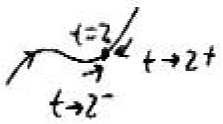
15.2: LIMITS, DERIVATIVES, and INTEGRALS

(A) Do them component-by-component, if the pieces exist.

$\vec{r}$ is continuous at a $\overset{\text{Def'n}}{\Longleftrightarrow}$ $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$

$\Longleftrightarrow$ $f, g,$ and h are cont. at a



($\vec{r}$ is cont. on an open interval I
$\Longleftrightarrow$ $\vec{r}$ is cont. at every # in I.
For non-open intervals, consider the appropriate one-sided limit(s) at the endpoint(s).)

Ex If $\vec{r}(t) = \langle e^{7t}, \cos(3t), t^2 + t - 1 \rangle$

$$\Rightarrow \vec{r}\,'(t) = \langle 7e^{7t}, -3\sin(3t), 2t+1 \rangle,$$

$$\int \vec{r}(t)\,dt = \langle \tfrac{1}{7}e^{7t}, \tfrac{1}{3}\sin(3t), \tfrac{t^3}{3} + \tfrac{t^2}{2} - t \rangle + \underset{\langle c_1, c_2, c_3 \rangle}{\vec{C}}$$

Ex (Extension of FTC, Fundamental Theorem of Calculus)
If $\vec{r}(t) = \langle t^3, e^{2t}, 1 \rangle \Rightarrow \int_2^3 \vec{r}(t)\,dt = \int_2^3 \underbrace{\langle t^3, e^{2t}, 1 \rangle}_{\text{cont. on } [2,3]} dt$

$$= \Big[\underbrace{\langle \tfrac{t^4}{4}, \tfrac{1}{2}e^{2t}, t \rangle}_{\text{an antiderivative, } \vec{R}(t)} \Big]_2^3$$

$$\overset{\text{FTC}}{=} \vec{R}(3) - \vec{R}(2)$$

$$= \langle \tfrac{(3)^4}{4}, \tfrac{1}{2}e^{2(3)}, 3 \rangle - \langle \tfrac{(2)^4}{4}, \tfrac{1}{2}e^{2(2)}, 2 \rangle$$

$$= \boxed{\langle \tfrac{65}{4}, \tfrac{1}{2}(e^6 - e^4), 1 \rangle}$$

Ex (Differential Equations)

Find $\vec{r}(t)$ if $\vec{r}\,'(t) = \langle 1, t, e^{\frac{t}{2}} \rangle$ subject to the initial condition (IC) $\vec{r}(0) = \langle 1, 3, 5 \rangle$.

(1,3,5) "starting point"

Sol'n

$$\vec{r}(t) = \int \langle 1, t, e^{\frac{t}{2}} \rangle \, dt \quad \text{(one member)}$$

$$= \langle t, \tfrac{t^2}{2}, 2e^{\frac{t}{2}} \rangle + \vec{C}$$

Use the IC to find $\vec{C}$.

$$t=0 \Rightarrow \vec{r}(0) = \langle 0, 0, 2 \rangle + \vec{C}$$

$$\langle 1,3,5 \rangle = \langle 0, 0, 2 \rangle + \vec{C}$$

$$\vec{C} = \langle 1, 3, 3 \rangle$$

$$\vec{r}(t) = \langle t, \tfrac{t^2}{2}, 2e^{\frac{t}{2}} \rangle + \langle 1,3,3 \rangle$$

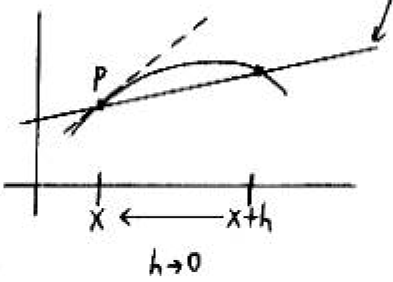
$$\boxed{\vec{r}(t) = \langle t+1, \tfrac{t^2}{2}+3, 2e^{\frac{t}{2}}+3 \rangle}$$

(B) Geometry of $\vec{r}\,'(t)$

Calc I

$y = f(x)$

$$f'(x) = \lim_{h \to 0} \underbrace{\frac{f(x+h) - f(x)}{h}}_{=\text{slope of secant line}}$$



