$

Let $u = \sqrt{2}t$

$\Rightarrow du = \sqrt{2}dt$

$\Rightarrow dt = \frac{1}{\sqrt{2}}du$

$t=0 \Rightarrow u=0$ $t=2 \Rightarrow u=2\sqrt{2}$
--

$$= \int_0^{2\sqrt{2}} 4\sqrt{1+u^2} \cdot \frac{1}{\sqrt{2}} du$$

Note: $\frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$

$$= 2\sqrt{2} \int_0^{2\sqrt{2}} \sqrt{1+u^2} du$$

Formula #21 on p. A21 (I'll give) - derived using Method 1

$$\int \sqrt{a^2 + u^2} du = \frac{u}{2} \sqrt{a^2 + u^2} + \frac{a^2}{2} \ln |u + \sqrt{a^2 + u^2}| + C$$

$a=1$ here

$$= 2\sqrt{2} \left[\frac{u}{2} \sqrt{1+u^2} + \frac{1}{2} \ln |u + \sqrt{1+u^2}| \right]_0^{2\sqrt{2}}$$

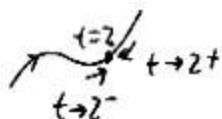
$$\begin{aligned}
 &= 2\sqrt{2} \left(\left[\frac{\frac{2\sqrt{2}}{2} \sqrt{1+(2\sqrt{2})^2} + \frac{1}{2} \ln |2\sqrt{2} + \sqrt{1+(2\sqrt{2})^2}| }{2\sqrt{2}} \right] \right. \\
 &\quad \left. - \left[0 + \frac{1}{2} \ln 1 \right] \right) \\
 &\quad \underbrace{- \left[0 + \frac{1}{2} \ln 1 \right]}_{=0} \\
 &= 2\sqrt{2} \left(\sqrt{2}(3) + \frac{1}{2} \ln (2\sqrt{2} + 3) \right) \\
 &= \boxed{12 + \sqrt{2} \ln (2\sqrt{2} + 3)}
 \end{aligned}$$

15.2: LIMITS, DERIVATIVES, and INTEGRALS

(A) Do them component-by-component, if the pieces exist.

$$\vec{r} \text{ is continuous at } a \iff \lim_{t \rightarrow a} \vec{r}(t) = \vec{r}(a)$$

$\iff f, g, \text{ and } h \text{ are}$
cont. at a



$\left(\begin{array}{l} \vec{r} \text{ is cont. on an open interval } I \\ \iff \vec{r} \text{ is cont. at every } \# \text{ in } I. \\ \text{For non-open intervals, consider the appropriate} \\ \text{one-sided limit(s) at the endpoint(s).} \end{array} \right)$

Ex If $\vec{r}(t) = \langle e^{7t}, \cos(3t), t^2 + t - 1 \rangle$

$$\Rightarrow \vec{r}'(t) = \langle 7e^{7t}, -3\sin(3t), 2t + 1 \rangle,$$

$$\int \vec{r}(t) dt = \langle \frac{1}{7}e^{7t}, \frac{1}{3}\sin(3t), \frac{t^3}{3} + \frac{t^2}{2} - t \rangle + \vec{C}$$

$\langle \vec{c}_1, \vec{c}_2, \vec{c}_3 \rangle$

Ex (Extension of FTC, Fundamental Theorem of Calculus)

$$\text{If } \vec{r}(t) = \langle t^3, e^{2t}, 1 \rangle \Rightarrow \int_2^3 \vec{r}(t) dt = \int_2^3 \underbrace{\langle t^3, e^{2t}, 1 \rangle}_{\substack{\text{cont. on } [2, 3]}} dt$$

$$= \left[\underbrace{\langle \frac{t^4}{4}, \frac{1}{2}e^{2t}, t \rangle}_{\substack{\text{an antiderivative, } \vec{R}(t)}} \right]_2^3$$

$$\begin{aligned}
 & \stackrel{\text{FTC}}{=} \vec{R}(3) - \vec{R}(2) \\
 & = \left\langle \frac{(3)^4}{4}, \frac{1}{2}e^{2(3)}, 3 \right\rangle - \left\langle \frac{(2)^4}{4}, \frac{1}{2}e^{2(2)}, 2 \right\rangle \\
 & = \boxed{\left\langle \frac{81}{4}, \frac{1}{2}(e^6 - e^4), 1 \right\rangle}
 \end{aligned}$$

Ex (Differential Equations)

Find $\vec{r}(t)$ if $\vec{r}'(t) = \langle 1, t, e^{\frac{t}{2}} \rangle$ subject to
the initial condition (IC) $\vec{r}(0) = \langle 1, 3, 5 \rangle$.

$\langle 1, 3, 5 \rangle$ "starting point"

Sol'n

$$\begin{aligned}\vec{r}(t) &= \int \langle 1, t, e^{\frac{t}{2}} \rangle dt \quad (\text{one member}) \\ &= \left\langle t, \frac{t^2}{2}, 2e^{\frac{t}{2}} \right\rangle + \vec{C}\end{aligned}$$

Use the IC to find \vec{C} .

$$\begin{aligned}t=0 \Rightarrow \vec{r}(0) &= \langle 0, 0, 2 \rangle + \vec{C} \\ \langle 1, 3, 5 \rangle &= \langle 0, 0, 2 \rangle + \vec{C} \\ \vec{C} &= \langle 1, 3, 3 \rangle\end{aligned}$$

$$\vec{r}(t) = \left\langle t, \frac{t^2}{2}, 2e^{\frac{t}{2}} \right\rangle + \langle 1, 3, 3 \rangle$$

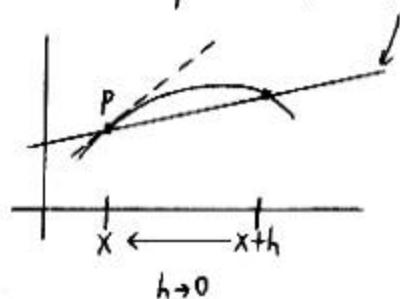
$$\boxed{\vec{r}(t) = \left\langle t+1, \frac{t^2}{2}+3, 2e^{\frac{t}{2}}+3 \right\rangle}$$

(B) Geometry of $\vec{r}'(t)$

Calc I

$$y = f(x)$$

$$f'(x) = \lim_{h \rightarrow 0} \underbrace{\frac{f(x+h) - f(x)}{h}}_{= \text{slope of secant line}}$$



= slope of tangent line (f') at P
(if it exists)

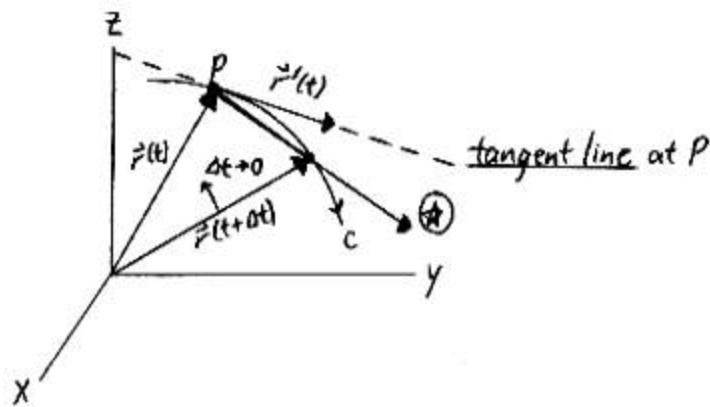
Calc III

$z = h(t)$
Don't use " $h \rightarrow 0$ "

$$\vec{r}'(t) = \lim_{\Delta t \rightarrow 0} \underbrace{\frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}}_{\textcircled{*}} \leftarrow \text{This lengthens } \text{ if } 0 < \Delta t < 1$$

= a tangent vector at P
direction consistent with
orientation of C at P

(if it exists)



Ex $\vec{r}(t) = \langle 4t, 3, t^2 \rangle$.

Find parametric eqs. for the tangent line to C at $P(8, 3, 4)$.

Sol'n

What's t at P ?

$$4t = 8$$

$$t = 2$$

Check: $\vec{r}(2) = \langle 8, 3, 4 \rangle \checkmark$

Find a tangent vector at P .

$$\vec{r}'(t) = \langle 4, 0, 2t \rangle$$

$$\vec{r}'(2) = \langle 4, 0, 4 \rangle$$

Answer

$$\boxed{\begin{cases} x = 8 + 4t \\ y = 3 \\ z = 4 + 4t \end{cases}, t \text{ in } \mathbb{R}}$$

The curve
could self-
intersect

③ Derivative Rules for VVF_s

When \vec{u}, \vec{v}, f are differentiable (diff'c)...

