15.4: CURVATURE

(A) TNB Frame

When $\vec{r}$ is diff'e, and $\vec{r}\,' \neq \vec{0}$,

Unit tangent vector to C

$$\boxed{\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}}$$

When $\vec{T}$ is diff'e, and $\vec{T}' \neq \vec{0}$,

[Principal] unit normal vector to C

$$\boxed{\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}}$$

$$\boxed{\vec{N}(t) \perp \vec{T}(t)}$$

Proof $\vec{T}$ is diff'e, and $\|\vec{T}(t)\| = 1$ for all relevant t

$\Rightarrow \vec{T}'(t) \perp \vec{T}(t)$ (")

by the Sphere Thm. (15.2.6)

$\Rightarrow \vec{N}(t) \perp \vec{T}(t)$ (")

because $\vec{N}(t)$ points in the same direction as $\vec{T}'(t)$

If we're in V_3 $(\mathbb{R}^3)$,

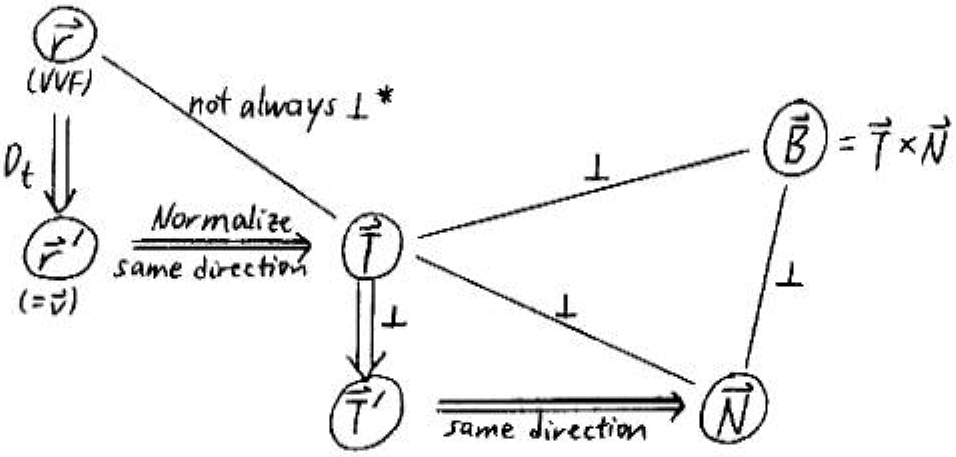
In Stewart 867 but not Larson or Swok.

<u>Binormal vector</u> to C

$$\boxed{\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)}$$

$\vec{B}(t)$ is a unit vector. (Proof: HW!)

<u>Schematic</u>



* If Sphere Thm. applies $\Rightarrow \vec{r}(t) \perp \vec{T}(t)$ for all relevant t.

<u>Note</u> Although $\vec{r}\,'' = \vec{a}$, don't expect $\vec{a} \parallel \vec{T}\,'$.

$\vec{r}\,' \parallel \vec{T}$, but often $\underset{D_t[\vec{r}\,'(t)]}{\vec{r}\,''} \nparallel \underset{D_t\left[\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\right]}{\vec{T}\,'}$.

This is because magnitude $\|\vec{r}\,'(t)\|$ is often a nonconstant function of t (see Ex).

TNB Frame

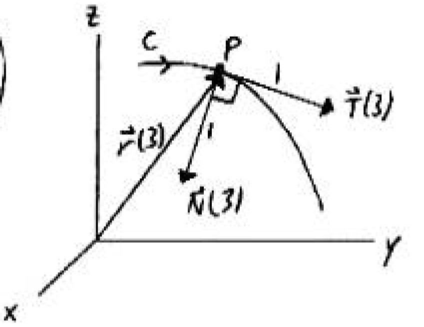
always unit vectors,
mutually $\perp$

(Important in differential geometry, spacecraft motion)

Imagine this rotating as it follows P along C.

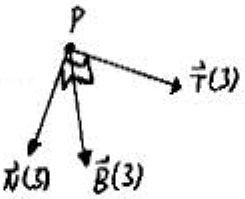
Face board
Right
3
thumb
2
Spock?

$t=3$, say

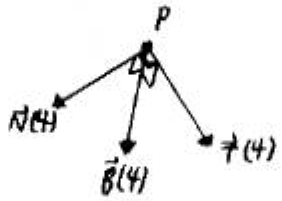


$\vec{N}$ points inside the concave side of C.

$\vec{N}$ indicates the direction in which C is turning.