= slope of tangent line (-‐-) at P
(if it exists)

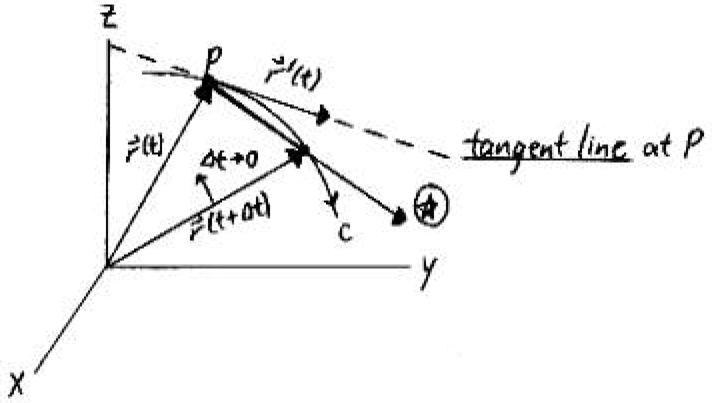
Calc III

$z = h(t)$
Don't use "$h \to 0$"

$$\vec{r}\,'(t) = \lim_{\Delta t \to 0} \underbrace{\frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}}_{(*)}$$ ← This lengthens if $0 < \Delta t < 1$

= a <u>tangent vector</u> at P
direction consistent with orientation of C at P

(if it exists)



<u>Ex</u> $\vec{r}(t) = \langle 4t, 3, t^2 \rangle$.
Find parametric eqs. for the tangent line to C at P(8,3,4).

<u>Sol'n</u>

What's t at P?

The curve could self-intersect

$$4t = 8$$
$$t = 2$$

Check: $\vec{r}(2) = \langle 8, 3, 4 \rangle$ ✓

Find a tangent vector at P.

$$\vec{r}'(t) = \langle 4, 0, 2t \rangle$$
$$\vec{r}'(2) = \langle 4, 0, 4 \rangle$$

Answer

$$\boxed{\begin{cases} x = 8 + 4t \\ y = 3 \\ z = 4 + 4t \end{cases}, \; t \text{ in } \mathbb{R}}$$

Up to 33

© Derivative Rules for VVFs

When $\vec{u}, \vec{v}, f$ are differentiable (diff'c)...

D_t is a linear operator

(i) $D_t[\vec{u}(t) + \vec{v}(t)] = \vec{u}\,'(t) + \vec{v}\,'(t)$	Can D_t term-by-term
(ii) $D_t[c\vec{u}(t)] = c\vec{u}\,'(t)$	Constant scalar factors pop out.

Product Rule analogs

(iii) $D_t[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}\,'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}\,'(t)$

(iv) $\times \quad \times \quad \times$ ← if $\vec{u}, \vec{v}$ in V_3

(v) $D_t[\underbrace{f(t)}_{\text{scalar func.}}\vec{u}(t)] = f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$

in Exercises

Chain Rule

(vi) $D_t\,\vec{u}(f(t)) = [\vec{u}\,'(f(t))][f'(t)]$ | $D_t\,\vec{u}(\star) = [\vec{u}\,'(\star)][D_t(\star)]$

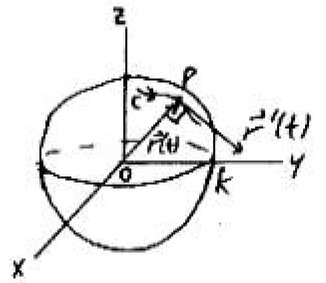
(D) "Sphere Theorem"

If $\vec{r}$ is diff'e and $\|\vec{r}(t)\|$ is constant on I, then $\vec{r}\,'(t) \perp \vec{r}(t)$ for every t in I.

<u>Note</u> $\|\vec{r}(t)\| = k \quad (k \geq 0)$

$$\|\langle x, y, z\rangle\| = k$$
$$\sqrt{x^2+y^2+z^2} = k$$
$$x^2+y^2+z^2 = k^2$$

i.e., If C lies entirely on a sphere centered at O, then for any point P on C, $\overrightarrow{OP} \perp$ tangent vector at P.



($\mathbb{R}^2$) Circles