D_t is a linear operator

$$(i) D_t [\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t) \quad \left| \begin{array}{l} \text{Can } D_t \text{ term-by-term} \end{array} \right.$$

$$(ii) D_t [c\vec{u}(t)] = c\vec{u}'(t) \quad \left| \begin{array}{l} \text{Constant scalar factors} \\ \text{pop out.} \end{array} \right.$$

Product Rule analogs

$$(iii) D_t [\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$$

$$(iv) \quad \times \quad \quad \quad \times \quad \quad \quad \left| \begin{array}{l} \leftarrow \text{if } \vec{u}, \vec{v} \text{ in } V_3 \end{array} \right.$$

$$(v) D_t [f(t) \vec{u}(t)] = \underbrace{f'(t)}_{\text{scalar func.}} \vec{u}(t) + f(t) \vec{u}'(t)$$

in
Exercises

Chain Rule

$$(vi) D_t \vec{u}(f(t)) = \underbrace{[\vec{u}'(f(t))] [f'(t)]}_{\text{Chain Rule}} \quad \left| \begin{array}{l} D_t \vec{u}(A) = [\vec{u}'(A)] [D_t(A)] \end{array} \right.$$

① "Sphere Theorem"

If \vec{r} is diff'e and $\|\vec{r}(t)\|$ is constant on I,
then $\vec{r}'(t) \perp \vec{r}(t)$ for every t in I.

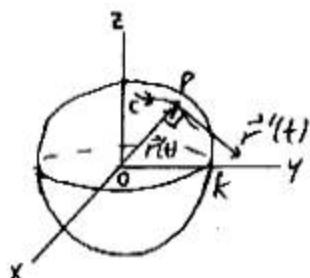
Note $\|\vec{r}(t)\| = k$ ($k \geq 0$)

$$\|(x, y, z)\| = k$$

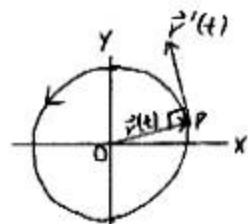
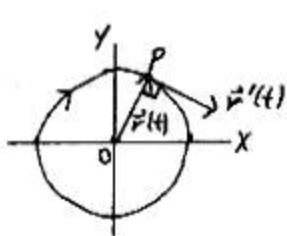
$$\sqrt{x^2 + y^2 + z^2} = k$$

$$x^2 + y^2 + z^2 = k^2$$

i.e., If C lies entirely on a sphere centered at O,
then for any point P on C, $\overline{OP} \perp$ tangent vector at P.



(R²) Circles



15.3: MOTION

When \vec{r} is twice diff'e,

$\vec{r}(t)$ = the position vector of a point P at t

Imagine P moving along C as t \nearrow .

$$\vec{r}'(t) = \vec{v}(t)$$

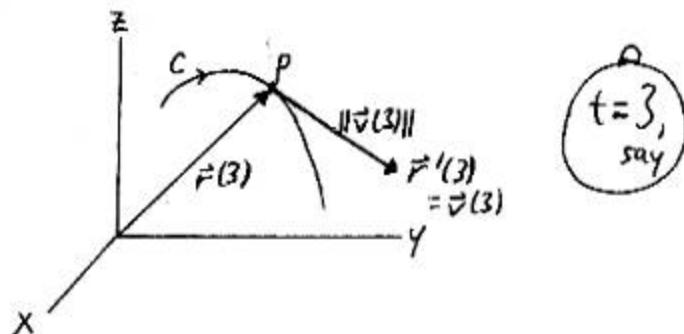
= the velocity [vector] of P at t

A vector is
characterized
by...

It's a tangent vector that indicates
direction of motion.

You can't
just look at
C and
determine
the length
of $\vec{v}(t)$.

Its length depends on how C is
parameterized. What is the significance
of its length?



$$\|\vec{r}'(t)\| = \|\vec{v}(t)\|$$

= the speed of P at t

Why?

$$\|\vec{v}(t)\| = \|\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle\|$$

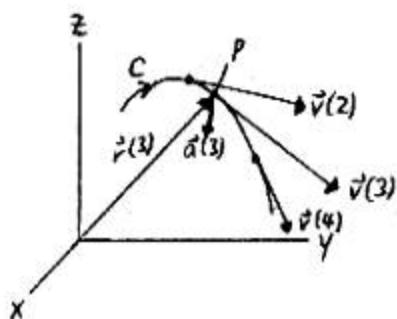
$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

= $\frac{ds}{dt}$, the rate of change of
arc length with respect to time } SPEED!!

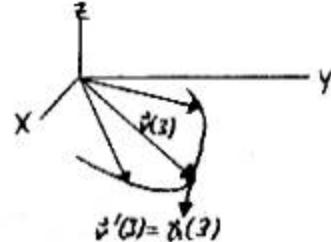
$$\vec{r}''(t) = \vec{v}'(t) = \vec{\alpha}(t)$$

= the acceleration [vector] of P at t

Graph of \vec{r}



Graph of \vec{v}



From P, $\vec{\alpha}(t)$ typically
points into the
concave side of C.

twisted cubic
 $\langle at, bt^2, ct^3 \rangle$
a, b, c all
non-0

Ex Let $\vec{r}(t) = \langle t, t^2, t^3 \rangle \Rightarrow \vec{r}(3) = \langle 3, 9, 27 \rangle$
 $\vec{v}(t) = \langle 1, 2t, 3t^2 \rangle \Rightarrow \vec{v}(3) = \langle 1, 6, 27 \rangle *$
 $\vec{\alpha}(t) = \langle 0, 2, 6t \rangle \Rightarrow \vec{\alpha}(3) = \langle 0, 2, 18 \rangle$

$$* \|\vec{v}(3)\| = \sqrt{(1)^2 + (6)^2 + (27)^2}$$

$$= \sqrt{766}$$

$$\approx 27.7$$

Note C lies on the intersection of the cylinders $y=x^2$, $z=x^3$. (p. 750)

Ex $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, t in \mathbb{R} .
 C is the helix from 15.1.3.

$$\begin{aligned} \text{Speed} &= \|\vec{v}(t)\| \\ &= \|\langle -\sin t, \cos t, 1 \rangle\| \\ &= \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2} \\ &= \sqrt{2} \quad (\text{constant}) \end{aligned}$$

$\vec{v}(t)$ always
has length $\sqrt{2}$

The particle travels $\sqrt{2}$ units
for every unit increase in t .

Ex $\vec{r}(t) = \langle \cos(t^3), \sin(t^3), t^3 \rangle$, t in \mathbb{R} .
 Same C , but...

$$\begin{aligned} \text{Speed} &= \|\vec{v}(t)\| \\ &= \|\langle -3t^2 \sin(t^3), 3t^2 \cos(t^3), 3t^2 \rangle\| \\ &= \|3t^2 \langle -\sin(t^3), \cos(t^3), 1 \rangle\| \\ &= \underbrace{|3t^2|}_{\geq 0} \sqrt{\underbrace{[-\sin(t^3)]^2 + [\cos(t^3)]^2 + 1}_{=1}} \\ &= 3t^2 \sqrt{2} \\ &= 3\sqrt{2}t^2 \end{aligned}$$

We lose smoothness at $t=0$, since $\vec{r}(0)=\vec{0}$

$\vec{v}(t)$ lengthens shrinks

\uparrow Speeds up ($t \rightarrow \infty$)
 \uparrow Slows down ($t \rightarrow 0^-$)

Ex Given $\vec{r}(0)$, $\vec{v}(t)$ \Rightarrow Find $\vec{r}(t)$ (See 15.2.2)
 ("starting") ("describes")

15.4: CURVATURE(A) TNB Frame

When \vec{r} is diff'e, and $\vec{r}' \neq \vec{0}$,

Unit tangent vector to C

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

When \vec{T} is diff'e, and $\vec{T}' \neq \vec{0}$,

[Principal] unit normal vector to C

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

$$\vec{N}(t) \perp \vec{T}(t)$$

Proof \vec{T} is diff'e, and
 $\|\vec{T}(t)\| = 1$ for all relevant t

$$\Rightarrow \vec{T}'(t) \perp \vec{T}(t)$$

by the Sphere Thm. (15.2.6)

$$\Rightarrow \vec{N}(t) \perp \vec{T}(t)$$

because $\vec{N}(t)$ points in the same direction as $\vec{T}'(t)$

If we're in $V_3(\mathbb{R}^3)$,

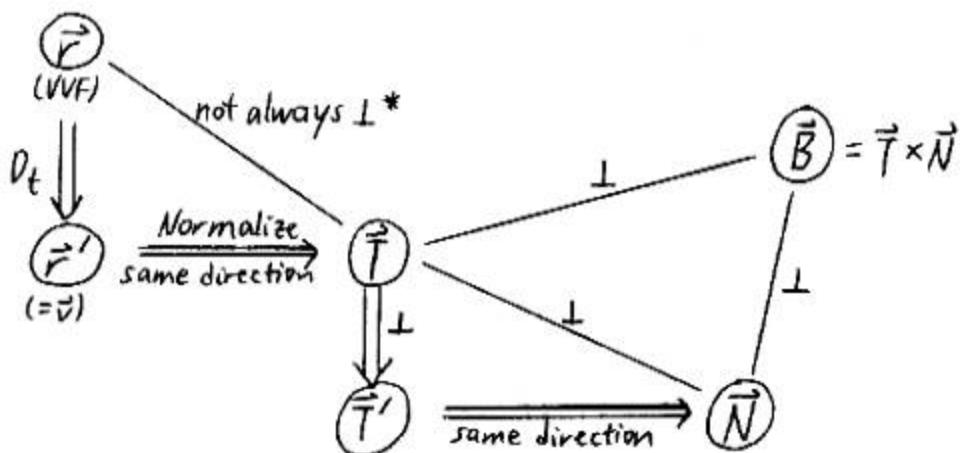
In Stewart 867
but not
Larson or
Swok.

