CH. 15: VECTOR-VALUED FUNCTIONS (VVFs)

15.1: VVFs and SPACE CURVES

Ⓐ Exs

Parametric eqs. for a line segment C.

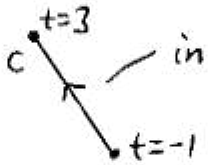
$$x = \underbrace{1 + 3t}_{f(t)}\text{, a } \underline{\text{scalar (or real-valued) function}} \text{ of } t$$

$$y = \underbrace{6 - 4t}_{g(t)}$$

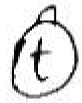
$$z = \underbrace{-1 + 2t}_{h(t)}$$

$$t \text{ in } \underbrace{[-1, 3]}_{\text{Domain "D"}}$$

No jumps ⇒ We can call this "I" for Interval.



indicates <u>orientation</u>
(direction in which t increases)



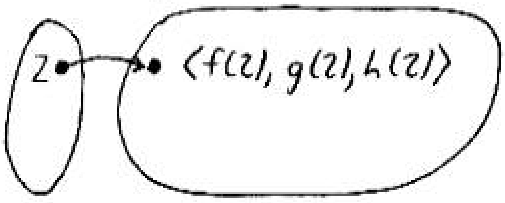
Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$= \langle 1+3t, 6-4t, -1+2t \rangle$$

$$t \text{ in } [-1, 3]$$

Then, $\vec{r}$ is a VVF that maps D into V_3.

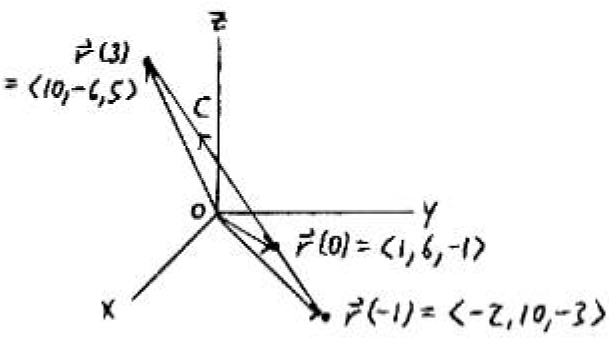
$$\vec{r} : D \to V_3$$



We'll typically assume:

The domain "D" is an interval ("I").
f, g, h are continuous on I.

The <u>curve C determined by $\vec{r}$</u> consists of the endpoints of the position vectors for $\vec{r}(t)$ for all t in I.

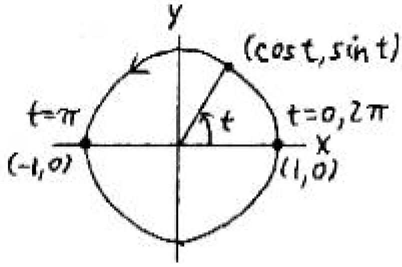


C is a <u>smooth</u> curve (has no breaks, corners, cusps) if it has a smooth parameterization (optional) on I in which f', g', h' are continuous on I, and there is no t-value in I at which $\underbrace{\text{all are } 0}_{\text{i.e., } \vec{r}\,'(t) = \vec{0}}$, except maybe at endpoints.

<u>Ex</u> $\vec{r}(t) = \langle \underset{x}{\cos t}, \underset{y}{\sin t}, \underset{z}{t} \rangle$, t in $\mathbb{R}$ (radians implied). Draw C.

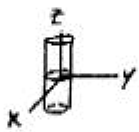
Book → i.e., $\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + t\vec{k}$

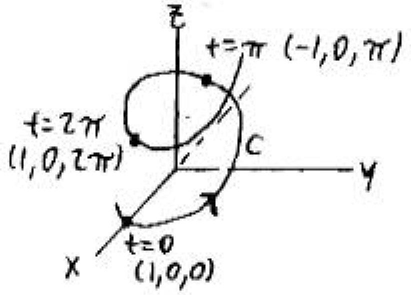
($\mathbb{R}^2$) Consider $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$



Slinky? ($\mathbb{R}^3$) $z = t \Rightarrow$ spring effect along cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$

Circular helix

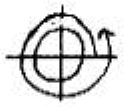




Ⓑ Arc Length

Let C have a smooth parameterization
$\langle f(t), g(t), h(t) \rangle$, $a \le t \le b$.

NO self-overlaps



Then, C has length "L":

$$\boxed{\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt \\ &= \int_a^b \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}_{=\frac{ds}{dt}\text{, where } s \text{ represents arc length.}}\, dt \end{aligned}}$$

from Pyth. Thm.

Ex (#24) Find the length of C if
$\vec{r}(t) = \langle 1-2t^2, 4t, 3+2t^2 \rangle$, $0 \le t \le 2$.

Sol'n

$$\begin{array}{ccc} 1-2t^2 & 4t & 3+2t^2 \\ D_t \Downarrow & \Downarrow & \Downarrow \\ -4t & 4 & 4t \end{array}$$

$$L = \int_0^2 \sqrt{(-4t)^2 + (4)^2 + (4t)^2}\, dt$$

$$= \int_0^2 \sqrt{32t^2 + 16}\, dt$$

$$= \int_0^2 \sqrt{16(2t^2+1)}\, dt$$

$$= \int_0^2 4\sqrt{2t^2+1}\, dt$$

$$= \int_0^2 4\sqrt{1+2t^2}\, dt$$

Method 1 (Trig Sub)

Want: $u^2 = 2t^2$
Let $u = \sqrt{2}t$

Form $\sqrt{1+u^2} \Rightarrow$ Use $u = \sqrt{2}t = \tan\theta$, etc.

Method 2 (Table of Integrals - p. A21)

Want: $u^2 = 2t^2$
Let $u = \sqrt{2}t$

$\Rightarrow du = \sqrt{2}\,dt$
$\Rightarrow dt = \frac{1}{\sqrt{2}}\,du$

$t=0 \Rightarrow u=0$
$t=2 \Rightarrow u=2\sqrt{2}$

$$= \int_0^{2\sqrt{2}} 4\sqrt{1+u^2} \cdot \frac{1}{\sqrt{2}}\,du$$

Note: $\frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$

$$= 2\sqrt{2} \int_0^{2\sqrt{2}} \sqrt{1+u^2}\,du$$

Formula #21 on p. A21 (I'll give) - derived using Method 1

$$\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left|u+\sqrt{a^2+u^2}\right| + C$$

a=1 here

$$= 2\sqrt{2}\left[\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left|u+\sqrt{1+u^2}\right|\right]_0^{2\sqrt{2}}$$

$$= 2\sqrt{2}\Big(\Big[\tfrac{2\sqrt{2}}{2}\sqrt{1+(2\sqrt{2})^2}+\tfrac{1}{2}\ln\left|2\sqrt{2}+\sqrt{1+(2\sqrt{2})^2}\right|\Big] \underbrace{-\Big[0+\tfrac{1}{2}\underbrace{\ln 1}_{=0}\Big]}_{=0}\Big)$$

$$= 2\sqrt{2}\left(\sqrt{2}(3)+\tfrac{1}{2}\ln(2\sqrt{2}+3)\right)$$

$$= \boxed{12+\sqrt{2}\ln(2\sqrt{2}+3)}$$