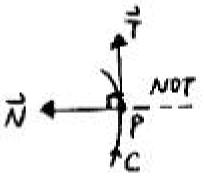


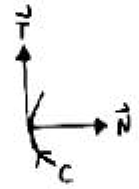
$t=4$, say



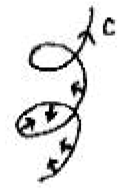
More on $\vec{N}$

$\mathbb{R}^2$





$\mathbb{R}^3$ Helix



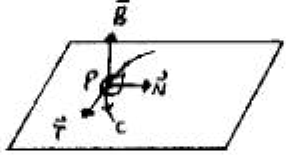
In $\mathbb{R}^3$,

$P, \vec{T}, \vec{N}$ determine the osculating plane of C at P.

Latin: osculum ("to kiss")

It's the plane that comes closest to containing the part of C near P.

$\vec{B} \perp$ this plane.



Note $P, \vec{N}, \vec{B}$ determine the normal plane of C at P.
$\vec{T} \perp$ this plane.

Ex (#2) If $\vec{r}(t) = \langle -t^2, 2t \rangle$ in $\mathbb{R}^2$ (V_2),
find $\vec{T}(t)$ and $\vec{N}(t)$.

Sol'n

$$\vec{r}\,'(t) = \langle -2t, 2 \rangle$$
$$= 2\langle -t, 1 \rangle$$

$$\|\vec{r}\,'(t)\| = |2|\sqrt{(-t)^2 + (1)^2}$$
$$= 2\sqrt{t^2+1}$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

$$= \frac{\cancel{2}\langle -t, 1 \rangle}{\cancel{2}\sqrt{t^2+1}}$$

$$= \boxed{\left\langle -\frac{t}{\underbrace{\sqrt{t^2+1}}_{(t^2+1)^{1/2}}}, \frac{1}{\sqrt{t^2+1}} \right\rangle}$$

Alternative: Use 15.2.5(v) Product Rule analog on $\underbrace{\frac{1}{\sqrt{t^2+1}}}_{f(t)} \underbrace{\langle -t, 1 \rangle}_{\vec{u}(t)}$

$$\vec{T}\,'(t) = \left\langle -\frac{(\sqrt{t^2+1})(1) - t\left[\cancel{\tfrac{1}{2}}(t^2+1)^{-\frac{1}{2}}(\cancel{2}t)\right]}{t^2+1}, \right.$$
$$\left. -\frac{\cancel{\tfrac{1}{2}}(t^2+1)^{-\frac{1}{2}}(\cancel{2}t)}{t^2+1} \right\rangle$$

by Quotient/Reciprocal Rules

$$= \left\langle -\underbrace{\frac{\sqrt{t^2+1} - \frac{t^2}{\sqrt{t^2+1}}}{t^2+1}}_{\cdot \frac{\sqrt{t^2+1}}{\sqrt{t^2+1}}}, \; -\frac{t}{(t^2+1)(t^2+1)^{1/2}} \right\rangle$$

$$= \left\langle -\frac{\cancel{t^2}+1-\cancel{t^2}}{(t^2+1)^{3/2}}, \; -\frac{t}{(t^2+1)^{3/2}} \right\rangle$$

$$= \left\langle -\frac{1}{(t^2+1)^{3/2}}, \; -\frac{t}{(t^2+1)^{3/2}} \right\rangle$$

$$\|\vec{T}'(t)\| = \sqrt{\left(-\frac{1}{(t^2+1)^{3/2}}\right)^2 + \left(-\frac{t}{(t^2+1)^{3/2}}\right)^2}$$

$$= \sqrt{\frac{1}{(t^2+1)^3} + \frac{t^2}{(t^2+1)^3}}$$

$$= \sqrt{\frac{t^2+1}{(t^2+1)^3}}$$

$$= \sqrt{\frac{1}{(t^2+1)^2}}$$

$$= \frac{1}{\underbrace{|t^2+1|}_{>0 \text{ always}}}$$

$$= \frac{1}{t^2+1}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

$$= \frac{\left\langle -\frac{1}{(t^2+1)^{3/2}}, -\frac{t}{(t^2+1)^{3/2}} \right\rangle}{\frac{1}{t^2+1}}$$

$$= (t^2+1)\left\langle -\frac{1}{(t^2+1)^{3/2}}, -\frac{t}{(t^2+1)^{3/2}} \right\rangle$$

Up to 5a

$$= \boxed{\left\langle -\frac{1}{\sqrt{t^2+1}}, -\frac{t}{\sqrt{t^2+1}} \right\rangle}$$

Note Easier way to find $\vec{N}(3)$, say:

Find $\vec{T}'(3)$

$\Rightarrow$ Find $\|\vec{T}'(3)\|$

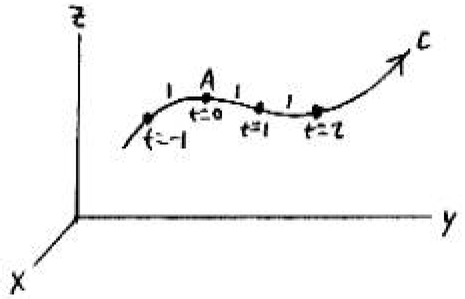
$\Rightarrow$ Find $\vec{N}(3) = \frac{\vec{T}'(3)}{\|\vec{T}'(3)\|}$