Binormal vector to C

$$\boxed{\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)}$$

$\vec{B}(t)$ is a unit vector. (Proof: HW!)

Schematic



* If Sphere Thm. applies $\Rightarrow \vec{r}'(t) \perp \vec{T}(t)$ for all relevant t .

Note Although $\vec{r}'' = \vec{a}$, don't expect $\vec{a} \parallel \vec{T}'$.

$\vec{r}' \parallel \vec{T}$, but often $\vec{r}'' \nparallel \vec{T}'$.

$$D_t[\vec{r}'(t)] \quad D_t\left[\frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}\right]$$

This is because magnitude $\|\vec{r}'(t)\|$ is often a nonconstant function of t (see Ex).

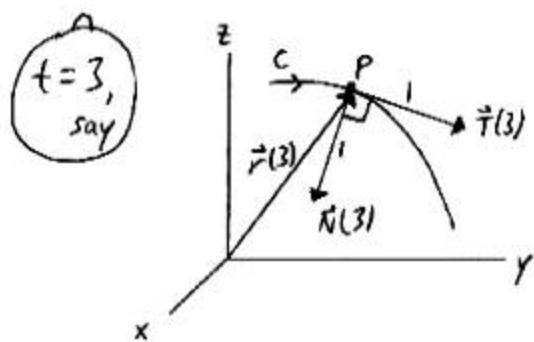
INB Frame

always unit vectors,
mutually \perp

(Important in differential geometry,
spacecraft motion)

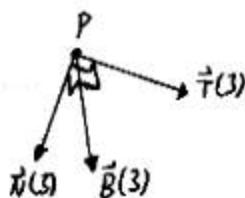
Imagine this rotating as it follows P along C .

Face board
Right
3 thumb
2
Spock?

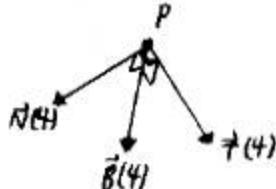


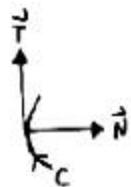
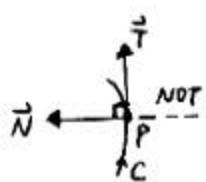
\vec{N} points inside the concave side of C .

\vec{N} indicates the direction in which C is turning.



$t=4$,
say

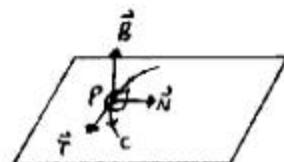


More on \vec{N} (R²)(R³) HelixIn (R³),

P, \vec{T}, \vec{N} determine the osculating plane of C at P .
 Latin: osculum ("to kiss")

It's the plane that comes closest to containing the part of C near P .

$\vec{B} \perp$ this plane.



Note P, \vec{N}, \vec{B} determine the normal plane of C at P .
 $\vec{T} \perp$ this plane.

Ex (#2) If $\vec{r}(t) = \langle -t^2, 2t \rangle$ in $\mathbb{R}^2 (V_2)$,
find $\vec{T}(t)$ and $\vec{N}(t)$.

Sol'n

$$\begin{aligned}\vec{r}'(t) &= \langle -2t, 2 \rangle \\ &= 2 \langle -t, 1 \rangle\end{aligned}$$

$$\begin{aligned}\|\vec{r}'(t)\| &= |2| \sqrt{(-t)^2 + 1^2} \\ &= 2 \sqrt{t^2 + 1}\end{aligned}$$

$$\begin{aligned}\vec{T}(t) &= \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \\ &= \frac{2 \langle -t, 1 \rangle}{2 \sqrt{t^2 + 1}} \\ &= \boxed{\left\langle -\frac{t}{\sqrt{t^2 + 1}}, \frac{1}{\sqrt{t^2 + 1}} \right\rangle}\end{aligned}$$

Alternative:
Use 15.2.5(c)
Product Rule
analog on
 $\underbrace{\frac{1}{\sqrt{t^2+1}}}_{f(t)}$ $\underbrace{\langle -t, 1 \rangle}_{g(t)}$

$$\begin{aligned}\vec{T}'(t) &= \left\langle -\frac{(\sqrt{t^2+1})(1) - t[\cancel{\frac{1}{2}}(t^2+1)^{-\frac{1}{2}}(2t)]}{t^2+1}, \right. \\ &\quad \left. - \cancel{\frac{1}{2}}(t^2+1)^{-\frac{1}{2}}(2t) \right\rangle\end{aligned}$$

by Quotient / Reciprocal Rules

$$= \left\langle -\underbrace{\frac{\sqrt{t^2+1} - \frac{t^2}{\sqrt{t^2+1}}}{t^2+1}}, -\frac{t}{(t^2+1)(t^2+1)^{1/2}} \right\rangle$$

$\cdot \frac{\sqrt{t^2+1}}{\sqrt{t^2+1}}$

$$= \left\langle -\frac{t^2+1 - t^2}{(t^2+1)^{3/2}}, -\frac{t}{(t^2+1)^{3/2}} \right\rangle$$

$$= \left\langle -\frac{1}{(t^2+1)^{3/2}}, -\frac{t}{(t^2+1)^{3/2}} \right\rangle$$

$$\|\vec{T}'(t)\| = \sqrt{\left(-\frac{1}{(t^2+1)^{3/2}}\right)^2 + \left(-\frac{t}{(t^2+1)^{3/2}}\right)^2}$$

$$= \sqrt{\frac{1}{(t^2+1)^3} + \frac{t^2}{(t^2+1)^3}}$$

$$= \sqrt{\frac{t^2+1}{(t^2+1)^3}}$$

$$= \sqrt{\frac{1}{(t^2+1)^2}}$$

$$= \frac{1}{\underbrace{|t^2+1|}_{>0 \text{ always}}}$$

$$= \frac{1}{t^2+1}$$

$$\begin{aligned}
 \vec{N}(t) &= \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} \\
 &= \frac{\left\langle -\frac{1}{(t^2+1)^{3/2}}, -\frac{t}{(t^2+1)^{3/2}} \right\rangle}{\frac{1}{t^2+1}} \\
 &= (t^2+1) \left\langle -\frac{1}{(t^2+1)^{3/2}}, -\frac{t}{(t^2+1)^{3/2}} \right\rangle \\
 &= \boxed{\left\langle -\frac{1}{\sqrt{t^2+1}}, -\frac{t}{\sqrt{t^2+1}} \right\rangle}
 \end{aligned}$$

Up to Sa

Note Easier way to find $\vec{N}(3)$, say:

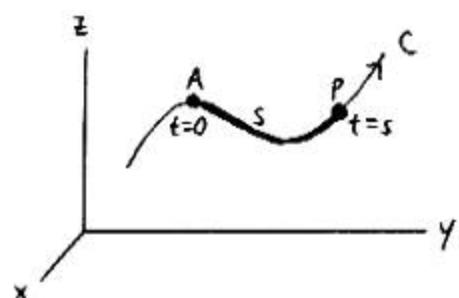
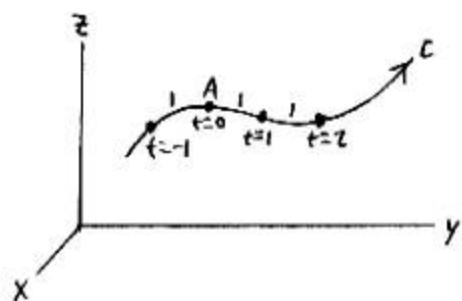
$$\begin{aligned}
 &\text{Find } \vec{T}'(3) \\
 \Rightarrow &\text{Find } \|\vec{T}'(3)\| \\
 \Rightarrow &\text{Find } \vec{N}(3) = \frac{\vec{T}'(3)}{\|\vec{T}'(3)\|}
 \end{aligned}$$

(B) Arc Length Parametrizations (ALPs) of C

Not written
on board

Let A be a fixed point on C corresponding to $t=0$.

If, for any point P on C , the t -value at P equals the signed length of arc \widehat{AP}
(i.e., the signed distance of P from A along C),
then we have an ALP of C .



If we have an ALP, we usually use " s " instead of " t " as our parameter. s is an arc length parameter that increases in the direction of orientation.

$$\text{ALP: } \vec{r}(s) = \begin{matrix} f(s) \\ g(s) \\ h(s) \end{matrix}$$

Con: It's often hard to find ALPs (see p. 779).

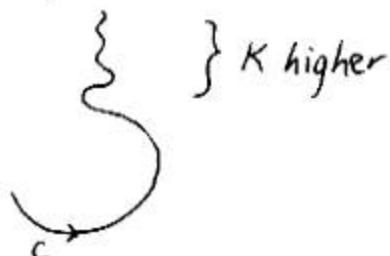
Pro: Speed = 1 always along C .

⑥ Idea of Curvature (K)

$\backslash \text{Kappa}$

(See How to Ace the Rest of Calculus.)

Winding road



K measures "nausea" if speed is constant along C .
 K should depend on shape, not speed.

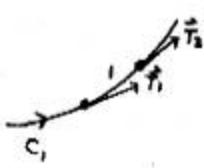
What kind
of param'n
gives us
constant
speed?

Let's use a smooth ALP, $\vec{r}(s)$, such that $\vec{r}''(s)$ exists.

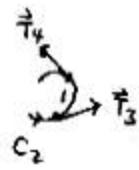
$$\text{Then, } \underbrace{\vec{T}(s)}_{\substack{\text{unit} \\ \text{tangent} \\ \text{vector}}} = \frac{\vec{r}'(s)}{\|\vec{r}'(s)\|} \Leftarrow \text{speed} = 1$$

$$= \vec{r}'(s) \quad \textcircled{*}$$

Its rate of change with respect to s ,
 $\vec{T}'(s)$ or $\vec{r}''(s)$, only reflects its rate
of change of direction, because its
length is constant.



Assuming 2D,
same scales
for graphs
(\vec{T}_i 's are unit)



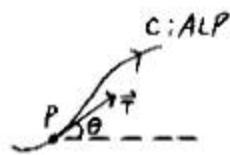
K higher
 \vec{T} changes more dramatically

Let $K(s) = \|\vec{r}''(s)\|$
 $\nwarrow \nearrow$
 so we get
 a scalar

$$\stackrel{\textcircled{4}}{=} \|\vec{T}'(s)\|$$

$$= \left\| \frac{d\vec{T}}{ds} \right\|$$

In \mathbb{R}^2 , $K(s) = \left| \frac{d\theta}{ds} \right|$
 is consistent with the above.



D Radius of Curvature, ρ (for plane curves in \mathbb{R}^2)

$$\rho = \frac{1}{K}$$

Why? Ex 7 on pp. 775-6

 K at every point on C
 $= \frac{1}{\rho}$

If $K \neq 0$ at $P \Rightarrow$ The circle of curvature

(osculating circle) at P has the same tangent line and K as C does at P . It's the circle that best models C near P .





o K higher \Rightarrow
 ρ lower

(ρ) K lower \Rightarrow
 ρ higher

— line: $K=0$
 $(\vec{T}$ constant $\Rightarrow \frac{d\vec{T}}{ds} = \vec{0})$

E) Practical Formulas for K

What if $\vec{r}(t)$ is not an ALP for C ? (When \vec{r} smooth, \vec{r}'' exists...)

$$K(t) = \left\| \frac{d\vec{T}}{ds} \right\|$$

$$= \left\| \frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} \right\| \quad \text{by Chain Rule for VVF}$$

$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \frac{ds}{dt}$

$$= \frac{\left\| \frac{d\vec{T}}{dt} \right\|}{\left\| \frac{ds}{dt} \right\|} \quad \leftarrow \text{speed} = \left\| \vec{v}(t) \right\| = \left\| \vec{r}'(t) \right\|$$

$$K(t) = \frac{\left\| \vec{T}'(t) \right\|}{\left\| \vec{r}'(t) \right\|}$$

★

Rate of change
of arc length
wrt time.
Sound familiar?

Not in Swok.~

Stewart
865Even better (after a tricky proof) if we're in \mathbb{R}^3

$$\Rightarrow K(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} \quad (15.5 \text{ in book})$$

$$\text{Think: } K = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|^3}$$

though we
can modify
for \mathbb{R}^2 ,
see Ex' below.

though we
don't normally
like thinking
about \vec{v}, \vec{a}
in the context
of K .

Even better for a plane curve: $y = g(x)$
some function

$$\Rightarrow K(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} \quad (\text{**** give.})$$

Proof Use Thm. 15.19 on p. 775 (see HW 15.5, #16),

or ~~****~~ with $\vec{r}(t) = \langle \begin{matrix} t \\ x \\ y \end{matrix} \rangle$.

We lose
smoothness
at $t=0$.

Ex In \mathbb{R}^2 , $C: x = t^3, y = t^2, t > 0$.
Find a K formula for C , and find K and ρ at $(8, 4)$.

Method 1 Thm. 15.19 on p. 775

Method 2 Use ~~****~~.

Method 3 Use ~~****~~.

Let $\vec{r}(t) = \langle t^3, t^2, 0 \rangle$ in \mathbb{R}^3

$$\Rightarrow \vec{v}(t) = \langle 3t^2, 2t, 0 \rangle$$

$$\Rightarrow \vec{a}(t) = \langle 6t, 2, 0 \rangle$$

$$\vec{v}(t) \times \vec{a}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3t^2 & 2t & 0 \\ 6t & 2 & 0 \end{vmatrix}$$

$$= \langle 0, 0, -6t^2 \rangle$$
$$\|\vec{v}(t) \times \vec{a}(t)\| = 6t^2$$

$$\|\vec{v}(t)\|^3 = (\sqrt{(3t^2)^2 + (2t)^2 + (0)^2})^3$$

$$= (9t^4 + 4t^2)^{3/2}$$

$$K(t) = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|^3}$$

$$= \boxed{\frac{6t^2}{(9t^4 + 4t^2)^{3/2}}}$$

Find K, ρ at $(8, 4)$

$t = 2$ at $(8, 4)$.

$$K(2) = \frac{6(2)^2}{[9(2)^4 + 4(2)^2]^{3/2}} = \boxed{\frac{24}{(160)^{3/2}} \approx 0.0119}$$

$$\rho(2) = \frac{1}{K(2)} = \boxed{\frac{(160)^{3/2}}{24} \approx 84.3}$$

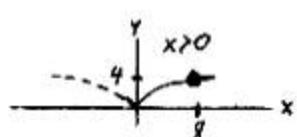
Method 4 Use ~~AAA~~.

Eliminate the parameter, t.

$$x = t^3 \implies t = \sqrt[3]{x} \quad | \quad t > 0 \iff x > 0$$

$$y = t^2 \implies y = (\sqrt[3]{x})^2$$

$$(y = x^{2/3}, x > 0)$$



Find $K(x)$.

$$K(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$

$$y = x^{2/3} \implies y' = \frac{2}{3}x^{-1/3} \implies y'' = -\frac{2}{9}x^{-4/3}$$

$$= -\frac{2}{9x^{4/3}}$$

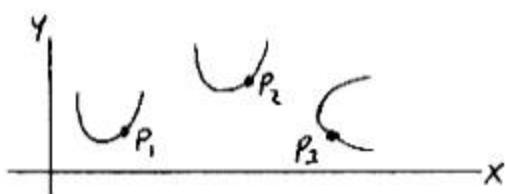
$$= \frac{1}{\left[1 + \left(\frac{2}{3x^{1/3}} \right)^2 \right]^{3/2}}$$

$$= \frac{2}{q_X^{4/3} \left[1 + \frac{4}{q_X^{4/3}} \right]^{3/2}}$$

Find K, p at (8, 4).

$$K(8) = \frac{1}{\rho(8)} \approx 0.0119 \quad \text{as for Method 3}$$

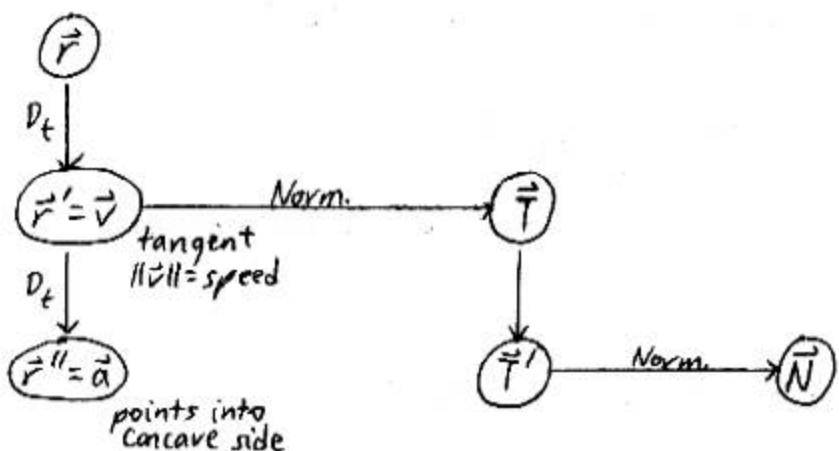
(F) K is Invariant under Translations and Rotations



K is the same at P_1 , P_2 , and P_3 .