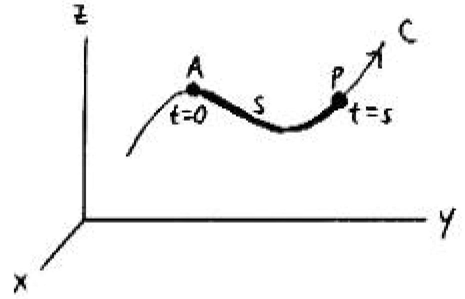
Ⓑ Arc Length Parametrizations (ALPs) of C

Not written on board

Let A be a fixed point on C corresponding to $t=0$.

If, for any point P on C, the t-value at P equals the signed length of arc $\widehat{AP}$ (i.e., the signed distance of P from A along C), then we have an ALP of C.





If we have an ALP, we usually use "s" instead of "t" as our parameter. s is an <u>arc length parameter</u> that increases in the direction of orientation.

$$\text{ALP}: \vec{r}(s) = \langle \underset{x}{f(s)}, \underset{y}{g(s)}, \underset{z}{h(s)} \rangle$$

Con: It's often hard to find ALPs (see p. 779).

Pro: Speed $=1$ always along C.

Ⓒ Idea of Curvature (κ)

Kappa

(See <u>How to Ace the Rest of Calculus.</u>)

Winding road

} κ higher

c

κ measures "nausea" <u>if</u> speed is constant along C.
κ should depend on shape, not speed.

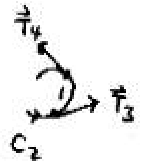
What kind of param'n gives us constant speed?

Let's use a smooth ALP, $\vec{r}(s)$, such that $\vec{r}\,''(s)$ exists.

$$\text{Then, } \underbrace{\vec{T}(s)}_{\substack{\text{unit}\\ \text{tangent}\\ \text{vector}}} = \frac{\vec{r}\,'(s)}{\|\vec{r}\,'(s)\|} \leftarrow = \text{speed} = 1$$

$$= \vec{r}\,'(s) \quad ✪$$

Its rate of change with respect to s, $\vec{T}\,'(s)$ or $\vec{r}\,''(s)$, only reflects its rate of change of <u>direction</u>, because its length is constant.



Assuming 2D, same scales for graphs ($\vec{T}_i$'s are unit)

κ higher
$\vec{T}$ changes more dramatically

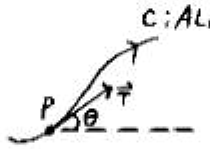
Let $\kappa(s) = \|\vec{r}\,''(s)\|$

so we get a scalar

$\overset{\circledast}{=} \|\vec{T}\,'(s)\|$

$= \left\| \dfrac{d\vec{T}}{ds} \right\|$

In $\mathbb{R}^2$, $\kappa(s) = \left|\frac{d\theta}{ds}\right|$ is consistent with the above.



Ⓓ <u>Radius of Curvature, ρ</u> (for plane curves in $\mathbb{R}^2$)

rho

$\boxed{\rho = \frac{1}{\kappa}}$

<u>Why? Ex 7 on pp. 775-6</u>

κ at every point on C $= \frac{1}{\rho}$

If $\kappa \neq 0$ at P $\Rightarrow$ The <u>circle of curvature</u> <u>(osculating circle)</u> at P has the same tangent line and κ as C does at P. It's the circle that best models C near P.

K higher $\Rightarrow$ ρ lower

K lower $\Rightarrow$ ρ higher

line: $K = 0$

($\vec{T}$ constant $\Rightarrow \frac{d\vec{T}}{ds} = \vec{0}$)

(E) Practical Formulas for K

What if $\vec{r}(t)$ is not an ALP for C? (When $\vec{r}$ smooth, $\vec{r}''$ exists...)

$$K(t) = \left\| \frac{d\vec{T}}{ds} \right\|$$

$$= \left\| \frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} \right\|$$

by Chain Rule for VVFs 15.2.5 (vi)

$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt}$

$$= \frac{\left\| \frac{d\vec{T}}{dt} \right\|}{\left| \frac{ds}{dt} \right|}$$

$\leftarrow$ speed $= \|\vec{v}(t)\| = \|\vec{r}\,'(t)\|$

Rate of change of arc length wrt time. Sound familiar?

$$\boxed{K(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|}}$$ (☆)

Not in Swok. $\rightarrow$

Stewart 865

Even better (after a tricky proof) if we're in $\mathbb{R}^3$

though we can modify for $\mathbb{R}^2$, see Ex below.

$$\Rightarrow \boxed{\begin{array}{l} \kappa(t) = \dfrac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} \\ \\ \underline{\text{Think}}:\ \kappa = \dfrac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|^3} \end{array}}$$

(15.5 in book)

☆☆

though we don't normally like thinking about $\vec{v}, \vec{a}$ in the context of κ.