Our K formulas didn't depend directly on \vec{r} .

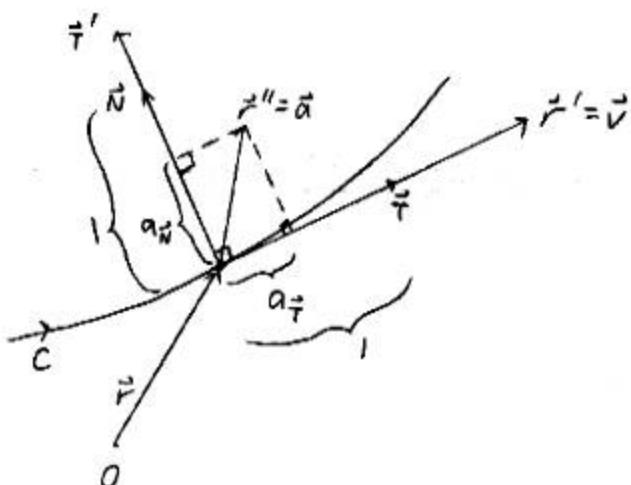
15.5 : UNDERSTANDING \vec{a} (Bonus)



\vec{T}', \vec{N} also do
✓ indicate
direction of
turn

Larson 6ed
p. 797: \vec{a} lies
in plane of
 \vec{T}, \vec{N}

$$\vec{a}(t) = a_T \vec{T}(t) + a_N \vec{N}(t)$$



$\|\vec{v}\| = \text{speed} = \frac{ds}{dt} = \text{rate of change of arc length wrt } t$

What gives us rate of change of speed, $\frac{d^2s}{dt^2}$?
(What plays the role that "acceleration" played in Calc I?)

$$\|\ddot{\vec{a}}\| = K = \|\vec{r}''(s)\|$$

if ALP

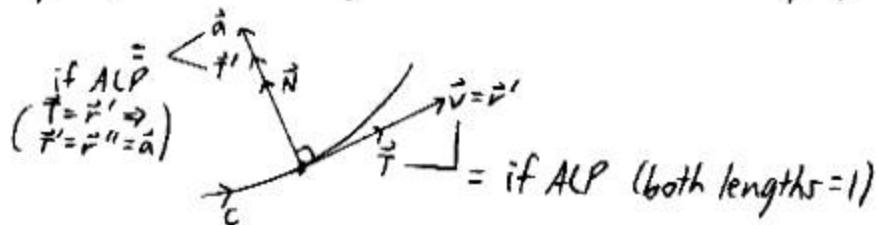
Not $\|\ddot{\vec{a}}\|$

Is $\text{comp}_{\vec{T}} \vec{a} = a_T = \text{tangential component of acc.}$

$$= \frac{\vec{a} \cdot \vec{T}}{\|\vec{T}\|} \leftarrow = 1$$

$$= \vec{a} \cdot \vec{T}$$

depends
on speed,
not K

See 15.4.2
NoteIf speed is constant, $\|\vec{v}\| = c \Rightarrow \vec{v} \perp \vec{a}$ by Sphere Thm.

$$\vec{a} \perp \vec{r}$$

$$a_r = \vec{a} \cdot \vec{r}$$

$$a_r = 0 \text{ (reflects constant speed)}$$

What is a_N ? $a_N = \text{normal (centripetal) component of acc.}$

$$= \vec{a} \cdot \vec{N}$$

$$\stackrel{\text{turns out}}{=} K \left(\frac{ds}{dt} \right)^2$$

↑ speed

If speed, K ,
driver are
high.

a_N is the "barf factor," even if no ALP.
High if crazy driver (goes fast when C curvy).

Note 1: $a_N \geq 0$, because ① \vec{a}, \vec{N} both point into concave side
② $K \geq 0, \left(\frac{ds}{dt} \right)^2 \geq 0$

K is barf factor if ALP

Note 2: If ALP, $\|\vec{a}\| = a_N = K$ (We got some
 \vec{a}, \vec{N} point in
same direction (see
figure))

Ch. 1616.1: FUNCTIONS OF SEVERAL VARIABLES(A) Intro

How to Ace

Ch. 15

Domain $D \subseteq \mathbb{R}$
is a \uparrow subset of

VVF $f: D \rightarrow V_n$ (or \mathbb{R}^n)

$$\vec{r}(1) = \langle 10, 20 \rangle$$

vs. Now

 $D \subseteq \mathbb{R}^n$ $f: D \rightarrow \mathbb{R}$ 

$$f(1, 2) = 30$$

Calc I

vs. Now

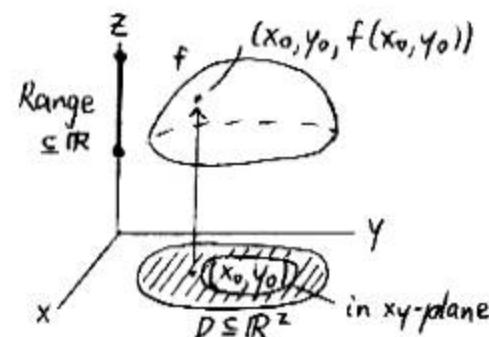
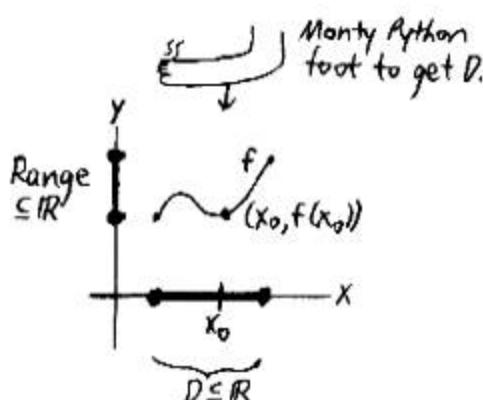
$$y = f(x)$$

$$z = f(x, y)$$

dependent variable independent variables

$$x \Rightarrow [f] \Rightarrow y$$

$$x \Rightarrow [f] \Rightarrow z$$



It is usually assumed that
 $D = \{(x, y) \mid f(x, y) \text{ is defined (as a real #)}\}$

How would you represent $w = f(x, y, z)$?

Ex Let $f(x, y, z) = \frac{\sqrt{4-z}}{x+3} + \ln(x+y) - \tan z$.

a) Find D.

$$\begin{array}{l} \underbrace{4-z \geq 0}_{z \leq 4}, \\ \underbrace{x+y > 0}_{x \neq -3} \\ \underbrace{\tan z \text{ undefined}}_{z \neq \frac{\pi}{2} + \pi n \text{ (n integer)}} \end{array}$$

$$D = \{(x, y, z) \mid x \neq -3, x+y > 0, z \leq 4, \\ z \neq \frac{\pi}{2} + \pi n \text{ (n integer)} \}$$

b) Find $f(1, 2, 3)$

$$= \frac{\sqrt{4-3}}{1+3} + \ln(1+2) - \tan 3$$

$$= \boxed{\frac{1}{4} + \ln 3 - \tan 3}$$

B) Level Curves (LCs)

Idea

Graph $z = f(x, y)$ in \mathbb{R}^3 .

Take traces in planes $z=k$.

(The k -values are preferably evenly spread out.)

(Think: Different colors on different transparencies.)

Project them onto the xy -plane, and indicate their k -values.

(Think: Overlaying the transparencies. 

In Practice

The LC of f for $z=k$ is the graph of

$k = f(x, y)$ in \mathbb{R}^2 .

(Replace z with k .)

$$\text{Ex } f(x, y) = \sqrt{9 - x^2 - y^2}$$

Find D

$$\begin{aligned} 9 - x^2 - y^2 &\geq 0 \\ 9 &\geq x^2 + y^2 \\ x^2 + y^2 &\leq 9 \end{aligned}$$

$$D = \{(x, y) \mid x^2 + y^2 \leq 9\}$$



Idea of LCs

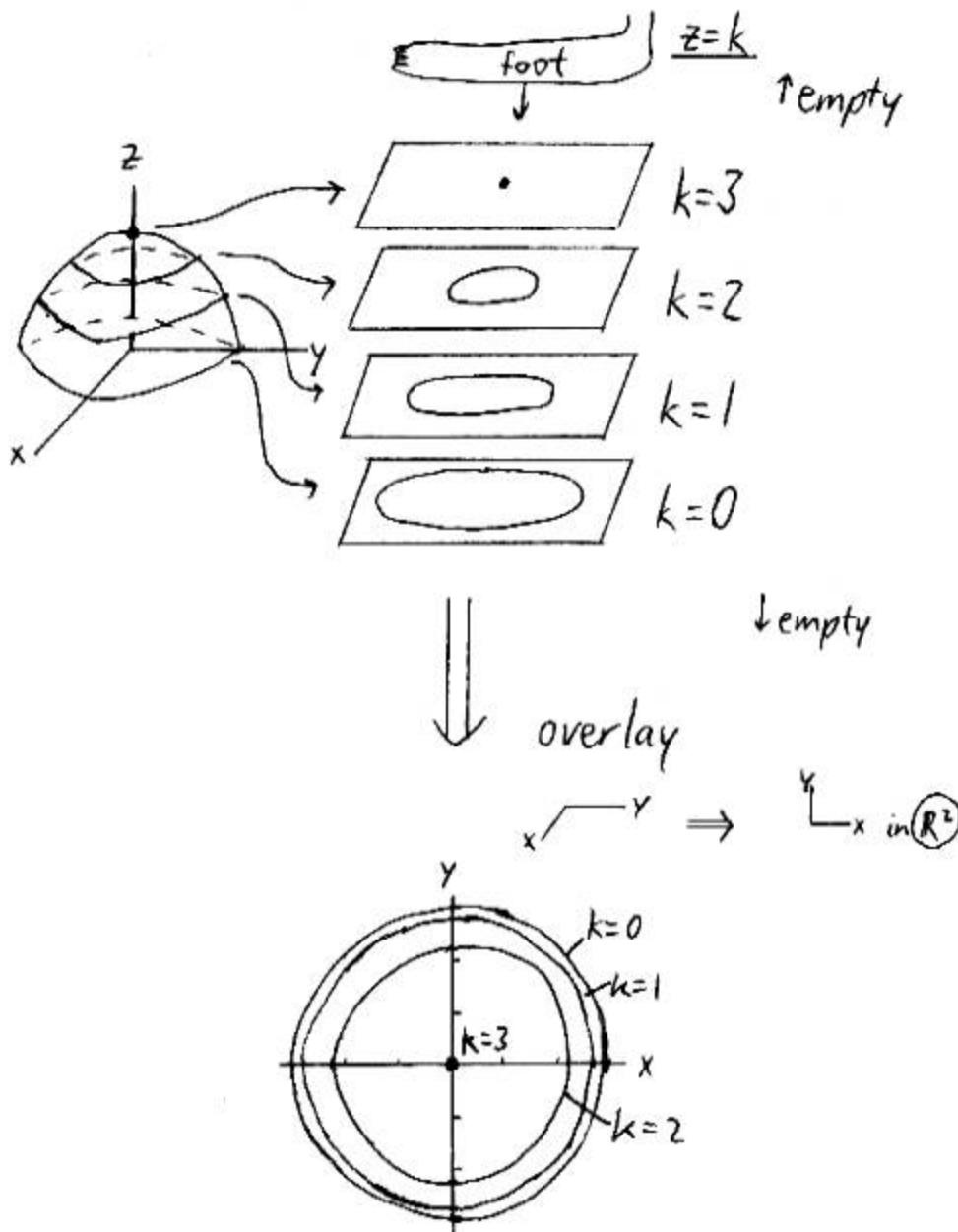
$$\text{Graph } z = \sqrt{9 - x^2 - y^2}$$

$$z^2 = 9 - x^2 - y^2, z \geq 0$$

$$x^2 + y^2 + z^2 = 9, z \geq 0$$

Upper half of the sphere of radius 3
centered at $(0, 0, 0)$.

Sideshow
Bob?



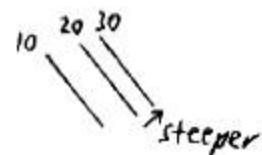
Radius:

$$\begin{aligned} 3 \\ \sqrt{8} \approx 2.8 \\ \sqrt{5} \approx 2.2 \\ 0 \end{aligned}$$

You can try to reconstruct the surface from its LCs.
Lift/drop a LC by $|k|$ units.

If the k -values are evenly spaced out, usually,
closer LCs \Rightarrow faster \nearrow in f -value.

"steeper climb"



In Practice

LC for $z = k$:

$$\begin{aligned} k &= \sqrt{9-x^2-y^2} \\ k^2 &= 9-x^2-y^2, \quad k \geq 0 \\ x^2+y^2 &= 9-k^2, \quad k \geq 0 \end{aligned}$$

If $k < 0 \Rightarrow$ empty

If $0 \leq k < 3 \Rightarrow$ circle of radius $\sqrt{9-k^2}$
centered at $(0,0)$

If $k = 3 \Rightarrow (0,0)$ only
 $x^2+y^2=0$

If $k > 3 \Rightarrow$ empty
 $9-k^2 < 0$

C) Applications

Used in topographic/contour maps; Mountain climbing
(bird's-eye view), Lake depths

Weather map - LCs are isotherms.
same temperature

$\sim 70^\circ F$

$\sim 60^\circ F$

(D) Level Surfaces (LSs)

Let $w = f(x, y, z)$ be a temperature function, say.

The LS for $w=k$ is often a 3D surface in \mathbb{R}^3 consisting of points with the same temperature

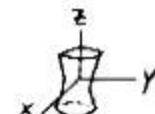
Ex 6, p.799
axis: y-axis

Ex (#40) $f(x, y, z) = \underbrace{x^2 + y^2 - z^2}_w$

LS for $w=k$

$$k=1: 1 = x^2 + y^2 - z^2$$

Hyp. of 1 sheet



$$k=0: 0 = x^2 + y^2 - z^2$$

$$z^2 = x^2 + y^2$$

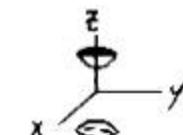
Cone



$$k=-1: -1 = x^2 + y^2 - z^2$$

$$z^2 = x^2 + y^2 + 1$$

Hyp. of 2 sheets



All with axes
along the z-axis.

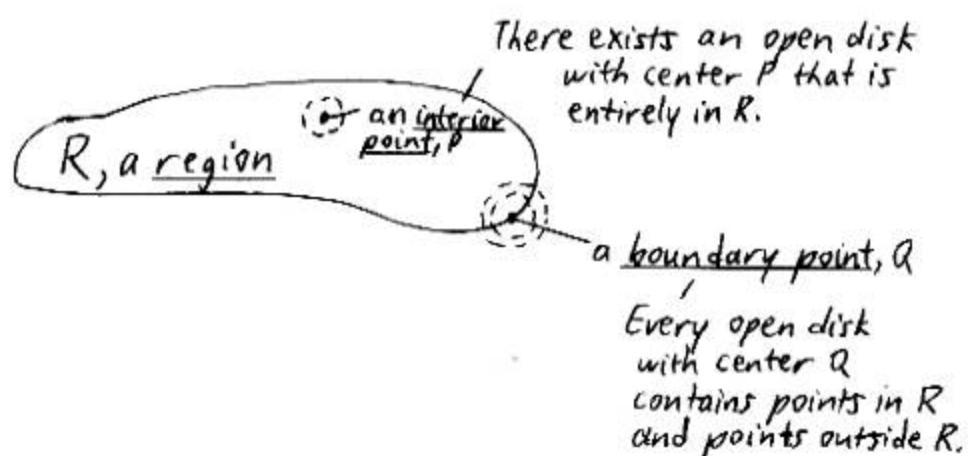
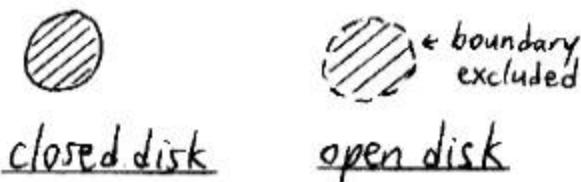
No shared points!

LSs never intersect, and
LSs

Think: Morphing as k changes.

16.2: LIMITS and CONTINUITY

(A) Point Set Topology



The Village

For limit and continuity analyses, always stay in R !

R is open \iff Every point in R is an interior point.

R is closed \iff R includes all its boundary points.