Even better for a plane curve : $y = g(x)$

some function

$$\Rightarrow \boxed{\kappa(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}}}$$

☆☆☆ (I'll give.)

<u>Proof</u> Use Thm. 15.19 on p. 775 (see HW 15.5, #16),

<u>or</u> ☆☆ with $\vec{r}(t) = \langle \underset{x}{t}, \underset{y}{g(t)}, \underset{z}{0} \rangle$.

We lose smoothness at $t=0$.

<u>Ex</u> In $\mathbb{R}^2$, $C: x = t^3, y = t^2, t > 0$.
Find a κ formula for C, and find κ and ρ at $(8,4)$.

<u>Method 1</u> Thm. 15.19 on p. 775

<u>Method 2</u> Use ☆.

<u>Method 3</u> Use ☆☆.

Let $\vec{r}(t) = \langle t^3, t^2, 0 \rangle$ in $\mathbb{R}^3$

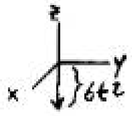
$\Rightarrow \vec{v}(t) = \langle 3t^2, 2t, 0 \rangle$

$\Rightarrow \vec{a}(t) = \langle 6t, 2, 0 \rangle$

15.4.13

$$\vec{v}(t) \times \vec{a}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3t^2 & 2t & 0 \\ 6t & 2 & 0 \end{vmatrix}$$

$$= \langle 0, 0, -6t^2 \rangle$$



$$\|\vec{v}(t) \times \vec{a}(t)\| = 6t^2$$

$$\|\vec{v}(t)\|^3 = \left(\sqrt{(3t^2)^2 + (2t)^2 + (0)^2}\right)^3$$
$$= (9t^4 + 4t^2)^{3/2}$$

$$K(t) = \frac{\|\vec{v}(t) \times \vec{a}(t)\|}{\|\vec{v}(t)\|^3}$$

$$= \boxed{\frac{6t^2}{(9t^4 + 4t^2)^{3/2}}}$$

<u>Find K, ρ at $(8,4)$</u>

$t = 2$ at $(8,4)$.

$$K(2) = \frac{6(2)^2}{[9(2)^4 + 4(2)^2]^{3/2}} = \boxed{\frac{24}{(160)^{3/2}} \approx 0.0119}$$

$$\rho(2) = \frac{1}{K(2)} = \boxed{\frac{(160)^{3/2}}{24} \approx 84.3}$$

15.4.14

Method 4 Use [illegible].

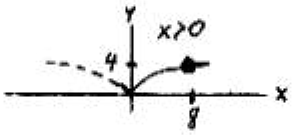
Eliminate the parameter, t.

$$x = t^3 \implies t = \sqrt[3]{x} \qquad \Big| \quad t > 0 \iff x > 0$$

$$\Downarrow$$

$$y = t^2 \implies y = (\sqrt[3]{x})^2$$

$$\boxed{y = x^{2/3}, \quad x > 0}$$



Find K(x).

$$K(x) = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$

$$y = x^{2/3} \implies y' = \tfrac{2}{3}x^{-1/3} \implies y'' = -\tfrac{2}{9}x^{-4/3}$$

$$y' = \frac{2}{3x^{1/3}} \qquad y'' = -\frac{2}{9x^{4/3}}$$

$$= \frac{\left| \overset{\text{can kill}}{-} \overbrace{\frac{2}{9x^{4/3}}}^{>0 \text{ for } x>0} \right|}{\left[1 + \left(\frac{2}{3x^{1/3}}\right)^2\right]^{3/2}}$$

$$= \boxed{\frac{2}{9x^{4/3}\left[1 + \frac{4}{9x^{2/3}}\right]^{3/2}}}$$

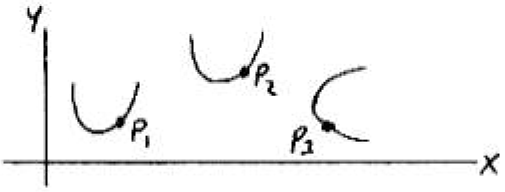
Find K, ρ at $(8,4)$.

$$\underset{x=8}{\Downarrow}$$

$$K(8) = \ldots \approx 0.0119$$
$$\rho(8) = \frac{1}{K(8)} \approx 84.3$$

} as for Method 3

(F) K is Invariant under Translations and Rotations



K is the same at P_1, P_2, and P_3.

Our K formulas didn't depend directly on $\vec{r}$.