(B) Notation

Relate to (A)
Erdős kid

\forall - for every
 \exists - there exists

ϵ - epsilon
 δ - delta (lowercase)

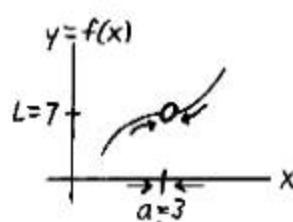
⑥ Defining Limits - Calc I (See Section 2.2 Notes.)

" $\lim_{x \rightarrow a} f(x) = L$ "

(or " $f(x) \rightarrow L$ as $x \rightarrow a$ ")

means that

Ex $\lim_{x \rightarrow 3} f(x) = 7$



We can ignore $x=a$.

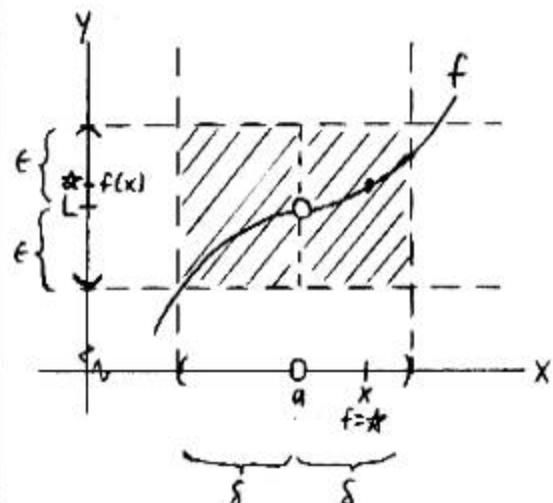
$\forall \epsilon > 0, \exists \delta > 0$ such that

if $0 < |x-a| < \delta$

$x \neq a$ distance between x, a

then $|f(x) - L| < \epsilon$

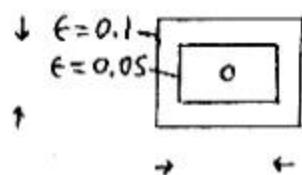
distance between $f(x), L$



Given ϵ , find a δ so that the box traps the graph ($x \neq a$).

As ϵ shrinks, δ tends to shrink.

"Walls closing in" / "Zooming in"



① Defining Limits - Calc III

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$$

(or " $f(x,y) \rightarrow L$ as $(x,y) \rightarrow (a,b)$ ")

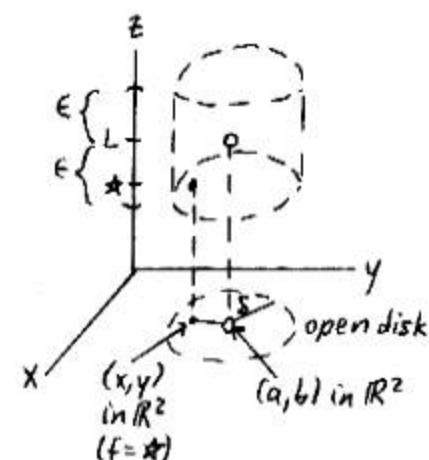
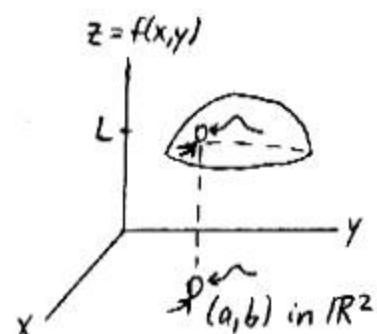
means that

$\forall \epsilon > 0, \exists \delta > 0$, such that

if $0 < \underbrace{\sqrt{(x-a)^2 + (y-b)^2}}_{\substack{(x,y) \neq (a,b) \\ \text{distance between } (x,y) \text{ and } (a,b)}} < \delta$

then $\underbrace{|f(x,y) - L|}_{\substack{\text{distance between} \\ f(x,y) \text{ and } L}} < \epsilon$

Ex

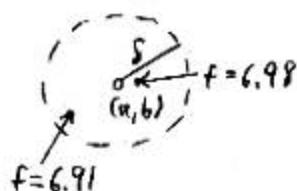


Given ϵ , find a δ so that the cylinder traps the surface $((x,y) \neq (a,b))$.

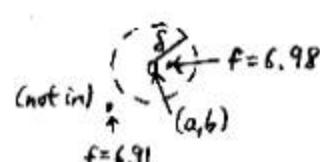
As ϵ shrinks, δ tends to shrink.

Ex $L=7$

$$\epsilon = 0.1$$



$$\epsilon = 0.05$$



(E) Computing Limits

Basic Limit Laws from pp. 60-64 (2.3) extend naturally,

The limit of a(n) sum = The sum of the limits (if they exist)	
difference	difference
product	product
quotient	quotient (if denom. $\neq 0$)
[nice] power	power
odd root	odd root
even root	even root (if radicand > 0)

unwritten
in class

Ex $\lim_{(x,y) \rightarrow (1,-2)} \frac{4y + \sqrt{x^2 + y^2}}{x^3 y^2 - 1}$ ← polynomial in x and y

$f(x,y)$ is algebraic (rational but $\sqrt[3]{0}$ OK)
If there's no problem with $\sqrt[n]{0}$ or $\sqrt[n]{\leq 0}$,
use direct sub to find the limit.

$$= \frac{4(-2) + \sqrt{(1)^2 + (-2)^2}}{(1)^3(-2)^2 - 1}$$

$$= \boxed{\frac{\sqrt{5} - 8}{3}}$$

If $\frac{0}{0}$ limit form \Rightarrow Factor and Cancel?

Rationalize the denom. or numer.?

L'Hôpital's Rule? (If one variable...)

When
Animals
Attack

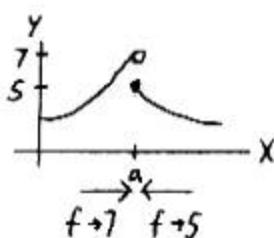
F) When Limits Do Not Exist (DNE)

includes $\infty, -\infty$

Calc I

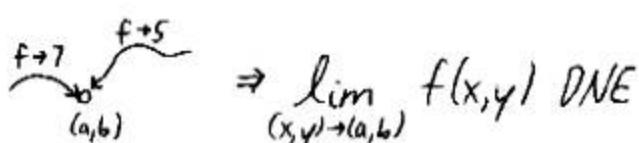
If $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$, then $\lim_{x \rightarrow a} f(x)$ DNE.

or if either DNE \Rightarrow

ExCalc IIITwo-Path Rule

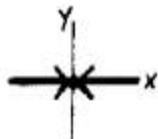
If two different paths to (a, b) yield different limit values for f , then $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$ DNE.

(Also, if any path yields an undefined limit \Rightarrow)

Ex

Ex Show $\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 + 3y^2}{4x^2 - 5xy + 6y^2}$ DNE.

i) $(x,y) \rightarrow (0,0)$ along the x -axis ($y=0$)



$$\begin{aligned} & \lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 + 3(0)^2}{4x^2 - 5x(0) + 6(0)^2} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{2x^2}{4x^2} \\ &= \left(\frac{1}{2}\right) \end{aligned}$$

ii) $(x,y) \rightarrow (0,0)$ along the y -axis ($x=0$)



$$\begin{aligned} & \lim_{(x,y) \rightarrow (0,0)} \frac{2(0)^2 + 3y^2}{4(0)^2 - 5(0)y + 6y^2} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{3y^2}{6y^2} \\ &= \left(\frac{1}{2}\right) \quad \text{WARNING: This does not prove} \end{aligned}$$

Waste
of time,
 $\textcircled{2}$
as it
turns out

that $\textcircled{1} = \frac{1}{2}$.

iii) $(x,y) \rightarrow (0,0)$ along the line $y=x$ (say)



$$\begin{aligned} & \lim_{(x,y) \rightarrow (0,0)} \frac{2x^2 + 3(x)^2}{4x^2 - 5x(x) + 6(x)^2} \\ &= \lim_{(x,y) \rightarrow (0,0)} \frac{5x^2}{5x^2} \\ &= \left(1\right) \end{aligned}$$

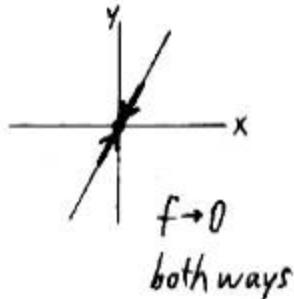
 By the Two-Path Rule, ~~DNE~~ DNE.

Ex 4 (p. 809)

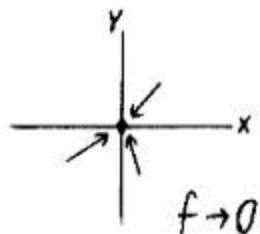
$$\lim_{(x,y) \rightarrow (0,0)} \frac{f(x,y)}{\frac{x^2y}{x^4+y^2}} \text{ DNE}$$

Book literally
throws you
a curve
Idea: As $m \rightarrow 0$
($\theta \rightarrow 0$ or π)
 f changes
more rapidly
($\lim \frac{mx}{x^2+m^2}$)
along
the line

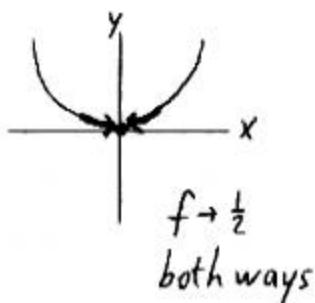
Along $y=mx$ ($m \neq 0$)



Let m vary:



Along $y=x^2$



Up to 11

(G) Using Polar Coordinates to Find Limits

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^4}{x^2+y^2} \quad (0) = \lim_{r \rightarrow 0} \frac{(r \cos \theta)^4}{r^2}$$

Any way you approach $(0,0)$
 $\Rightarrow r \rightarrow 0$

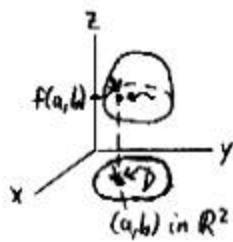
$$= \lim_{r \rightarrow 0} \frac{r^4 \cos^4 \theta}{r^2}$$

$$= \lim_{r \rightarrow 0} r^2 \underbrace{\cos^4 \theta}_{\text{in } [-1,1] \text{ always}}$$

$$= 0 \quad (\theta \text{ is irrelevant})$$

(H) Continuity

f is continuous at an interior point (a,b) of its domain, D , if
 $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$ (and these exist).



How to Ace:
 Mountain
 hazards
 at disconts.

(For a boundary point, make sure paths stay in D).

f is cont. on $D \Leftrightarrow f$ is cont. at every (a,b) in D .

Rules from pp. 80-83 (2.S) extend naturally.

If f, g cont. at $(a,b) \Rightarrow$ so are $f \pm g$ and $\frac{f}{g}$ if $g(a,b) \neq 0$.

If g cont. at (a,b) , f cont. at $g(a,b) \Rightarrow f \circ g$ cont. at (a,b) .

Domain of $f \subseteq \mathbb{R}$

REVIEW: CH. 15, 16.1, 16.215.1: VVF_s

Smooth curve determined by VVF $\vec{r}: D \rightarrow V_n$
 $D \subseteq \mathbb{R}$

\vec{r}' cont., never $\vec{0}$ on I
 (except maybe at endpoints)

$$\vec{r}(t) = \langle f(t), g(t), \underbrace{h(t)}_{\text{if in } V_3} \rangle$$

$$\text{Arc Length "L"} = \int_a^b \|\vec{r}'(t)\| dt$$

$$= \|\vec{v}(t)\| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

"speed"

$$= \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

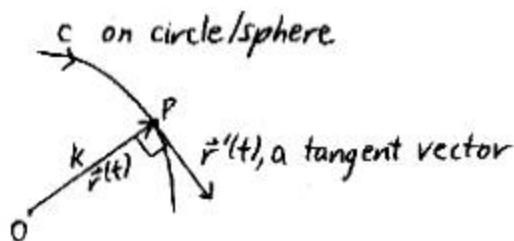
Use Table of Integrals?

15.2: LIMITS, D_t , $\int dt$, CONTINUITY for VVEs

Do component-by-component.
 Apply FTC
 Solve Differential Eqs.

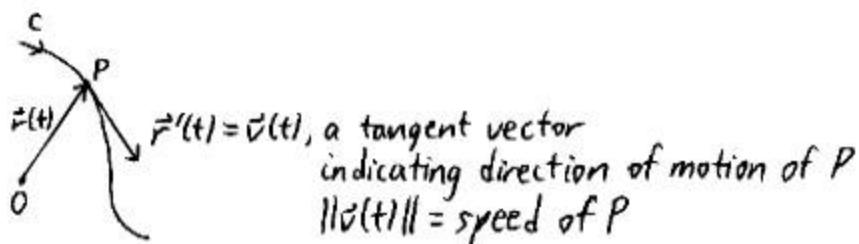
 D_t Rules

Linearity
 Product Rules for $\cdot, x, f\dot{u}$
 Chain Rule for $\ddot{u}(f(t))$

Sphere Theorem

If $\|\vec{r}(t)\| = k$ on I, r diff're

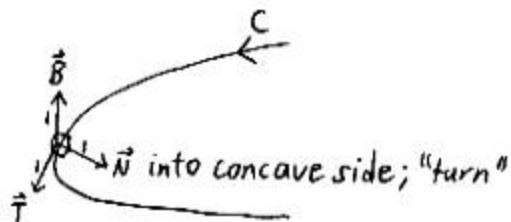
$\Rightarrow \vec{r}'(t) \perp \vec{r}(t)$ on I

15.3: MOTION

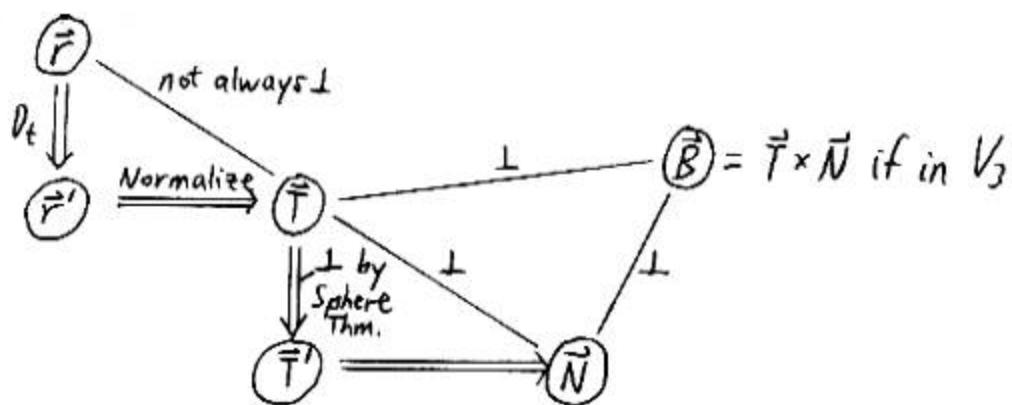
$$\vec{r}''(t) = \vec{v}'(t) = \vec{a}(t)$$

15.4/15.5

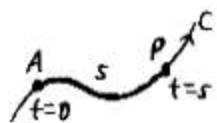
TNB Frame



Point $P, \vec{T}, \vec{N} \Rightarrow$
Get osculating plane at P .

ALPs

$$\vec{r}(s)$$



$$\begin{aligned} \text{Speed} &= \|\vec{v}(s)\| \\ &= \|\vec{r}'(s)\| \\ &= 1 \end{aligned}$$

$$\Rightarrow \vec{T}(s) = \vec{r}'(s)$$

K

$$\rho = \frac{1}{K} \quad \textcircled{2} \quad \text{for } \mathbb{R}^2$$

If ALP,

$$\begin{aligned} K(s) &= \|\vec{r}''(s)\| \\ &= \|\vec{T}'(s)\| \\ &= \left\| \frac{d\vec{T}}{ds} \right\| \end{aligned} \quad \left. \begin{array}{l} \vec{\tau} = \vec{r}' \\ \text{How quickly is } \vec{T} \text{ changing direction?} \end{array} \right.$$

If not ALP,

$$\begin{aligned} K(t) &= \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} \quad \left(= \left\| \frac{d\vec{T}/dt}{ds/dt} \right\| \right) \\ &= \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|^3} \quad \text{or} \quad \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} \end{aligned}$$

If a plane curve: $y = g(x)$,

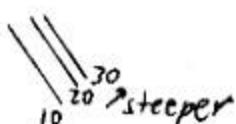
$$K(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}} \quad (\text{I'll give.})$$

16.1: $z = f(x, y)$, etc.

$f: D \rightarrow \mathbb{R}$, where Domain $D \subseteq \mathbb{R}^n$
 Find D .

Level Curves

Graph $k = f(x, y)$ in \mathbb{R}^2 for several k -values.

Level Surfaces

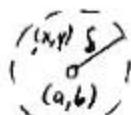
Graph $k = f(x, y, z)$ in \mathbb{R}^3
 Identify them.

16.2: LIMITS and CONTINUITY

" $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$ " means that

$\forall \epsilon > 0, \exists \delta > 0$ such that

if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$



then $|f(x,y) - L| < \epsilon$



Computing Limits

Basic Laws, Direct Sub
Dealing with $\frac{0}{0}$
Polar Coordinates

When Limits DNE

Two-Path Rule to show \exists

Ex $\downarrow \neq$ different limits
 (a,b)

f is cont. at (a,b) if $\lim_{\substack{(x,y) \rightarrow (a,b) \\ \text{through } D}} f(x,y) = f(a,b)$ [and these exist].